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Best Coapproximation in Fuzzy Anti-n-Normed Spaces

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Abstract—The main purpose of this paper is to consider the new kind of approximation which is called as t-best coapproximation in fuzzy n-normed spaces. The set of all t-best coapproximation define the t-coproximinal, t-co-Chebyshev and F-best coapproximation and then prove several theorems pertaining to this sets.

Keywords—Fuzzy-n-normed space, best coapproximation, co-proximinal, co-Chebyshev, F-best coapproximation, orthogonality

I. Introduction

The concept of best coapproximation was introduced by Franchetti and Furi [2], in order to study some characteristic properties of real Hilbert spaces, and such problems were considered further by Papini and Singer, [12] and Rao and Saravanan [13].The concept of n-norm on a linear space has been introduced and developed by Gähler in [3], [4]. Following Misiak [10], Malčeski [9] and Gunawan and Mashadi [5] developed the theory of n-normed space.The concept of fuzzy norm was initiated by Katsaras in [7] and further, Narayanan and Vijayabalaji [11] introduced the concept of fuzzy n-normed linear space. Moreover, Vijayabalaji and Thillaigovindan [17] introduced the notion of convergent sequence and Cauchy sequence in fuzzy n-normed linear space. In [6]Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [1] and investigated their important properties. In [8] Kavikumar et. al. introduced the notion of fuzzy anti-n-normed linear space. Further, Surender Reddy [15] introduced the notion of convergent sequence and Cauchy sequence in fuzzy anti-n-normed linear space. The set of all t-best approximations on fuzzy normed linear spaces was initiated and studied by Vaezpour and Karimi [16].The set of all t-best approximations on fuzzy anti-n-normed linear space was introduced in [14]. In this paper we consider the set of all t-best coapproximation in fuzzy anti-n-normed spaces and then prove several theorems pertaining to this set.

II. Preliminaries

Definition 1: [17]. Let $n \in \mathbb{N}$ (natural numbers) and X be a real linear space of dimension $d \geq n$. (Here we allow d to be infinite). A real valued function $\| \bullet, \bullet, \cdots, \bullet \|$ on $X \times X \times ... \times X$ (n times)=X^n satisfying the following four properties:

- $\| x_1, x_2, ..., x_n \| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.
- $\| x_1, x_2, ..., x_n \|$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
- $\| x_1, x_2, ..., cx_n \| = | c | \, \| x_1, x_2, ..., x_n \|$, for any real c.
- $\| x_1, x_2, ..., x_{n-1}, y + z \| \leq \| x_1, x_2, ..., x_{n-1}, y \| + \| x_1, x_2, ..., x_{n-1}, z \|$

is called an n-norm on X and the pair $(X, \| \bullet, ..., \bullet \|)$ is called an n-normed linear space.

Definition 2: [17]. Let X be a linear space over a real field $\mathbb{F}$. A fuzzy subset N of $X^n \times [0, \infty)$ is called a fuzzy n-norm on X if and only if:

- $N(x_1, x_2, ..., x_n, t) > 0$.
- $N(x_1, x_2, ..., x_n, t) = 1 \Leftrightarrow x_1, x_2, ..., x_n$ are linearly dependent.
- $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
- $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0, c \in \mathbb{F}$(field)
- $N(x_1, x_2, ..., x_n + x'_n, s + t) \geq N(x_1, x_2, ..., x_n, t) * N(x_1, x_2, ..., x'_n, t)$ for all $s, t \in \mathbb{R}$
- $N(x_1, x_2, ..., x_n, \cdot)$ is left continuous and non-decreasing function of $\mathbb{R}$ such that $\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1$.

Then (X, N) is called a fuzzy n-normed linear space.

