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Intuitionistic Fuzzy Subalgebras (Ideals) with Thresholds (λ, μ) of BCI-Algebras

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***Abstract*—**Based on the theory of intuitionistic fuzzy sets, the concepts of intuitionistic fuzzy subalgebras with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ) of BCI-algebras are introduced and some properties of them are discussed.

***Keywords*—**BCI-algebra, intuitionistic fuzzy set, intuitionistic fuzzy subalgebra with thresholds (λ, μ), intuitionistic fuzzy ideal with thresholds (λ, μ).

I. Introduction

THE notions of BCK/BCI-algebras were introduced by Iséki [1], [2] and were extensively investigated by many researchers. They are two important classes of logical algebras. The concept of fuzzy sets was introduced by Zadeh [3], which had been extensively applied to many mathematical fields. In 1991, Xi [4] applied the concept to BCK-algebras. From then on Jun, Meng, E. H. Roh, H. S. Kim [5]-[9] applied the concept to the ideals theory of BCK-algebras. K. Atannassov [10] later introduced the concept of intuitionistic fuzzy sets, with the development of this theory; it was applied to algebras by several researchers recently. K. Hur [11] investigated intuitionistic fuzzy subgroups and subrings, some meaningful results are obtained.

In this paper, we introduce the notions of intuitionistic fuzzy subalgebras with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ) of BCI-algebras and investigate their properties. We discuss the relations between intuitionistic fuzzy subalgebras with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ). The necessary and sufficient conditions about that an intuitionistic fuzzy set on BCI-algebra is an intuitionistic fuzzy ideal with thresholds (λ,μ) on it are given, the intersection and Cartesian product of intuitionistic fuzzy ideals with thresholds (λ,μ) on BCI-algebra are still intuitionistic fuzzy ideals with thresholds (λ,μ) of it are proved.

II. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is called a BCI- algebra if it satisfies the following axioms:

(BCI-1) $((x*y)*(x*z))*(z*y)=0,$

(BCI-2) $(x*(x*y))*y=0,$

(BCI-3) $x*x=0,$

(BCI-4) $x*y=0$ and $y*x=0$ imply $x=y$,

for all $x,y,z\in X$. In a BCI-algebra X, we can define a partial ordering $\le$ by putting $x\le y$ if and only if $x*y=0.$

If a BCI-algebra X satisfies the identity: $0*x=0$, for all $x\in X$, then X is called a BCK-algebra.

In any BCI-algebra X, the following hold:

(1) $(x*y)*z=(x*z)*y,$

(2) $x*0=x,$

(3) $0*(x*y)=(0*x)*(0*y),$

for all $x,y,z\in X$.

In this paper, X always means a BCI-algebra unless otherwise specified. For more details of BCI-algebras we refer the reader to Meng [7]. A nonempty subset I of X is called an ideal of X if $(I_1): 0\in I$, $(I_2): x*y\in I$ and $y\in I$ imply $x\in I$. An ideal I of X is called a closed ideal if (I_3): $x\in I$ imply $0*x\in I$. A nonempty subset S of X is called a subalgebra of X if the constant 0 of X is in S, and (I_4): $x*y\in S$ for any $x,y\in S$.

Definition 1 [10] Let S be any set. An intuitionistic fuzzy subset A of S is an object of the following form $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$, where $\mu_A: S\to[0,1]$ and $\nu_A: S\to[0,1]$ define the degree of membership and the degree of non-membership of the element $x\in S$ respectively and for every $x\in S$, $0\le\mu_A(x)+\nu_A(x)\le 1$..

Denote $\langle I\rangle=\{\langle a,b\rangle: a,b\in[0,1]\}$.

Definition 2 Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$ be an intuitionistic fuzzy set in a set S. For $\langle\alpha,\beta\rangle\in\langle I\rangle$, the set $A_{\langle\alpha,\beta\rangle}=\{x\in S:\mu_A(x)\ge\alpha,\nu_A(x)\le\beta\}$ is called a cut set of A.

Proposition 1 [8] Every fuzzy ideal A of X is order reversing.

Proposition 2 [9] Let A be a fuzzy idea of X . Then $x*y\le z$ implies $A(x)\ge A(y)\wedge A(z)$ for all $x,y,z\in X$.

