Study of Explicit Finite Difference Method in One Dimensional System

Azizollah Khormali, Seyyed Shahab Tabatabaee Moradi, Dmitry Petrakov

***Abstract*—**One of the most important parameters in petroleum reservoirs is the pressure distribution along the reservoir, as the pressure varies with the time and location. A popular method to determine the pressure distribution in a reservoir in the unsteady state regime of flow is applying Darcy's equation and solving this equation numerically. The numerical simulation of reservoirs is based on these numerical solutions of different partial differential equations (PDEs) representing the multiphase flow of fluids. Pressure profile has obtained in a one dimensional system solving Darcy's equation explicitly. Changes of pressure profile in three situations are investigated in this work. These situations include section length changes, step time changes and time approach to infinity. The effects of these changes in pressure profile are shown and discussed in the paper.

***Keywords*—**Explicit solution, Numerical simulation, Petroleum reservoir, Pressure distribution.

I. Introduction

FOR real petroleum reservoirs, the multiphase flow of reservoir fluids are expected. These multiphase flow equations are so complex and can be solved analytically, i.e., numerically.

The numerical simulation of reservoirs is based on these numerical solutions of different partial differential equations (PDEs) representing the multiphase flow of fluids. Numerical simulation has become a common means of predicting and understanding complex performance of oil and gas reservoirs in the petroleum industry. It has been widely used for simulations in primary, secondary, and tertiary oil and gas recovery processes [1].

The most common forms of numerical methods are based on finite difference approximation of the flow equations. Finite different method (FDM) is more common due to several factors (robustness, ease of programming, etc.) [2]. FDM approach will be followed in this paper too.

The unknown parameters of flow equations (i.e., pressure, saturation, etc.) are a function of both time and space, which gives the flow equations a partial differential appearance. One of these equations is experimentally derived Darcy's equation which describes the fluid flow through a porous medium.

In this work a linear one dimensional system is described by Darcy's equation and then the PDE is solved based on the finite difference method explicitly, i.e., the partial derivatives are replaced by finite difference quotients and then the solutions of the resulting system of algebraic equations are obtained [3].

Different quantities, which are involved in the explicit finite difference solution of PDE, are changed to study the sensitivity and the applicability of the explicit solution.

Azizollah Khormali and Dmitry Petrakov are in the Department of Oil and Gas Field Development and Operation, Oil and Gas Faculty, National Mineral Resources University, Saint-Petersburg, Russia (phone: +79111443828; e-mail: aziz.khormaly.put@gmail.com, petrakovdg@mail.ru).

Seyyed Shahab Tabatabaee Moradi is in the Department of Oil Well Drilling, Oil and Gas Faculty, National Mineral Resources University, Saint-Petersburg, Russia (e-mail: s.sh.tabatabaee@gmail.com).

II. Methodology

To obtain best results, accurate numerical simulation and an appropriate mathematical model are crucial [4]. In transient flow through porous media, the general PDE to be solved is obtained by combining appropriate forms of Darcy's law and the equation of mass conservation [5], which has the form of (1), for an unsteady state regime of flow in a three dimensional linear system.

$$\frac{\partial P}{\partial t} = \frac{k}{c\phi\mu}\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right) \tag{1}$$

where
P is the pressure (MPa)
t is the total time (s)
k is the permeability (mD)
c is the compressibility (1/MPa)
Φ is the porosity
μ is the viscosity (cP)
x is the dimension in x axis (m)
y is the dimension in y axis (m)
z is the dimension in z axis (m)

In this paper pressure distribution has been studied in a one dimensional system, therefore (1) simplifies to (2):

$$\frac{\partial P}{\partial t} = \frac{mk}{c\phi\mu}\left(\frac{\partial^2 P}{\partial x^2}\right) \tag{2}$$

In which, $k/(c\phi\mu)$ is a constant and is called hydraulic diffusivity. For further simplification, this constant will be assumed equal to one, transforming (2) to (3):

$$\frac{\partial P}{\partial t} = \left(\frac{\partial^2 P}{\partial x^2}\right) \tag{3}$$

Equation (3) has been solved by finite difference method. The finite difference method is the most popular and has been used widely in modeling [6], [7]. For solving (3), by using a numerical method, the length of the system (L) is divided into several sections, length of each section equal to Δx (spatial discretization). Applying finite differences to both sides of (3), taking Δt as time step, can be written as:

$$\left(\frac{\partial P}{\partial t}\right)_j \approx \frac{P_m{}^{\tau+1} - P_m{}^{\tau}}{\Delta t} \quad (4)$$

$$\left(\frac{\partial^2 P}{\partial x^2}\right)_j \approx \frac{P^{\tau}{}_{j+1} + P^{\tau}{}_{j-1} - 2P^{\tau}{}_j}{\Delta x^2} \quad (5)$$

Here, subscript j shows the section number (i.e., 1, 2, …, n) and superscript τ shows time in seconds.