Definition 3: [8] Let X be a linear space over a real field $\mathbb{F}$. A fuzzy subset N of $X^n \times [0, \infty)$ is called a fuzzy anti n-norm on X if and only if:

- for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 1$.
- for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 0 \Leftrightarrow x_1, x_2, ..., x_n$ are linearly dependent.
- $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
- for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0, c \in \mathbb{F}$(field)
- for all $s, t \in \mathbb{R}$, $N(x_1, x_2, ..., x_n + x'_n, s + t) \leq \max\{N(x_1, x_2, ..., x_n, s), N^*(x_1, x_2, ..., x'_n, t)\}$
- $N(x_1, x_2, ..., x_n, \cdot)$ is right continuous and non-increasing function of $\mathbb{R}$ such that

$$\lim_{t \to \infty} N(x_1, x_2, ..., x_n) = 0$$

Then (X, N) is called a fuzzy anti n-normed linear space.
To strengthen the above definition, we present the following example.

Example 1: [8] Let $(X, \| \bullet, \bullet, \cdots, \bullet \|)$ be a n-normed linear space
Define,

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} 1 - \frac{t}{t + \| x_1, x_2, ..., x_n \|} & \text{when } t (> 0) \in \mathbb{R}, \forall x \in X \\ 1 & \text{when } t (\leq 0) \in \mathbb{R}, \forall x \in X \end{cases}$$

Then (X, N) is a fuzzy anti n-normed linear space.

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Definition 4: [15]A sequence $\{x_k\}$ in a fuzzy anti-n-normed linear space (X, N) is said to be convergent to $x \in X$ ig given $t > 0, 0 < r < 1$, there eixsts an integer $n_0 \in \mathbb{N}$ such that

$$N(x_1, x_2, \cdots, x_{n_1}, x_k - x, t) < r, \forall k \geq n_0.$$

Theorem 1: [15]In a fuzzy anti-n-normed linear space (X, N), a sequence $\{x_k\}$ converges to $x \in X$ if and only if

$$\lim_{k\to\infty} N(x_1, x_2, \cdots, x_{n_1}, x_k - x, t) = 0, \forall t > 0.$$

Definition 5: [15] Let (X, N) be a fuzzy anti-n-normed linear space. Let $\{x_k\}$ be a sequence in X then $\{x_k\}$ is said to be a Cauchy sequence if

$$\lim_{k\to\infty} N(x_1, x_2, \cdots, x_{n_1}, x_{k+p} - x_k, t) = 0, \forall t > 0$$

and $p = 1, 2, 3, \cdots$. A fuzzy anti-n-normed linear space (X, N) is said to be complete if every Cauchy sequence in X is convergent. A complete fuzzy anti-n-normed space (X, N) is called a fuzzy anti-n-Banach space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$B(x, r, t) = \{y \in X : N(x_1, x_2, \cdots, x_{n_1}, x - y, t) < r\}$,
$B[x, r, t] = \{y \in X : N(x_1, x_2, \cdots, x_{n_1}, x - y, t) \leq r\}$.

A subset A of X is said to be open if there exists $r \in (0, 1)$ such that $B(x, r, t) \subset A$ for all $x \in A$ and $t > 0$. A subset A of X is said to be closed if for any sequence $\{x_k\}$ in A converges to $x \in A$. i.e., $\lim_{k\to\infty} N(x_1, x_2, \cdots, x_{n_1}, x_k - x, t) = 0$, for all $t > 0$ implies that $x \in A$.

Corollary 1: [15] Let (X, N) be a fuzzy anti-n-normed linear space. Then N is a continuous function on

$$\underbrace{X \times X \times ... \times X}_{n} \times \mathbb{R}.$$

III. t-Best Coapproximation

Definition 6: Let A be a nonempty subset of fuzzy anti-n-normed space (X, N) and $t > 0$. For $x \in X$, an element $y_0 \in A$ is said to be a t-best coapproximation of x from A if $N(x_1, x_2, \cdots, x_{n-1}, y_0 - y, t) \leq N(x_1, x_2, \cdots, x_{n-1}, x - y, t)$, for all $y \in A$.The set of all elements of t-best coapproximation of x from A is denoted by $R_A^t(x)$; i.e., $R_A^t(x) = \{y_0 \in A : N(x_1, x_2, \cdots, x_{n-1}, y_0 - y, t) \leq N(x_1, x_2, \cdots, x_{n-1}, x - y, t), \forall y \in A\}$.

