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On the Positive Definite Solutions of Nonlinear Matrix Equation

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***Abstract*—**In this paper, the nonlinear matrix equation is investigated. Based on the fixed-point theory, the boundary and the existence of the solution with the case $r > -\delta_i$ are discussed. An algorithm that avoids matrix inversion with the case $-1 < \delta_i < 0$ is proposed.

***Keywords*—**Nonlinear matrix equation, Positive definite solution, The maximal-minimal solution, Iterative method, Free-inversion.

I. Introduction

In this paper, we consider the nonlinear matrix equation

$$X^r + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = I \qquad (1)$$

where r, m are positive integers, I is an $n \times n$ identity matrix, $A_i (i = 1, 2, \cdots, m)$ are $n \times n$ nonsingular complex matrices and A_i^* is the conjugate transpose of A_i. As nonlinear matrix equations have applications in control theory, dynamic programming ladder networks and other fields, similar kinds of nonlinear matrix equations have been widely studied [1]-[4]. The case when $m = 1$, $\delta_i = -t_i$ is one of the important study fields of the numerical algebra [5], [6]. To solve such matrix equations, numerical solutions are usually taken into consideration.

Different iterative methods including a kind of inversion-free method have been put forward and improved [7]-[9].

A. M. Sarhan et al. have studied the extremal positive definite solutions of (1) when $-1 < \delta_i < 0$ holds [10]. They have given a necessary condition and a sufficient condition for the existence of the solution and also, several algorithms are derived to compute the extremal (maximal or minimal) positive definite solutions.

In this paper, we first continue to discuss (1) with $r > -\delta_i (\delta_i < 0)$ and then we have several results for the definite solution of the equation. We also give an iterative algorithm that avoids matrix inversion, and according to this method we obtain the maximal solution when $-1 < \delta_i < 0$. Finally, we use some numerical examples to illustrate our algorithm.

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The notations used in this paper are summarized as follows. $A \geq 0 (A > 0)$ means that matrix A is Hermitian positive semi-definite (definite); the maximal and minimal eigenvalues of A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively; $\|A\|$ and $\|A\|_F$ represent the spectral norm and the Frobenius norm of A, respectively. For matrix $A = (a_1, a_2, \cdots a_n) = (a_{ij})$, $vec(A)$ is a vector defined by $vec(A) = (a_1^T, a_2^T, \cdots a_n^T)^T$; $A \otimes B = (a_{ij} B)$ is a Kronecker product.

Lemma 1 If $A > B > 0$ (or $A \geq B > 0$), then $A^\alpha > B^\alpha$ (or $A^\alpha \geq B^\alpha > 0$) for all $\alpha \in (0,1]$, and $A^\alpha < B^\alpha$ (or $0 < A^\alpha \leq B^\alpha$) for all $\alpha \in [-1, 0)$ [11].

Lemma 2 If C and P are Hermitian matrices of the same order with $P > 0$, then $CPC + P^{-1} \geq 2C$ [12].

Lemma 3 If $0 < \alpha \leq 1$, and P and Q are positive definite matrices of the same order with $P, Q \geq bI > 0$ then $\|P^\alpha - Q^\alpha\| \leq \alpha b^{\alpha-1} \|P - Q\|$. Here $\|\bullet\|$ stands for one kind of matrix norm[11].

II. The Main Results

Now, we consider the following nonlinear matrix equation

$$X + \sum_{i=1}^{m} A_i^* X^{-t_i} A_i = I, \ 0 < t_i < 1 \qquad (2)$$

Theorem 1 If $r > -\delta_i (\delta_i < 0, i = 1, 2, \cdots m)$, then (1) is equivalent to (2).

Proof. Consider $X^r + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = I$. let $Y = X^r$, then $X = Y^{1/r}$, $X^{\delta_i} = Y^{\delta_i / r}$. The equation can be rewritten as $Y + \sum_{i=1}^{m} A_i^* Y^{\delta_i / r} A_i = I$. Since $r > -\delta_i$, $i = 1, 2, \cdots m$, we have $-1 < \frac{\delta_i}{r} < 0$, that is $0 < -\frac{\delta_i}{r} < 1$. Replace δ_i / r with $-t_i$, then $Y + \sum_{i=1}^{m} A_i^* Y^{-t_i} A_i = I$, $0 < t_i < 1$. So (1) is equivalent to (2).

