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Common Fixed Point Theorems for Co-cyclic Weak Contractions in Compact Metric

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Abstract—In this paper, we prove some common fixed point theorems for co-cyclic weak contractions in compact metric spaces.

Keywords—Cyclic weak contraction, Co-cyclic weak contraction, Co-cyclic representation, Common fixed point.

I. Introduction

ALBER and Guerre-Delabriere in [2] defined weakly contractive mappings and they proved some fixed point theorems in the Hilbert spaces. In [10], Rhoades extended some results of [2] to complete metric spaces.

Beg et. al. [4] and Babu et. al. [3] proved common fixed point theorems for a pair of weakly contractive map in complete metric space.

In 2003, Kirk et. al. [9] introduced the notion of Cyclic contraction and established some related fixed point theorems for mappings satisfying such contraction conditions. Suggested by the consideration in [9], Rus [11] introduced the following concept of cyclic representation and proved some fixed point theorems.

***Definition** 1:* [11] Let X be a nonempty set, m a positive integer and $T : X \to X$ a selfmap. $X = \cup_{i=1}^{m} A_i$ is said to be a cyclic representation of X with respect to the map T if the following conditions hold:

1) $A_i, i = 1, 2, \cdots, m$ are nonempty subsets of X;
2) $T(A_1) \subset A_2, \cdots, T(A_{m-1}) \subset A_m, T(A_m) \subset A_1$.

In [8], Karapinar proves a fixed point theorem for a mapping T defined on a complete metric space X when X has a cyclic representation with respect to T.

***Example** 1:* [5] Let $X = [0, 2], A_1 = [0, 1]$, $A_2 = [\frac{1}{2}, \frac{3}{2}]$ and $A_3 = [1, 2]$. Now, we define a selfmap T on X by

$$T(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0,\ \frac{1}{2}] \\ 1 & \text{if } x \in (\frac{1}{2},\ \frac{3}{2}] \\ x - 1 & \text{if } x \in (\frac{3}{2},\ 2]. \end{cases}$$

Then we observe that $T(A_1) = [\frac{1}{2}, 1] \subset [\frac{1}{2}, \frac{3}{2}] = A_2$, $T(A_2) \subset [1, 2] = A_3$ and $T(A_3) = (\frac{1}{2}, 1] \subset [0, 1] = A_0$. Therefore, $X = \cup_{i=1}^{3} A_i$ is a cyclic representation of X with respect to T.

Throughout this paper, we denote $R_+ = [0, \infty)$ and

$$\mathfrak{J} = \{\varphi |\ \varphi : R_+ \to R_+ \ is\ nondecreasing, \varphi(0) = 0,\ \varphi(t) > 0 \text{ for } t > 0\}.$$

Recently, Harjani et.al. [6] established the following fixed point theorem for a continuous selfmap.

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***Theorem** 1:* Let $(X,\ d)$ be a compact metric space and $T : X \to X$ a continuous operator. Suppose that m is a positive integer, $A_1,\ A_2, \cdots, A_m$ nonempty subsets of X, $X = \cup_{i=1}^{m} A_i$ satisfying

1) $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T;
2) $d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$ for any $x \in A_i$ and $y \in A_{i+1}$, where $\varphi \in \mathfrak{J}$.

Then T has a unique fixed point.

Note that to guarantee the existence and uniqueness of common fixed points of a pair of maps, we need an additional condition, called weak compatibility, which is defined as follows.

***Definition** 2:* [7] Let X be a nonempty set. Two selfmaps $S,\ T : X \to X$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $x \in X$ such that $Sx = Tx$, then $STx = TSx$.

The purpose of this paper is to establish a common fixed point theorem for a co-cyclic weak contraction defined in compact metric spaces. Our result extends the result of Harjani et. al. [6] to a co-cyclic weak contraction.

II. Preliminaries

***Definition** 3:* [5] Let X be a nonempty set, m a positive integer and $T,\ f : X \to X$ be two selfmaps. $X = \cup_{i=1}^{m} A_i$ is said to be a co-cyclic representation of X between f and T if the following conditions are satisfied:

1) $A_i, i = 1, 2, \cdots, m$ are nonempty subsets of X;
2) $T(A_1) \subset f(A_2), \cdots, T(A_{m-1}) \subset f(A_m), and$ $T(A_m) \subset f(A_1)$.