For $t > 0$ putting

$$\begin{aligned} \check{A}_x^t &= \{x \in X; N(x_1, x_2, \cdots, x_{n-1}, y, t) \\ &\leq N(x_1, x_2, \cdots, x_{n-1}, y - x, t) \forall y \in A\} \\ &= (R_A^t)^{-1}(\{0\}). \end{aligned}$$

It is clear $y_0 \in R_A^t(x)$ if and only if $x - y_0 \in \check{A}_x^t$.

Definition 7: Let A be a nonempty subset of a fuzzy anti-n-normed space (X, N). If for $t > 0$ and each $x \in X$ has at least (respectively exactly) one t-best coapproximation in A, then A is called a t-best coproximinal (respectively t-co-Chebyshev) set. Also A is called t-quasi-co-Chebyshev set if $R_A^t(x)$ is a compact set.

Theorem 2: Let (X, N) be a fuzzy anti-n-normed space and A be a subspace of X and $t > 0$. Then for each $x \in X$

(a) A is a t-coproximinal if and only if $X = A + \check{A}_x^t$.
(b) A is a t-co-Chebyshev subspace if and only if $X = A \oplus \check{A}_x^t$.

Proof: (a)($\Rightarrow$) Assume that A is t-coproximinal, $x \in X$ and $y_0 \in R_A^t(x)$.Then, $x - y_0 \in \check{A}_x^t$. Now, $x = y_0 + (x - y_0) \in A + \check{A}_x^t$. Hence $X = A + \check{A}_x^t$.
($\Leftarrow$) Let $x \in X = A + \check{A}_x^t$. $x = y_0 + \overline{y}$, $y_0 \in A$, $\overline{y} \in \check{A}_x^t$ and so $0 \in R_A^t(\overline{y}) = R_A^t(x - y_0)$. Since, $N(x_1, x_2, \cdots, x_{n-1}, 0 - (x - y_0), t) \leq N(x_1, x_2, \cdots, x_{n-1}, y - (x - y_0), t)$, so $N(x_1, x_2, \cdots, x_{n-1} y_0 - x, t) \leq N(x_1, x_2, \cdots, x_{n-1}, (y + y_0) - x, t)$ where $y + y_0 \in A$; hence $y_0 \in R_A^t(x)$. Therefore A is t-coproximinal.
(b)($\Rightarrow$) Suppose that A is t-co-Chebyshev subspace, $x \in X$, and $x = y_1 + \overline{y}_1 = y_2 + \overline{y}_2$, where $y_1, y_2 \in A$ and $\overline{y}_1, \overline{y}_2 \in A_x^t$. We show that $y_1 = y_2$ and $\overline{y}_1 = \overline{y}_2$. Since $x = y_1 + \overline{y}_1 = y_2 + \overline{y}_2$, then $x - y_1 = \overline{y}_1, x - y_2 = \overline{y}_2$, this implies that $y_1, y_2 \in R_A^t(x)$. Therefore $y_1 = y_2$, it follows that $\overline{y}_1 = \overline{y}_2$. Thus $X = A \oplus \check{A}_x^t$.
($\Leftarrow$) Let $X = A \oplus \check{A}_x^t$, and suppose for $x \in X$, there exist $y_1, y_2 \in R_A^t(x)$. Then $x - y_1, x - y_2 \in \check{A}_x^t$ and therefore, $x = y_1 + \overline{y}_1 = y_2 + \overline{y}_2$, where $\overline{y}_1 = x - y_1, \overline{y}_2 = x - y_2$. It follows that $y_1 = y_2$ and $\overline{y}_1 = \overline{y}_2$. ■

Theorem 3: Let A be a nonempty subset of a fuzzy anti-n-normed space (X, N). The for $t > 0$ and for each $x \in X$.

(a) $R_{A+y}^t(x + y) = R_A^t(x) + y$, for every $x, y \in X$.
(b) $R_{\alpha A}^{|\alpha|t}(\alpha x) = \alpha R_A^t$ for every $x \in X$ and $\alpha \in \mathbb{R}\backslash\{0\}$.
(c) A is t-coproximinal (respectively t-co-Chebyshev) if and only if $A + y$ is t-coproximinal (respectively t-co-Chebyshev), for any $y \in X$.
(d) A is t-coproximinal (respectively t-co-Chebyshev) if and only if αA is $\mid \alpha \mid t$-coproximinal (respectively $\mid \alpha \mid t$-co-Chebyshev), for any $y \in X$, for any given $\alpha \in \mathbb{R}\backslash\{0\}$.