III. Intuitionistic Fuzzy Subalgebras (Ideals) with Thresholds (λ,μ) of BCI- Algebras

Definition 3 Let $\lambda,\mu\in(0,1]$ and $\lambda<\mu$. An intuitionistic fuzzy set A in X is said to be an intuitionistic fuzzy

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subalgebra with thresholds (λ, μ) of X if the following are satisfied:

$\mu_A(x*y) \vee \lambda \geq \mu_A(x) \wedge \mu_A(y) \wedge \mu,$

$\nu_A(x*y) \wedge \mu \leq \nu_A(x) \vee \nu_A(y) \vee \lambda,$

for all $x, y \in X$.

Definition 4 Let $\lambda, \mu \in (0,1]$ and $\lambda < \mu$. An intuitionistic fuzzy set A in X is said to be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X if the following are satisfied:

(IF_1) $\mu_A(0) \vee \lambda \geq \mu_A(x) \wedge \mu,$

(IF_2) $\mu_A(x) \vee \lambda \geq \mu_A(x*y) \wedge \mu_A(y) \wedge \mu,$

(IF_3) $\nu_A(0) \wedge \mu \leq \nu_A(x) \vee \lambda,$

(IF_4) $\nu_A(x) \wedge \mu \leq \nu_A(x*y) \vee \nu_A(y) \vee \lambda,$

for all $x, y \in X$.

Definition 5 Let $\lambda, \mu \in (0,1]$ and $\lambda < \mu$. An intuitionistic fuzzy ideal A with thresholds (λ, μ) in X is said to be an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of X if the following are satisfied:

$\mu_A(0*x) \vee \lambda \geq \mu_A(x) \wedge \mu,$

$\nu_A(0*x) \wedge \mu \leq \nu_A(x) \vee \lambda,$

for all $x \in X$.

Proposition 3 Let A be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X. If $x \leq y$ holds in X, then $\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \mu,\ \nu_A(x) \wedge \mu \leq \nu_A(y) \vee \lambda.$

Proof. For all $x, y \in X$, if $x \leq y$, then $x*y = 0$, so by Definition 4,

$$\begin{aligned}\mu_A(x) \vee \lambda &\geq \mu_A(x*y) \wedge \mu_A(y) \wedge \mu \\ &= \mu_A(0) \wedge \mu_A(y) \wedge \mu,\end{aligned}$$

$$\begin{aligned}\nu_A(x) \wedge \mu &\leq \nu_A(x*y) \vee \nu_A(y) \vee \lambda \\ &= \nu_A(0) \vee \nu_A(y) \vee \lambda.\end{aligned}$$

Since $\mu_A(x) \vee \lambda \geq \lambda,\ \nu_A(x) \wedge \mu \leq \mu,$ therefore

$$\begin{aligned}\mu_A(x) \vee \lambda &\geq (\mu_A(0) \wedge \mu_A(y) \wedge \mu) \vee \lambda \\ &= (\mu_A(0) \vee \lambda) \wedge (\mu_A(y) \vee \lambda) \wedge (\mu \vee \lambda) \\ &\geq (\mu_A(y) \wedge \mu) \wedge \mu_A(y) \wedge \mu \\ &= \mu_A(y) \wedge \mu,\end{aligned}$$

$$\begin{aligned}\nu_A(x) \wedge \mu &\leq (\nu_A(0) \vee \nu_A(y) \vee \lambda) \wedge \mu \\ &= (\nu_A(0) \wedge \mu) \vee (\nu_A(y) \wedge \mu) \vee (\lambda \wedge \mu) \\ &\leq (\nu_A(y) \vee \lambda) \vee \nu_A(y) \vee \lambda \\ &= \nu_A(y) \vee \lambda.\end{aligned}$$

Proposition 4 Let A be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X. If the inequality $x*y \leq z$ holds in X, then for all $x, y, z \in X$,

$\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \mu_A(z) \wedge \mu,$

$\nu_A(x) \wedge \mu \leq \nu_A(y) \vee \nu_A(z) \vee \lambda$.