***Example** 2:* Let $X = [0, 1]$, and $A_1 = [0, \frac{1}{2}]$ and $A_2 = [\frac{1}{2}, 1]$. We define a selfmap T and f on X by

$$T(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0,\ \frac{1}{2}] \\ 1 - x & \text{if } x \in (\frac{1}{2},\ 1] \end{cases}$$

and

$$f(x) = \begin{cases} x & \text{if } x \in [0,\ \frac{1}{2}] \\ 2x - 1 & \text{if } x \in (\frac{1}{2},\ 1]. \end{cases}$$

Then we observe that $T(A_1) = [\frac{1}{2}, 1] \subset [0,\ 1] = f(A_2)$, $T(A_2) = [0, \frac{1}{2}] = f(A_1)$. Therefore, $X = \cup_{i=1}^{2} A_i$ is a co-cyclic representation of X between f and T.

We now introduce the following definitions.

***Definition** 4:* Let $(X,\ d)$ be a metric space, m is a positive integer, $A_1,\ A_2, \cdots, A_m$ a closed nonempty subsets of X, and $X = \cup_{i=1}^{m} A_i$. An operator $T : X \to X$ is said to be co-cyclic weak contraction if there is an operator $f : X \to X$ such that

1) $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X between f and T;
2) $d(Tx, Ty) \le d(fx, fy) - \varphi(d(fx, fy))$ for any $x \in A_i$ and $y \in A_{i+1}$, where $A_{m+1} = A_1$ and $\varphi \in \mathfrak{J}$.

The purpose of this paper is to prove the following theorem.

III. Main Results

***Theorem** 2:* Let $(X,\ d)$ be a compact metric space and $f,\ T : X \to\ X$ be two continuous operators. Suppose that m is a positive integer, $A_1,\ A_2, \cdots, A_m$ are nonempty subsets of X, and $X = \cup_{i=1}^{m} A_i$ satisfying

1) $X = \cup_{i=1}^{m} A_i$ is a co-cyclic representation of X between f and T;
2) $d(Tx, Ty) \le d(fx, fy) - \varphi(d(fx, fy))$ for any $x \in A_i$ and $y \in A_{i+1}$, where $A_{m+1} = A_1$ and $\varphi \in \mathfrak{J}$.

If the pair of operators $(f,\ T)$ are weakly compatible on X, then f and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $T(A_i) \subset f(A_{i+1})$ for each $i = 1, 2, \cdots, m-1$ and $T(A_m) \subset f(A_1)$, there exists $x_1 \in X$ such that $Tx_0 = fx_1$. On continuing the process, inductively we get a sequence x_n in X such that $Tx_n = fx_{n+1}$ for each $n = 0, 1, 2, \cdots$.

If there exists $n_0 \in \mathbb{N}$ with $Tx_{n_0+1} = Tx_{n_0} = fx_{n_0+1}$ and, thus, f and T have coincidence point x_{n_0+1}.

Suppose that $x_{n+1} \ne x_n$ for all $n = 0, 1, 2, \cdots$. We now show that the sequence $\{d(fx_n, fx_{n+1})\}$ is a nonincreasing sequence. By (1) of Theorem 3.1, for each $n > 0$ there exists $i_n \in \{1, 2, \cdots, m\}$ such that $x_{n-1} \in A_{i_n-1}$ and $x_n \in A_{i_n}$ and using (2) of Theorem 1, we get