Proof: (i) For any $x, y \in X$, $t > 0$, $y_0 \in R_{A+y}^t(x + y)$ if and only if
$N(x_1, x_2, \cdots, x_{n-1}, y_0 - (a + y), t) \leq N(x_1, x_2, \cdots, x_{n-1}, x + y - (a + y), t)$ for all $(a + y) \in A + y$ if and only if,
$N(x_1, x_2, \cdots, x_{n-1}, (y_0 - y) - a, t) \leq N(x_1, x_2, \cdots, x_{n-1}, x - a), t)$ for all $a \in A$, if and only if, $(y_0 - y) \in R_A^t(x)$, i.e., $y_0 \in R_A^t(x) + y$.
(ii) For any $x \in X$, $\alpha \in \mathbb{R}\backslash\{0\}$, and $t > 0$, $y_0 \in R_{\alpha A}^{|\alpha|t}(\alpha x)$ if and only if,
$N(x_1, x_2, \cdots, x_{n-1}, (y_0 - \alpha a, \mid \alpha \mid t) \leq N(x_1, x_2, \cdots, x_{n-1}, \alpha x - \alpha a), \alpha t)$ for all $a \in A$
if and only if $N(x_1, x_2, \cdots, x_{n-1}, (\frac{1}{\alpha} y_0 - a, \mid \alpha \mid t) \leq N(x_1, x_2, \cdots, x_{n-1}, x - a), t)$ for all $a \in A$ if and only if, $\frac{1}{\alpha} y_0 \in R_A^t(x)$ if and only if, $y_0 \in \alpha R_A^t(x)$. Therefore $R_{\alpha A}^{|\alpha|t}(\alpha x) = \alpha R_A^t$
(iii) Is an immediate consequence of (i)
(iv) Is an immediate consequence of (ii). ■

Corollary 2: Let M be a nonempty subspace of a fuzzy anti-n-normed space X. Then for $t > 0$ and each $x \in X$.

(a) $R_M^t(x + y) = R_M^t(x) + y$, for every $x, y \in X$.
(b) $R_M^{|\alpha|t}(\alpha x) = \alpha R_M^t$ for every $x \in X$ and $\alpha \in \mathbb{R}\backslash\{0\}$.

Proof: The proof is an immediate consequence of theorem 3 and this fact that $M + y = M$ and $\alpha M = M$ for all $y \in M$ and $\alpha \in \mathbb{R}\backslash\{0\}$. ■

Definition 8: For $x \in X$, $a \in A$, $0 < r < 1$, and $t > 0$, define

$$e_a^t(x) = N(x_1, x_2, \cdots, x_{n-1}, x - a, t)$$

Theorem 4: Let (X, N) be a fuzzy anti n-normed space, A be a subset of X, $x \in X\backslash\overline{A}$ and $t > 0$. Then we have

$$R^t_A(x) = [\bigcap_{a\in A} B[a, e^t_a(x), t]] \cap A.$$

Proof: By definition of $R^t_A(x)$ for each $a \in A$ we have

$$R^t_A(x) \subseteq [B[a, e^t_a(x), t]] \cap A$$

Therefore

$$R^t_A(x) \subseteq [\bigcap_{a\in A} B[a, e^t_a(x), t]] \cap A.$$

Conversely, let $y \in [\bigcap_{a\in A} B[a, e^t_a(x), t]] \cap A$, then we have $y \in A$, and for each $a \in A$, $N(x_1, x_2, \cdots, x_{n-1}, a - y, t) \leq e^t_a = N(x_1, x_2, \cdots, x_{n-1}, x - a, t)$, which implies that $y \in R^t_A(x)$. So $[\bigcap_{a\in A} B[a, e^t_a(x), t]] \cap A \subseteq R^t_A(x)$, which completes the proof. ■

Corollary 3: Let (X, N) be a fuzzy anti-n-normed space, A be a subset of X, $x \in X\backslash\overline{A}$ and $t > 0$. Then

(a) The set $R^t_A(x)$ is t-bounded.
(b) If A is t-closed, then $R^t_A(x)$ is t-closed.