Proof. For all $x, y, z \in X$, if $x*y \leq z$, then

$$\begin{aligned}\mu_A(x) \vee \lambda &= (\mu_A(x) \vee \lambda) \vee \lambda \\ &\geq (\mu_A(x*y) \wedge \mu_A(y) \wedge \mu) \vee \lambda \\ &= (\mu_A(x*y) \vee \lambda) \wedge (\mu_A(y) \vee \lambda) \wedge (\mu \vee \lambda) \\ &\geq \mu_A(y) \wedge \mu_A(z) \wedge \mu,\end{aligned}$$

$$\begin{aligned}\nu_A(x) \wedge \mu &= (\nu_A(x) \wedge \mu) \wedge \mu \\ &\leq (\nu_A(x*y) \vee \nu_A(y) \vee \lambda) \wedge \mu \\ &= (\nu_A(x*y) \wedge \mu) \vee (\nu_A(y) \wedge \mu) \vee (\lambda \wedge \mu) \\ &\leq \nu_A(y) \vee \nu_A(z) \vee \lambda.\end{aligned}$$

Proposition 5 Let X be a BCK-algebra. Any intuitionistic fuzzy ideal A with thresholds (λ, μ) of X must be an intuitionistic fuzzy subalgebra with thresholds (λ, μ) of X.

Proof. Since $x*y \leq x$, by Proposition 3, we get

$$\begin{aligned}\mu_A(x*y) \vee \lambda &\geq \mu_A(x) \wedge \mu \\ &\geq \mu_A(x) \wedge \mu_A(y) \wedge \mu,\end{aligned}$$

$$\begin{aligned}\nu_A(x*y) \wedge \mu &\leq \nu_A(x) \vee \lambda \\ &\leq \nu_A(x) \vee \nu_A(y) \vee \lambda.\end{aligned}$$

It means that A is an intuitionistic fuzzy subalgebra with thresholds (λ, μ) of X.

Proposition 6 Let X be a BCK-algebra. An intuitionistic fuzzy subalgebra A with thresholds (λ, μ) of X is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X if and only if, for all $x, y, z \in X$, the inequality $x*y \leq z$ holds in X implies

$\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \mu_A(z) \wedge \mu,$

$\nu_A(x) \wedge \mu \leq \nu_A(y) \vee \nu_A(z) \vee \lambda$.

Proof. Necessity: It follows from Proposition 4. Sufficiency: Assume that A is an intuitionistic fuzzy subalgebra with thresholds (λ, μ) of X, and satisfying that $x*y \leq z$ implies

$\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \mu_A(z) \wedge \mu,$

$\nu_A(x) \wedge \mu \leq \nu_A(y) \vee \nu_A(z) \vee \lambda$.

Since $0*x \leq x$ and $x*(x*y) \leq y$, it follows that

$\mu_A(0) \vee \lambda \geq \mu_A(x) \wedge \mu,$

$\nu_A(0) \wedge \mu \leq \nu_A(x) \vee \lambda,$

$\mu_A(x) \vee \lambda \geq \mu_A(x*y) \wedge \mu_A(y) \wedge \mu,$

$\nu_A(x) \wedge \mu \leq \nu_A(x*y) \vee \nu_A(y) \vee \lambda$.

Hence A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X.

Proposition 7 Let A be a fuzzy set of X, then A is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X if and only if, for all $\langle\alpha,\beta\rangle\in\langle I\rangle$, where $\alpha,\beta\in(\lambda,\mu]$, $A_{\langle\alpha,\beta\rangle}$ is either empty or an ideal of X.

Proof. Suppose that A is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X and $A_{\langle\alpha,\beta\rangle}\neq\varnothing$, for any $\langle\alpha,\beta\rangle\in\langle I\rangle$. It is clear that $0\in A_{\langle\alpha,\beta\rangle}$. Let $x*y\in A_{\langle\alpha,\beta\rangle}$ and $y\in A_{\langle\alpha,\beta\rangle}$,

Then $\mu_A(x*y)\geq\alpha,\ \mu_A(y)\geq\alpha,\ \nu_A(x*y)\leq\beta,\ \nu_A(y)\leq\beta$. It follows from (IF_2) that

$\mu_A(x)\vee\lambda\geq\mu_A(x*y)\wedge\mu_A(y)\wedge\mu\geq\alpha$,

$\nu_A(x)\wedge\mu\leq\nu_A(x*y)\vee\nu_A(y)\vee\lambda\leq\beta$.