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\le d(fx_{n-1}, fx_n) - \varphi(d(fx_{n-1}, fx_n)) \\ &\le d(fx_{n-1}, fx_n) \end{aligned} \tag{1}$$

for each $n = 1, 2, \cdots$. Therefore,

$$d(fx_n, fx_{n+1}) \le d(fx_{n-1}, fx_n) \tag{2}$$

for all $n \ge 0$. Hence $\{d(fx_n, fx_{n+1})\}$ is a non-increasing sequence of nonnegative reals and hence converges to a limit $l \ge 0$. Letting $n \to \infty$ in (1), we obtain

$$l \le l - \lim_{n\to\infty} \varphi(d(fx_n, fx_{n+1})) \le l$$

and, hence

$$\lim_{n\to\infty} \varphi(d(fx_n,\ fx_{n+1})) = 0 \tag{3}$$

We claim that $l = 0$. Suppose $l > 0$.
Since $l = \inf\{d(fx_n,\ fx_{n+1}) : n \in \mathbb{N}\}$,

$$0 < l \le d(fx_n,\ fx_{n+1})$$

for $n = 0, 1, 2, \cdots$ and since φ is nondecreasing and $\varphi(t) > 0$ for $t \in (0,\ \infty)$, we obtain

$$0 < \varphi(l) \le \varphi(d(fx_n, fx_{n+1}))$$

for $n = 0, 1, 2, \cdots$, and hence letting $n \to \infty$, we get

$$0 < \varphi(l) \le \lim_{n\to\infty} \varphi(d(fx_n, fx_{n+1}))$$

which is a contradiction to (3). Therefore, $l = 0$.
Hence,

$$\lim_{n\to\infty} d(fx_n,\ fx_{n+1}) = 0 \tag{4}$$

Since $Tx_n = fx_{n+1}$ for each $n = 1, 2, \cdots$, from (4) it follows that

$$\inf\{d(fx,\ Tx) : x \in X\} = 0. \tag{5}$$

Now since the mapping $X \mapsto \mathbb{R}^+$ defined by $x \mapsto d(fx,\ Tx)$ is continuous and X is compact, we find $u \in X$ such that

$$d(fu,\ Tu) = \inf\{d(fx,\ Tx) : x \in X\}.$$

By (5), $d(fu,\ Tu) = 0$ and, consequently, $fu = Tu = z$ (say), which shows that the pair $(f,\ T)$ has a point of coincidence. Since the pair $(f,\ T)$ is weakly compatible,

$$Tz = Tfu = fTu = fz.$$

Hence,

$$Tz = fz. \tag{6}$$

We claim that $z = Tz$. Suppose $z \ne Tz$. Then,

$$\begin{aligned} d(z, Tz) &= d(Tu, Tz) \\ &\le d(fu, fz) - \varphi(d(fu, fz)) \\ &\le d(z, Tz) - \varphi(d(z, Tz)), \end{aligned}$$

which shows that

$$\varphi(d(z, Tz)) \le 0.$$

But

$$\varphi(d(z, Tz)) \ge 0.$$

Hence,

$$\varphi(d(z, Tz)) = 0$$

and since $\varphi \in \mathfrak{J}$, we have

$$d(z, Tz) = 0.$$

Hence,

$$Tz = z.$$

Hence, by (6), we obtain

$$fz = Tz = z.$$

For the uniqueness part, suppose that z and w are common fixed points of f and T. Since $X = \cup_{i=1}^{m} A_i$ is co-cyclic representation of X between f and T, we have $z, w \in \cap_{i=1}^{m} A_i$. By (2), we have

$$\begin{aligned} d(z, w) = d(Tz, Tw) &\le d(fz, fw) - \varphi(d(fz, fw)) \\ &\le d(z, w) - \varphi(d(z, w)) \end{aligned}$$

Therefore,

$$\varphi(d(z, w)) = 0.$$

Since $\varphi \in \mathfrak{J}$, $d(z,\ w) = 0$ and hence, $z = w$.

Since the identity map I_X defined on X is weakly compatible with any selfmap T defined on X, if we choose $f = I_X$, the identity map on X, we obtain the following result:

***Corollary** 1:* Let $(X,\ d)$ be a compact metric space and $T : X \to\ X$ be a continuous operator. Suppose that m is a positive integer, $A_1,\ A_2, \cdots, A_m$ are nonempty subsets of X, and $X = \cup_{i=1}^m A_i$ satisfying

1) $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to T;
2) $d(Tx, Ty) \le d(x, y) - \varphi(d(x, y))$ for any $x \in A_i$ and $y \in A_{i+1}$, where $A_{m+1} = A_1$ and $\varphi \in \mathfrak{J}$.