Theorem 5: Let (X, N) be a fuzzy anti-n-normed space. For each $x \in X$ and $t > 0$, if A is a convex subset of X, then $R^t_A(x)$ is a convex subset of A (for $R^t_A(x) \neq \emptyset$).

Proof: Let $z_1, z_2 \in R^t_A$, then for $t > 0$ and each $x \in X$, $N(x_1, x_2, \cdots, x_{n-1}, y - z_1, t) \leq N(x_1, x_2, \cdots, x_{n-1}, x - y, t)$ and $N(x_1, x_2, \cdots, x_{n-1}, y - z_2, t) \leq N(x_1, x_2, \cdots, x_{n-1}, x - y, t)$ for all $y \in A$. Now for each $\lambda \in (0, 1)$ we have

$$\begin{aligned}
&N(x_1, x_2, \cdots, x_{n-1}, y - (\lambda z_1 + (1-\lambda)z_2, t)\\
&= N(x_1, x_2, \cdots, x_{n-1}, \lambda y - \lambda z_1 + y - \lambda y - z_2 + \lambda z_2, t)\\
&= N(x_1, x_2, \cdots, x_{n-1}, \lambda(y - z_1) + (1-\lambda)(y - z_2),\\
&\qquad \lambda t + (1-\lambda)t)\\
&\leq \max\{N(x-1, x_2, \cdots, x_{n-1}, y - z_1, \frac{\lambda t}{\lambda}),\\
&\qquad N(x_1, x_2, \cdots, x_{n-1}, y - z_2, \frac{(1-\lambda)t}{(1-\lambda)})\}\\
&\leq \max\{N(x-1, x_2, \cdots, x_{n-1}, x - y, \frac{\lambda t}{\lambda}),\\
&\qquad N(x_1, x_2, \cdots, x_{n-1}, x - y, \frac{(1-\lambda)t}{(1-\lambda)})\}\\
&\leq N(x_1, x_2, \cdots, x_{n-1}, x - y, t)
\end{aligned}$$

So $\lambda z_1 + (1-\lambda)z_2 \in R^t_A(x)$ and $R^t_A(x)$ is convex. ■

Theorem 6: For $t > 0$ and each $x \in X$. let A be a t-coproximinal subspace of a fuzzy anti-n-normed space (X, N). Then

(a) If $\breve{A}^t_x$ is a t-compact set then A is t-quasi-co-Chebyshev.
(b) If $\breve{A}^t_x$ is a t-closed set then $R^t_A(x)$ is t-closed for every $x \in X$.

Proof: (i) Suppose $x \in X$ and $\{y_n\}$ is a sequence in $R^t_A(x)$. Since $x - y_n \in \breve{A}^t_x$ and $\breve{A}^t_x$ is a t-compact set, there exists a subsequence $\{x - y_{n_k}\}$ that t-convergence to $x - y_0 \in \breve{A}^t_x$. Consequently, $\{y_n\}$ has a subsequence $y_{n_k} \xrightarrow{t} y_0 \in R^t_A(x)$ and hence A is t-quasi-co-Chebyshev.
(ii) The proof is similar to (i). ■

Definition 9: A subset A of a fuzzy anti-n-normed space (X, N) is called to be t-boundedly compact if every t-bounded sequence in A has a subsequence t-converging to an element of X.

Theorem 7: Suppose for some $t > 0$ and each $x \in X$, A is a t-boundedly compact and t-closed subset of a fuzzy anti-n-normed space (X, N), then, A is t-quasi-co-Chebyshev.