Namely, $\mu_A(x)\geq\alpha,\ \nu_A(x)\leq\beta$ and $x\in A_{\langle\alpha,\beta\rangle}$. This shows that $A_{\langle\alpha,\beta\rangle}$ is an ideal of X.

Conversely, suppose that for each $\langle\alpha,\beta\rangle\in\langle I\rangle$, where $\alpha,\beta\in(\lambda,\mu]$, $A_{\langle\alpha,\beta\rangle}$ is either empty or an ideal of X.

For any $x\in X$, let $\alpha=\mu_A(x)\wedge\mu,\quad\beta=\nu_A(x)\vee\lambda$ .Then $\mu_A(x)\geq\alpha$,$\nu_A(x)\leq\beta$, hence $x\in A_{\langle\alpha,\beta\rangle}$ and $A_{\langle\alpha,\beta\rangle}$ is an ideal of X, therefore $0\in A_{\langle\alpha,\beta\rangle}$, i.e., $\mu_A(0)\geq\alpha$ and $\nu_A(0)\leq\beta$.

We get

$\mu_A(0)\vee\lambda\geq\mu_A(0)\geq\alpha=\mu_A(x)\wedge\mu,$

$\nu_A(0)\wedge\mu\leq\nu_A(0)\leq\beta=\nu_A(x)\vee\lambda,$

i.e., $\mu_A(0)\vee\lambda\geq\mu_A(x)\wedge\mu$ and $\nu_A(0)\wedge\mu\leq\nu_A(x)\vee\lambda$ for all $x\in X$. Now we only need to show that A satisfies (IF_2) and (IF_4).

Let $\alpha=\mu_A(x*y)\wedge\mu_A(y)\wedge\mu,\ \beta=\nu_A(x*y)\vee\nu_A(y)\vee\lambda.$

Then $\mu_A(x*y)\geq\alpha,\ \mu_A(y)\geq\alpha,\ \nu_A(x*y)\leq\beta,\ \nu_A(y)\leq\beta,$ hence $x*y\in A_{\langle\alpha,\beta\rangle}$ and $y\in A_{\langle\alpha,\beta\rangle}$. Since $A_{\langle\alpha,\beta\rangle}$ is an ideal of X , thus $x\in A_{\langle\alpha,\beta\rangle}$, i.e., $\mu_A(x)\geq\alpha,\ \nu_A(x)\leq\beta$. We get

$\mu_A(x)\vee\lambda\geq\mu_A(x)\geq\alpha=\mu_A(x*y)\wedge\mu_A(y)\wedge\mu,$

$\nu_A(x)\wedge\mu\leq\nu_A(x)\leq\beta=\nu_A(x*y)\vee\nu_A(y)\vee\lambda.$

i.e., $\mu_A(x)\vee\lambda\geq\mu_A(x*y)\wedge\mu_A(y)\wedge\mu,$

$\nu_A(x)\wedge\mu\leq\nu_A(x*y)\vee\nu_A(y)\vee\lambda.$

Summarizing the above arguments, A is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X.

Proposition 8 Let A be a fuzzy set of X, then A is an intuitionistic fuzzy closed ideal with thresholds (λ,μ) of X if and only if, for all $\langle\alpha,\beta\rangle\in\langle I\rangle$, where $\alpha,\beta\in(\lambda,\mu]$, $A_{\langle\alpha,\beta\rangle}$ is either empty or a closed ideal of X.

Proof. It is similar to the proof of Proposition 7 and omitted.

Definition 9 [10] Let S be any set.