Then T has a unique fixed point in X.

Proof. Follows from Theorem 1 by choosing $f = I_X$.

Remark We observe that Theorem 1 extends Theorem 2.1 of [6] to co-cyclic weak contraction.

***Definition** 5:* [1] Let $(X,\ d)$ be a metric space, and $\mathfrak{T}$ be a set of selfmappings of X. The common fixed points of the set $\mathfrak{T}$ is said to be well-posed if:

1) $\mathfrak{T}$ has a unique common fixed point in X (That is, z is the unique point in X such that $Tz = z$ for all $T \in \mathfrak{T}$);
2) For every sequence $\{z_n\}$ in X such that
$$\lim_{n\to\infty} d(z_n,\ Tz_{n+1}) = 0, \forall T \in \mathfrak{T},$$
we have
$$\lim_{n\to\infty} d(z_n,\ z) = 0.$$

Our second result is concerned with the well-posedness of the common fixed point problem for two mappings f and T satisfying the inequality (2) of Theorem 1.

***Theorem** 3:* Under the assumptions of Theorem 1, the common fixed point problem for f and T is well-posed; that is, if there is a sequence $\{z_n\}$ in X with $d(z_n,\ Tz_n) \to 0$ and $d(z_n,\ fz_n) \to 0$ as $n \to \infty$, then $z_n \to z$ as $n \to \infty$, where z is the unique common fixed point of f and T (whose existence is guaranteed by Theorem 1).

Proof. By Theorem 1, f and T have a unique common fixed point z. As z is common fixed point of f and T, by (2) of Theorem 1, $z \in \cap_{i=1}^m A_i$. Let $\{z_n\}$ be a sequence in X such that $d(z_n,\ Tz_n) \to 0$ and $d(z_n,\ fz_n) \to 0$ as $n \to \infty$. Now consider

$$d(z, Tz_n) \le d(z, fz_n) - \varphi(d(z, fz_n)) \tag{7}$$
$$\le d(z, z_n) + d(z_n, fz_n) - \varphi(d(z, fz_n)) \tag{8}$$

Also, from the triangle inequality, (2) of Theorem 1, Equation (8) and the fact that $z \in \cap_{i=1}^m A_i$, we have

$$\begin{aligned} d(z, z_n) &\le d(z, Tz_n) + d(Tz_n, z_n) \qquad (9)\\ &\le d(z, z_n) + d(z_n, fz_n)\\ &\quad -\varphi(d(z, fz_n)) + d(Tz_n, z_n) \end{aligned}$$

which implies

$$\varphi(d(z, fz_n)) \le d(z_n, fz_n) + d(Tz_n, z_n) \to 0$$

as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \varphi(d(fz_n,\ z)) = 0. \tag{10}$$

We now claim that $\lim_{n\to\infty} d(fz_n,\ z) = 0$. Suppose not. Then there exists $\varepsilon \ge 0$ such that for any $n \in \mathbb{N}$ we can find $k_n \ge n$ with $d(fz_{k_n}, z) \ge \varepsilon$. Since $\varphi \in \mathfrak{J}$ is nondecreasing and $\varphi(t) > 0$ for $t \in (0,\ \infty)$, we have

$$0 < \varphi(\varepsilon) \le \varphi(d(fz_{k_n}, z)). \tag{11}$$

Letting $n \to \infty$ in (11)

$$0 < \varphi(\varepsilon) \le \lim_{n\to\infty} \varphi(d(fz_{k_n}, z)).$$

which contradicts (10). Therefore,

$$\lim_{n\to\infty} d(fz_n,\ z) = 0$$

and hence letting $n \to \infty$ in (7), we obtain

$$\lim_{n\to\infty} d(z,\ Tz_n) = 0 \tag{12}$$

Consequently, letting $n \to \infty$ in (9), using (12) we obtain

$$\lim_{n\to\infty} d(z_n,\ z) = 0$$

Hence the common fixed point problem of f and T is well-posed.

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