As for $r > -\delta_i$, the solution of (1) can be obtained by (2). In the following theorems, we will discuss the solution of (2).

Theorem 2 If (2) has a HPD solution, then we have $X \in (T_1, T_2)$, where

$$T_1 = \max(M_1, M_2),\ T_2 = I - \sum_{i=1}^{m} A_i^* A_i,\ M_1 = \frac{1}{m} \sum_{i=1}^{m} (A_i A_i^*)^{1/t_i},\ M_2 = I - \sum_{i=1}^{m} A_i^* M_1^{-t_i} A_i \qquad (3)$$

Proof. If (2) has a HPD solution X, then $X = I - \sum_{i=1}^{m} A_i^* X^{-t_i} A_i < I$, i.e., $X < I$. Moreover, $X = I - \sum_{i=1}^{m} A_i^* X^{-t_i} A_i < I - \sum_{i=1}^{m} A_i^* A_i$. On the other hand, $\sum_{i=1}^{m} A_i^* X^{-t_i} A_i < I$ implies $A_i^* X^{-t_i} A_i < I$, hence $X > (A_i A_i^*)^{1/t_i}$ and $mX > \sum_{i=1}^{m} (A_i A_i^*)^{1/t_i}$, i.e., $X > \frac{1}{m}\sum_{i=1}^{m} (A_i A_i^*)^{1/t_i} = M_1$. Further, $X = I - \sum_{i=1}^{m} A_i^* X^{-t_i} A_i > I - \sum_{i=1}^{m} A_i^* M_1^{-t_i} A_i = M_2$. Let $T_1 = \max(M_1, M_2)$, $T_2 = I - \sum_{i=1}^{m} A_i^* A_i$, then we have $X \in (T_1, T_2)$.

Lemma 4 Let X and Y be positive definite matrices, satisfying $X \ge \alpha I$, $Y \ge \alpha I$ where α is a positive number. Then $\|Y^{-t} - X^{-t}\|_F \le t\alpha^{-t-1}\|Y - X\|_F$ where $0 < t < 1$ [10].

Theorem 3 If $M_1 + \sum_{i=1}^{m} A_i^* M_1^{-t_i} A_i < I$, then (2) has solutions in $\Omega = [T_1, T_2]$. And, if $\alpha I \le T_1 \le X \le T_2$ (where α is the minimum eigenvalue of T_1) and $\beta = \sum_{i=1}^{m} t_i \alpha^{-t_i-1}\|A_i\|_F^2 < 1$, then (2) has a unique HPD solution.

Proof. We consider the map $f(X) = I - \sum_{i=1}^{m} A_i^* X^{-t_i} A_i$, and $X \in \Omega = [T_1, T_2]$. Since $X < I$ and $X \ge T_1 \ge M_1$, $f(X) = I - \sum_{i=1}^{m} A_i^* X^{-t_i} A_i < I - \sum_{i=1}^{m} A_i^* A_i = T_2$ and $f(X) = I - \sum_{i=1}^{m} A_i^* X^{-t_i} A_i \ge I - \sum_{i=1}^{m} A_i^* M_1^{-t_i} A_i = M_2$. Since $M_1 + \sum_{i=1}^{m} A_i^* M_1^{-t_i} A_i \le I$, we have $f(X) \ge I - \sum_{i=1}^{m} A_i^* M_1^{-t_i} A_i \ge M_1$. Hence, $f(\Omega) \subseteq \Omega$, then by Brouwer's fixed point theory, (2) has solutions in Ω. For arbitrary $X, Y \in \Omega$, we have $X \ge \alpha I$, $Y \ge \alpha I$ and $f(X) - f(Y) = (I - \sum_{i=1}^{m} A_i^* X^{-t_i} A_i) - (I - \sum_{i=1}^{m} A_i^* Y^{-t_i} A_i) = \sum_{i=1}^{m} A_i^* (Y^{-t_i} - X^{-t_i}) A$. Then,

$$\|f(X) - f(Y)\|_F = \left\|\sum_{i=1}^{m} A_i^* (Y^{-t_i} - X^{-t_i}) A_i\right\|_F \le \sum_{i=1}^{m} \|A_i\|_F^2 \|Y^{-t_i} - X^{-t_i}\|_F .$$

By Lemma 4, we derive

$$\|f(X) - f(Y)\|_F \le \sum_{i=1}^{m} t_i \alpha^{-t_i-1}\|A_i\|_F^2 \|X - Y\|_F = \beta\|X - Y\|_F .$$

Since $\beta < 1$, by Banach's fixed point theory, (2) has an unique solution in Ω.