If $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle: x\in S\}$ and $B=\{\langle x,\mu_B(x),\nu_B(x)\rangle: x\in S\}$ be any two intuitionistic fuzzy subsets of S **,** then

$A\subseteq B$ iff $\forall x\in S,\mu_A(x)\leq\mu_B(x)$ and $\nu_A(x)\geq\nu_B(x)$,

$A=B$ iff $\forall x\in S,\mu_A(x)=\mu_B(x)$ and $\nu_A(x)=\nu_B(x)$,

$$A\cap B=\{\langle x,(\mu_A\cap\mu_B)(x),(\nu_A\cup\nu_B)(x)\rangle: x\in S\}$$
$$=\{\langle x,\mu_A(x)\wedge\mu_B(x),\nu_A(x)\vee\nu_B(x)\rangle: x\in S\},$$

$$A\cup B=\{\langle x,(\mu_A\cup\mu_B)(x),(\nu_A\cap\nu_B)(x)\rangle: x\in S\}$$
$$=\{\langle x,\mu_A(x)\vee\mu_B(x),\nu_A(x)\wedge\nu_B(x)\rangle: x\in S\}.$$

Proposition 10 Let A and B be two intuitionistic fuzzy ideals with thresholds (λ,μ) of X. Then $A\cap B$ is also an intuitionistic fuzzy ideal with thresholds (λ,μ) of X.

Proof. For all $x,y\in X$. Then

$$\begin{aligned}\mu_{A\cap B}(0)\vee\lambda&=(\mu_A(0)\wedge\mu_B(0))\vee\lambda\\&=(\mu_A(0)\vee\lambda)\wedge(\mu_B(0)\vee\lambda)\\&\geq(\mu_A(x)\wedge\mu)\wedge(\mu_B(x)\wedge\mu)\\&=(\mu_A(x)\wedge\mu_B(x))\wedge\mu\\&=\mu_{A\cap B}(x)\wedge\mu\end{aligned}$$

and

$$\begin{aligned}&\mu_{A\cap B}(x)\vee\lambda\\&=(\mu_A(x)\wedge\mu_B(x))\vee\lambda\\&=(\mu_A(x)\vee\lambda)\wedge(\mu_B(x)\vee\lambda)\\&\geq(\mu_A(x*y)\wedge\mu_A(y)\wedge\mu)\wedge(\mu_B(x*y)\wedge\mu_B(y)\wedge\mu)\\&=(\mu_A(x*y)\wedge\mu_B(x*y))\wedge(\mu_A(y)\wedge\mu_B(y))\wedge\mu\\&=\mu_{A\cap B}(x*y)\wedge\mu_{A\cap B}(y)\wedge\mu.\end{aligned}$$

Again

$$\begin{aligned}\nu_{A\cap B}(0)\wedge\mu&=(\nu_A(0)\vee\nu_B(0))\wedge\mu\\&=(\nu_A(0)\wedge\mu)\vee(\nu_B(0)\wedge\mu)\\&\leq(\nu_A(x)\vee\lambda)\vee(\nu_B(x)\vee\lambda)\\&=(\nu_A(x)\vee\nu_B(x))\vee\lambda\\&=\nu_{A\cap B}(x)\vee\lambda\end{aligned}$$

and

$$\begin{aligned}&\nu_{A\cap B}(x)\wedge\mu\\&=(\nu_A(x)\vee\nu_B(x))\wedge\mu\\&=(\nu_A(x)\wedge\mu)\vee(\nu_B(x)\wedge\mu)\\&\leq(\nu_A(x*y)\vee\nu_A(y)\vee\lambda)\vee(\nu_B(x*y)\vee\nu_B(y)\vee\lambda)\\&=(\nu_A(x*y)\vee\nu_B(x*y))\vee(\nu_A(y)\vee\nu_B(y))\vee\lambda\\&=\nu_{A\cap B}(x*y)\vee\nu_{A\cap B}(y)\vee\lambda.\end{aligned}$$

Hence $A \cap B$ is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X.

In the following, we give two Propositions, which display close relations between the intuitionistic fuzzy subalgebras with thresholds (λ, μ), intuitionistic fuzzy ideals with thresholds (λ, μ) and intuitionistic fuzzy closed ideals with thresholds (λ, μ) of BCI-algebras.

Proposition 11 Let A be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X, then A is an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of X if and only if, A is an intuitionistic fuzzy subalgebra with thresholds (λ, μ) of X.

Proof. Suppose that A is an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of X.