Proof: Let $\{y_n\}$ be any sequence in $R^t_A(x)$. Then $N(x_1, x_2, \cdots, x_{n-1}, y_n - y, t) \leq N(x_1, x_2, \cdots, x_{n-1}, x - y, t)$ for every $y \in A$. Since $R^t_A(x)$ is t-bounded, $\{y_n\}$ is a t-bounded sequence in A, and so $\{y_n\}$ has a t-convergent subsequence $\{y_{n_k}\}$, let $y_{n_k} \xrightarrow{t} y_0 \in A$, as A is t-closed. Consider

$$\begin{aligned}
&N(x-1, x_2, \cdots, x_{n-1}, y_0 - y, t)\\
&= \lim_k N(x_1, x_2, \cdots, x_{n-1}, y_{n_k} - y, t)\\
&\leq N(x_1, x_2, \cdots, x_{n-1}, x - y, t)
\end{aligned}$$

for every $y \in A$. So $y_0 \in R^t_A(x)$, which implies that A is t-quasi-co-Chebyshev. ■

Definition 10: Let (X, N) be a fuzzy anti-n-normed space and A be a subset of X. For $t > 0$ and an element $x \in X$ is said to be t-orthogonal to an element $y \in X$, and we denote it by $x\perp^t_x y$, if $N(x_1, x_2, \cdots, x_{n-1}, x + \lambda y, t) \geq N(x_1, x_2, \cdots, x_{n-1}, x, t)$ for all scalar $\lambda \in \mathbb{R}$, $\lambda \neq 0$. We say $A\perp^t_x y$ if $x\perp^t_x y$ for every $x \in A$.

Theorem 8: For $t > 0$ and each $x \in X$ and $y_0 \in A$, let (X, N) be a fuzzy anti-n-normed space and A be a subspace of X. If $A\perp^t_x x - y_0$ then $y_0 \in R^t_A(x)$.

Proof: Suppose $t > 0$, $x \in X$ and $A\perp^t_x x - y_0$. Then $N(x_1, x_2, \cdots, x_{n-1}, a + \lambda(x - y_0), t) \geq N(x_1, x_2, \cdots, x_{n-1}, a, t)$ for all $a \in A$ and all scalar $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then $N(x_1, x_2, \cdots, x_{n-1}, x - y_0 + \lambda^{-1}a, \frac{t}{|\lambda|}) \geq N(x_1, x_2, \cdots, x_{n-1}, \lambda^{-1}a, \frac{t}{|\lambda|})$. Hence $N(x_1, x_2, \cdots, x_{n-1}, x - a^{'}, \frac{t}{|\lambda|}) \geq N(x_1, x_2, \cdots, x_{n-1}, y_0 - a^{'}, \frac{t}{|\lambda|})$, where $a^{'} = y_0 - \lambda^{-1}a$. Now if $\lambda = 1$ then, $N(x_1, x_2, \cdots, x_{n-1}, y - y_0, t) \geq N(x_1, x_2, \cdots, x_{n-1}, x - y, t)$ for all $y \in A$, and so $y_0 \in R^t_A(x)$. ■

IV. F-BEST COAPPROXIMATION

Definition 11: Let A be a nonempty subset of a fuzzy anti-n-normed space (X, N). An element $y_0 \in A$ is said to be an F-best coapproximation of $x \in X$ from A if it is a t-best coapproximation of x from A, for every $t > 0$, i.e.,

$$y_0 \in \bigcap_{t\in(0,\infty)} R^t_A(x).$$

The set of all elements of F-best coapproximation of X from A is denoted by $FR^t_A(x)$, i.e.,

$$FR^t_A(x) = \bigcap_{t\in(0,\infty)} R^t_A(x).$$

If each $x \in X$ has at least (respectively exactly) one F-best coapproximation in A, then A is called F-coproximinal (respectively F-co-Chebyshev) set.

Example 2: Let $X = \mathbb{R}^3$. Define $N : X \times X \times X \times [0, \infty) \to [0, 1]$ by

$$\begin{aligned}
N(x_1, x_2, x_3, t) &= \frac{\|x_1, x_2, x_3\|}{t}, \quad if \quad t > 0, t \in \mathbb{R}, x_1, x_2, x_3 \in X\\
&= 1, \quad if \quad t \leq 0, t \in \mathbb{R}, x_1, x_2, x_3 \in X.
\end{aligned}$$

where $\|x_1, x_2, x_3\| = \underset{1\leq i\leq 3}{min} \sum_{j=1}^{3} \mid x_{ij} \mid$. Then (X, N) is a fuzzy anti-3-normed linear space. Let $A = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 \leq 1, 0 \leq c \leq a^2 + b^2\}$ and $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 4)$ are in$X$. Let $a_0 = (0, -1, 1)$ and $a_1 = (0, 1, 1)$ are in A. Then $(0, -1, 1), (0, 1, 1) \in FR_A^t(0, 0, 4)$. So A is not a F-co-Chebyshev set.