Algorithm 1 Consider the following algorithm

$$X_0 = \alpha I$$

$$X_{k+1} = I - \sum_{i=1}^{m} A_i^* X_k^{-t_i} A_i, k = 0,1,2,\cdots$$

Theorem 4 Suppose that $A_i (i = 1,2,\cdots)$ are nonsingular complex matrices, and we consider the sequence of positive definite matrices $\{X_k\}$ derived from Algorithm 1. If (2) has a PHD solution X and $\alpha > 1$, $\sum_{i=1}^{m} \alpha^{-t_i} A_i^* A_i > (1-\alpha) I$ (or $0 < \alpha < 1$, $\sum_{i=1}^{m} \alpha^{-t_i} A_i^* A_i < (1-\alpha) I$), then $\{X_k\}$ is monotonic decreasing (or increasing) and converges to the maximal solution (or the minimal solution).

Proof. Compare to the proof of Theorem 1 and Lemma 4 in [10], and replace δ_i with $-t_i$, $r = 1$, we can easily get the result.

III. An Iterative Method That Avoids Matrix Inversion for $-1 < \delta_i < 0$

We consider the following nonlinear matrix equation

$$X^r + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = I, \quad (-1 < \delta_i < 0, i = 1,2,\cdots,m) \tag{4}$$

Let $\eta_i = -\delta_i$, $i = 1,2,\cdots m$. Since $-1 < \delta_i < 0$, then $0 < \eta_i < 1$.

Algorithm 2 Consider the iterative algorithm

$$Y_0 = I$$

$$X_k = (I - \sum_{i=1}^{m} A_i^* Y_k^{\eta_i} A_i)^{1/r}$$

$$Y_{k+1} = Y_k(2I - X_k Y_k), \quad k = 0,1,2,\cdots$$

Theorem 5 Let $A_i (i = 1,2,\cdots,m)$ be nonsingular complex matrices and satisfy the condition $\sum_{i=1}^{m} A_i^* A_i \le I$. And if (4) has a positive definite solution X, then the sequence of positive definite matrices $\{X_k\}$ derived from Algorithm 2 is monotonic decreasing and converges to the maximal solution X_L.

Proof. Let X be a HPD solution of (4). Then $X \le I$ and hence $X = (I - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i)^{1/r} \le (I - \sum_{i=1}^{m} A_i^* A_i)^{1/r}$ that is, $X \le (I - \sum_{i=1}^{m} A_i^* A_i)^{1/r}$. Therefore, by algorithm 2, we have $X_0 = (I - \sum_{i=1}^{m} A_i^* A_i)^{1/r} \ge X, Y_0 = I \le X^{-1}$. By Lemma 2 and Lemma 3, we have $Y_1 = 2Y_0 - Y_0 X_0 Y_0 \le X_0^{-1} \le X^{-1}$ and $Y_1 - Y_0 = Y_0 - Y_0 X_0 Y_0 = Y_0(Y_0^{-1} - X_0)Y_0 = I - X_0 \ge 0$, that is, $Y_1 \le X^{-1}$ and $Y_1 \ge Y_0$. It follows from Lemma 1 that $X_1 = (I - \sum_{i=1}^{m} A_i^* Y_1^{\eta_i} A_i)^{1/r} \ge (I - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i)^{1/r} = X$ and $X_1 = (I - \sum_{i=1}^{m} A_i^* Y_1^{\eta_i} A_i)^{1/r} \le (I - \sum_{i=1}^{m} A_i^* Y_0^{\eta_i} A_i)^{1/r} = X_0$, that is, $X_0 \ge X_1 \ge X$ and $Y_0 \le Y_1 \le X^{-1}$. Assume that $X_{k-1} \ge X_k \ge X$ and $Y_{k-1} \le Y_k \le X^{-1}$, $k = 2,3,\cdots$, we have $Y_{k+1} = 2Y_k - Y_k X_k Y_k \le X_k^{-1} \le X^{-1}$ and $X_{k+1} = (I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{\eta_i} A_i)^{1/r} \ge (I - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i)^{1/r} = X$.