Therefore, for all $x, y \in X$,

$$\begin{aligned} \mu_A((x*y)*x) \vee \lambda &= \mu_A((x*x)*y) \vee \lambda \\ &= \mu_A(0*y) \vee \lambda \\ &\geq \mu_A(y) \wedge \mu \end{aligned}$$

and

$$\begin{aligned} \nu_A((x*y)*x) \wedge \mu &= \nu_A((x*x)*y) \wedge \mu \\ &= \nu_A(0*y) \wedge \mu \\ &\leq \nu_A(y) \vee \lambda . \end{aligned}$$

Since A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X, we get:

$$\begin{aligned} \mu_A(x*y) \vee \lambda &\geq \left(\mu_A((x*y)*x) \wedge \mu_A(x) \wedge \mu\right) \vee \lambda \\ &= \left(\mu_A((x*y)*x) \vee \lambda\right) \wedge \left(\mu_A(x) \vee \lambda\right) \wedge (\mu \vee \lambda) \\ &\geq \mu_A(x) \wedge \mu_A(y) \wedge \mu \end{aligned}$$

and

$$\begin{aligned} \nu_A(x*y) \wedge \mu &\leq \left(\nu_A((x*y)*x) \vee \nu_A(x) \vee \lambda\right) \wedge \mu \\ &= \left(\nu_A((x*y)*x) \wedge \mu\right) \vee \left(\nu_A(x) \wedge \mu\right) \vee (\lambda \wedge \mu) \\ &\leq \nu_A(x) \vee \nu_A(y) \vee \lambda . \end{aligned}$$

Hence A is an intuitionistic fuzzy subalgebra with thresholds (λ, μ) of X.

Conversely, Suppose that A is an intuitionistic fuzzy subalgebra with thresholds (λ, μ) of X.

Therefore, for all $x \in X$,

$$\begin{aligned} \mu_A(0*x) \vee \lambda &\geq \left(\mu_A(0) \wedge \mu_A(x) \wedge \mu\right) \vee \lambda \\ &= \left(\mu_A(0) \vee \lambda\right) \wedge \left(\mu_A(x) \vee \lambda\right) \wedge (\mu \vee \lambda) \\ &\geq \mu_A(x) \wedge \mu \end{aligned}$$

and

$$\begin{aligned} \nu_A(0*x) \wedge \mu &\leq \left(\nu_A(0) \vee \nu_A(x) \vee \lambda\right) \wedge \mu \\ &= \left(\nu_A(0) \wedge \mu\right) \vee \left(\nu_A(x) \wedge \mu\right) \vee (\lambda \wedge \mu) \\ &\leq \nu_A(x) \vee \lambda. \end{aligned}$$

We have known that A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X, hence A is an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of X.

Proposition 12 Let K be an intuitionistic fuzzy subalgebra with thresholds (λ, μ) of X. If A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X, then $K \cap A$ is an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of K.

Proof. For all $x, y \in K$, we have

$$\begin{aligned} \mu_{K\cap A}(x*y) \vee \lambda &\geq \mu_A(x*y) \vee \lambda \\ &\geq \mu_A(x) \wedge \mu_A(y) \wedge \mu \\ &= \mu_{K\cap A}(x) \wedge \mu_{K\cap A}(y) \wedge \mu \end{aligned}$$

and

$$\begin{aligned} \nu_{K\cap A}(x*y) \wedge \mu &\leq \nu_A(x*y) \wedge \mu \\ &\leq \nu_A(x) \vee \nu_A(y) \vee \lambda \\ &= \nu_{K\cap A}(x) \vee \nu_{K\cap A}(y) \vee \lambda. \end{aligned}$$

Thus

$$\begin{aligned} \mu_{K\cap A}(0) \vee \lambda &\geq \mu_{K\cap A}(x*x) \vee \lambda \\ &\geq \mu_{K\cap A}(x) \wedge \mu \end{aligned}$$

and

$$\begin{aligned} \nu_{K\cap A}(0) \wedge \mu &\leq \nu_{K\cap A}(x*x) \wedge \mu \\ &\leq \nu_{K\cap A}(x) \vee \lambda . \end{aligned}$$

Since A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X. Therefore, for all $x, y \in K$, we have

$$\begin{aligned} \mu_{K\cap A}(x) \vee \lambda &\geq \mu_A(x) \vee \lambda \\ &\geq \mu_A(x*y) \wedge \mu_A(y) \wedge \mu \\ &= \mu_{K\cap A}(x*y) \wedge \mu_{K\cap A}(y) \wedge \mu \end{aligned}$$

and

$$\begin{aligned} \nu_{K\cap A}(x) \wedge \mu &\leq \nu_A(x) \wedge \mu \\ &\leq \nu_A(x*y) \vee \nu_A(y) \vee \lambda \\ &= \nu_{K\cap A}(x*y) \vee \nu_{K\cap A}(y) \vee \lambda . \end{aligned}$$

This means that $K \cap A$ is an intuitionistic fuzzy ideal with thresholds (λ, μ) of K. Therefore, $K \cap A$ is an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of K.