Theorem 9: Let $\{\|., ., \cdots, .\|_\alpha^* : \alpha \in (0,]\}$ be a descending family of α-n-norm on X corresponding to the fuzzy anti-n-norm on X. Then $y_0 \in A$ is a best coapproximation to $x \in X$ in the descending family of α-n-norm on X corresponding to the fuzzy anti-n-norm on X if and only if y_0 is a F-best coapproximation to x in the fuzzy anti-n-normed space (X, N).

Proof: For each $x \in X$, y_0 is a best coapproximation to $x \in X$ in the descending family of α-n-norm on X corresponding to the fuzzy anti-n-norm on X. if and only if $\|x_1, x_2, \cdots, y - y_0\|_\alpha^* \leq \|x_1, x_2, \cdots, x - y\|_\alpha^*$, for every $y \in A$, if and only if $\frac{t}{t+\|x_1,x_2,\cdots,y-y_0\|_\alpha^*} \geq \frac{t}{t+\|x_1,x_2,\cdots,x-y\|_\alpha^*}$ for every $y \in A$ and $t \in (0, \infty)$, if and only if $N(x_1, x_2, \cdots, x_{n-1}, y - y_0, t) \leq N(x_1, x_2, \cdots, x_{n-1}, x - y, t)$ for every $y \in A$ and $t \in (0, \infty)$, if and only if $y_0 \in FR_A^t(x)$. ■

Definition 12: Let (X, N) be a fuzzy anti-n-normed space and A be a subset of X. For each element $x \in X$ is said to be F-orthogonal to an element $y \in X$ and we denote it by $x\perp_x^F y$, if for every $t > 0$, x$\perp_x^t y$. We say $A\perp_x^F y$ if $x\perp_x^F y$ for every $x \in A$.

Theorem 10: Let $\{\|., ., \cdots, .\|_\alpha^* : \alpha \in (0,]\}$ be a descending family of α-n-norm on X corresponding to the fuzzy anti-n-norm on X. Then $x \in X$ is Brikhoff orthogonal to $y \in X$ in the descending family of α-n-norm on X corresponding to the fuzzy anti-n-norm on X if and only if x is a F-orthogonal to y in the fuzzy anti-n-normed space (X, N).

Proof: For each $x \in X$, x is a Brikhoff orthogonal to $y \in X$ in the descending family of α-n-norm on X corresponding to the fuzzy anti-n-norm on X. if and only if $\|x_1, x_2, \cdots, x_{n-1}, x\|_\alpha^* \leq \|x_1, x_2, \cdots, x_{n-1}, x + \lambda y\|_\alpha^*$, for every scalar $\lambda \in \mathbb{R}\backslash\{0\}$, if and only if $\frac{t}{t+\|x_1,x_2,\cdots,x_{n-1},x\|_\alpha^*} \geq \frac{t}{t+\|x_1,x_2,\cdots,x_{n-1},x+\lambda y\|_\alpha^*}$ for every scalar $\lambda \in \mathbb{R}\backslash\{0\}$and $t > 0$, if and only if $N(x_1, x_2, \cdots, x_{n-1}, x, t) \leq N(x_1, x_2, \cdots, x_{n-1}, x + \lambda y, t)$ for every scalar $\lambda \in \mathbb{R}\backslash\{0\}$ and $t > 0$, if and only if $x\perp_x^F y$. ■

V. Conclusion

In this paper we introduced the concept of t-best coapproximation in and F-best coapproximation in fuzzy anti-n-normed spaces and also introduced t-coproximinal and t-co-Chebyshev in fuzzy anti-n-normed spaces. Then prove several theorems pertaining to this sets illustrate with example.

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