Further, since $Y_k \le X_{k-1}^{-1} \le X_k^{-1}$, i.e., $Y_k^{-1} \ge X_k$, we have $Y_{k+1} - Y_k = Y_k - Y_k X_k Y_k = Y_k(Y_k^{-1} - X_k)Y_k \ge 0$ and $X_{k+1} = (I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{\eta_i} A_i)^{1/r} \le (I - \sum_{i=1}^{m} A_i^* Y_k^{\eta_i} A_i)^{1/r} = X_k$. By induction, we derive the following results $X_0 \ge X_1 \ge \cdots \ge X_k \ge X$ and $Y_0 \le Y_1 \le \cdots Y_k \le X^{-1}$.

Apparently the limits of $\{X_k\}$ and $\{Y_k\}$ exist. Taking limit in the Algorithm 2 leads to $\lim_{k\to\infty} Y_k = (\lim_{k\to\infty} X_k)^{-1}$ and $(\lim X_k)^r + \sum_{i=1}^{m} A_i^* (\lim Y_k)^{\eta_i} A_i = I$, i.e., $(\lim X_k)^r + \sum_{i=1}^{m} A_i^* (\lim X_k)^{\delta_i} A_i = I$. So, we get $\lim_{k\to\infty} X_k$ is a HPD solution of (4). As $X_k \ge X, k = 0,1,2,\cdots$ holds for any HPD solution of (4), we derive $\lim_{k\to\infty} X_k = X_L$ where X_L stands for the maximal solution.

Theorem 6 After k iterative steps of Algorithm 2, if $\|I - X_k Y_k\| < \varepsilon$, then $\left\| X_k^r + \sum_{i=1}^{m} A_i^* X_k^{\delta_i} A_i - I \right\| < p\varepsilon$ where $p = \|T_1^{-1}\| \cdot \sum_{i=1}^{m} \eta_i \|A_i\|^2$.

Proof. Since $I = X_{k+1}^r + \sum_{i=1}^{m} A_i^* Y_{k+1}^{\delta_i} A_i$, then

$$X_k^r + \sum_{i=1}^{m} A_i^* X_k^{\delta_i} A_i - I = X_k^r - X_{k+1}^r + \sum_{i=1}^{m} A_i^* (X_k^{\delta_i} - Y_{k+1}^{\eta_i}) A_i$$

$$= \sum_{i=1}^{m} A_i^* (Y_{k+1}^{\eta_i} - Y_k^{\eta_i}) A_i + \sum_{i=1}^{m} A_i^* (X_k^{\delta_i} - Y_{k+1}^{\eta_i}) A_i$$

$$= \sum_{i=1}^{m} A_i^* (X_k^{\delta_i} - Y_k^{\eta_i}) A_i .$$

Use of Lemma 3, we have

$$\left\| X_k^r + \sum_{i=1}^{m} A_i^* X_k^{\delta_i} A_i - I \right\| = \left\| \sum_{i=1}^{m} A_i^* (X_k^{\delta_i} - Y_k^{\eta_i}) A_i \right\|$$

$$\le \sum_{i=1}^{m} \left\| A_i^* (X_k^{\delta_i} - Y_k^{\eta_i}) A_i \right\|$$

$$\le \sum_{i=1}^{m} \|A_i\|^2 \left\| X_k^{\delta_i} - Y_k^{\eta_i} \right\|$$

$$\le \left\| X_k^{-1} - Y_k \right\| \cdot \sum_{i=1}^{m} \|A_i\|^2 \eta_i .$$

Since $X_k \ge X \ge T_1$, which implies $\|X_k^{-1}\| \le \|T_1^{-1}\|$ and $\|X_k^{-1} - Y_k\| = \|X_k^{-1}(I - X_k Y_k)\| \le \|X_k^{-1}\| \cdot \|I - X_k Y_k\| \le \|T_1^{-1}\| \cdot \|I - X_k Y_k\|$.

Then, $\left\| X_k^r + \sum_{i=1}^{m} A_i^* X_k^{\delta_i} A_i - I \right\| \le \|I - X_k Y_k\| \cdot \|T_1^{-1}\| \cdot \sum_{i=1}^{m} \|A_i\|^2 \eta_i < p\varepsilon$.

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