Definition 6 Let A and B be two intuitionistic fuzzy sets of a set X. The Cartesian product of A and B is defined by $A \times B = \left\{ \left\langle \mu_{A\times B}(x, y), \nu_{A\times B}(x, y) \right\rangle : x, y \in X \right\}$ where

$$\mu_{A\times B}(x, y) = \mu_A(x) \wedge \mu_B(y), \nu_{A\times B}(x, y) = \nu_A(x) \vee \nu_B(y).$$

Proposition 13 Let A and B be two intuitionistic fuzzy ideals with thresholds (λ, μ) of X. Then $A \times B$ is also an intuitionistic fuzzy ideal with thresholds (λ, μ) of $X \times X$.

Proof. For all $(x,y) \in X \times X$, by Definition 4, we get

$$\mu_{A\times B}(0,0) \vee \lambda = \left(\mu_A(0) \wedge \mu_B(0)\right) \vee \lambda$$

$$= \left(\mu_A(0) \vee \lambda\right) \wedge \left(\mu_B(0) \vee \lambda\right)$$
$$\geq \left(\mu_A(x) \wedge \mu\right) \wedge \left(\mu_B(y) \wedge \mu\right)$$
$$= \mu_{A\times B}(x,y) \wedge \mu,$$

$$\nu_{A\times B}(0,0) \wedge \mu = \left(\nu_A(0) \vee \nu_B(0)\right) \wedge \mu$$
$$= \left(\nu_A(0) \wedge \mu\right) \vee \left(\nu_B(0) \wedge \mu\right)$$
$$\leq \left(\nu_A(x) \vee \lambda\right) \vee \left(\nu_B(y) \vee \lambda\right)$$
$$= \nu_{A\times B}(x,y) \vee \lambda,$$

for all $(x_1,x_2),(y_1,y_2) \in X \times X$, we have

$$\mu_{A\times B}(x_1,x_2) \vee \lambda$$
$$= \left(\mu_A(x_1) \wedge \mu_B(x_2)\right) \vee \lambda$$
$$= \left(\mu_A(x_1) \vee \lambda\right) \wedge \left(\mu_B(x_2) \vee \lambda\right)$$
$$\geq \mu_A(x_1 * y_1) \wedge \mu_A(y_1) \wedge \mu_B(x_2 * y_2) \wedge \mu_B(y_2) \wedge \mu$$
$$= \mu_A(x_1 * y_1) \wedge \mu_B(x_2 * y_2) \wedge \mu_A(y_1) \wedge \mu_B(y_2) \wedge \mu$$
$$= \mu_{A\times B}(x_1 * y_1, x_2 * y_2) \wedge \mu_{A\times B}(y_1,y_2) \wedge \mu,$$

$$\nu_{A\times B}(x_1,x_2) \wedge \mu$$
$$= \left(\nu_A(x_1) \vee \nu_B(x_2)\right) \wedge \mu$$
$$= \left(\nu_A(x_1) \wedge \mu\right) \vee \left(\nu_B(x_2) \wedge \mu\right)$$
$$\leq \nu_A(x_1 * y_1) \vee \nu_A(y_1) \vee \nu_B(x_2 * y_2) \vee \nu_B(y_2) \vee \lambda$$
$$= \nu_A(x_1 * y_1) \vee \nu_B(x_2 * y_2) \vee \nu_A(y_1) \vee \nu_B(y_2) \vee \lambda$$
$$= \nu_{A\times B}(x_1 * y_1, x_2 * y_2) \vee \nu_{A\times B}(y_1,y_2) \vee \lambda.$$

Hence $A \times B$ is an intuitionistic fuzzy ideal with thresholds (λ, μ) of $X \times X$.

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