To overcome the difficulties associated with the selection of discretization step, the explicit method is used. By combining (4) and (5), then rearrangements, the next equation will appear:

$$P^{\tau+1}{}_j = P^{\tau}{}_j + (P^{\tau}{}_{j+1} + P^{\tau}{}_{j-1} - 2P^{\tau}{}_j)\frac{\Delta t}{\Delta x^2} \quad (6)$$

The pressure term at a time ($\tau+1$) seconds in (6), can be calculated in each separated section (j) by knowing pressure in three sections ($j-1$), (j), and ($j+1$) at a time (τ) with specified section length (Δx) and time step (Δt). The boundary condition and initial conditions are as follows:

1) Pressure in the first block (P_1) is constant for all time.
2) Pressure in the last block (P_n) is constant for all time.
3) Pressure at t=0 second from the second block to the ($n-1$) block is constant and equal to the pressure in the last block (P_n).

The first and second conditions are boundary conditions and the last one is the initial condition.

According to (6), the changes in three parameters appearing in the solution of the pressure distribution equation are considered. These three parameters are as follows:

1) Section length (Δx).
2) Time step (Δt).
3) Time (τ).

To investigate the effect of changes in these three parameters, which are involved in solving of pressure distribution equations by numerical method, each parameter is changed while the other parameters are fixed.

III. Results and Discussion

Equation (6) has been solved for different values of section length (Δx), time step (Δt), and time (τ). Each one of these three parameters is changed while the other two parameters are kept constant to see how the changes in these parameters affect the final pressure distribution. The Matlab software was used to generate data for evaluation and plotting the graphs by (6). The text program which has been used is given in the Fig. 5. The pressures, which were gotten at the end of the process, were applied for plotting the graphs. Table I shows the results of pressure changes for length 10m, time step 0.1s and section length 1m in one second. Tables II-IV are resulted in 5, 10 and 100 seconds, respectively. The last row has been used form the tables for plotting. The following figures show the effect of changes in parameters respectively:

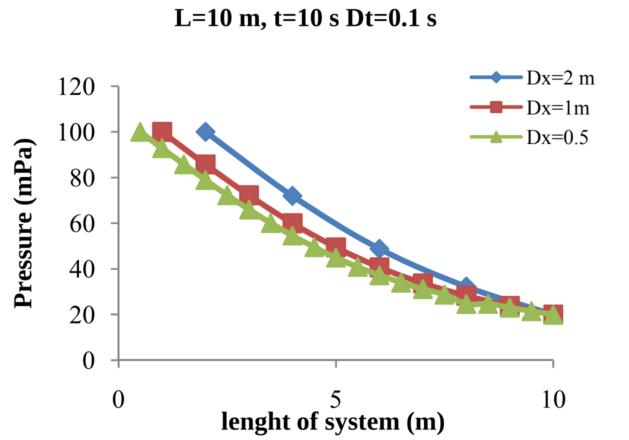

Fig. 1 Effect of changes in Δx values in pressure profile

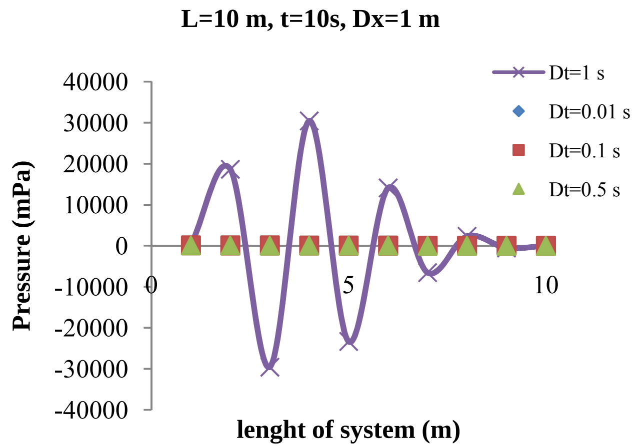

Fig. 2 Effect of increase in Δt values in pressure profile

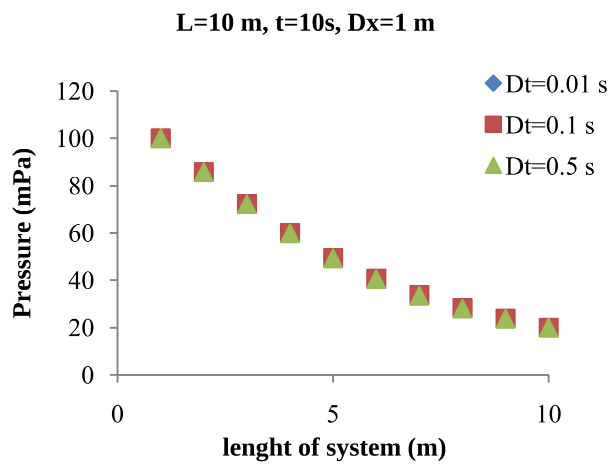

Fig. 3 Effect of a small increase in time step on pressure profile

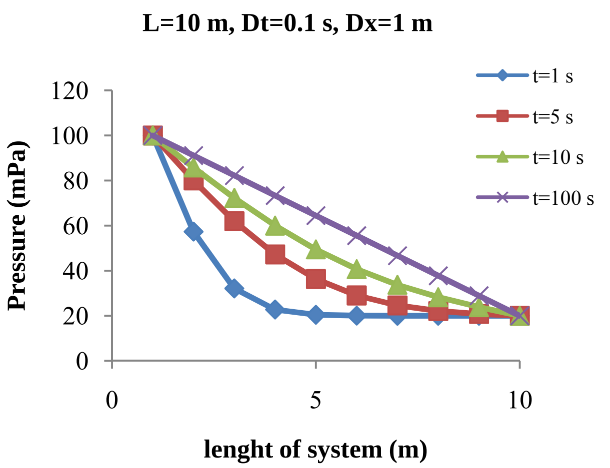

Fig. 4 Effect of changes in the values of time on pressure profile

As the length of each section (Δx) decreases, the accuracy of the solution increases (Fig. 1), although for very small values of Δx (Δx<0.25m in this system) the negative and incorrect solutions appear.

With the large increase in the time step (Δt), negative and wrong pressures are occurring, which are physically impossible (Fig. 2). In the Fig. 3, the pressure profile for the time step equal to one second is removed, which indicates that small increase in time step does not strongly affect the pressure profile along the length of the system.

Fig. 4 shows the long time behavior of explicit finite difference solutions of PDE with the time, which indicates that as the time increases; pressure profile tends to a straight line.

IV. Conclusion

Resulting from discussions, it can be concluded that although the explicit numerical solution of PDE is robust and easy to program, but also it does have some limitations in the parameters, which are involved in the solution. These limitations should be considered carefully by using an explicit numerical method. The implicit finite difference solution may be suggested for cases with multiple limitations.

Appendix

```
%Taking input data from user including the number of cells and time step
l=input('Entetr the lenght of reservoir: ');
q=input('Enter the number of grid blocks: ');
dt=input('Enter the time step: ');
p(1,1)=input('Enter the pressure at the first block: ');
p(1,q)=input('Enter the pressure at the last block: ');
timestep=input('Enter the number of time steps: ');
%Calculating the value of Dx and (DT/Dx^2)
dx=(1./q);
beta=dt./(dx.^2);

%Boumdary condition
for ii=2:timestep
    p(ii,1)=p(1,1);
    p(ii,q)=p(1,q);

end
%Initial condition
for jj=2:q-1
    p(1,jj)=p(1,q);
end

%Using EXPLICIT FINITE DIFFERENCE SOLUTION TO SOLVE THE PROBLEM
%Calculating pressure distribution
for ii=2:timestep
    for jj=2:q-1

        p(ii,jj)=p(ii-1,jj)+(beta*(p(ii-1,jj+1)+p(ii-1,jj-1)-2*p(ii-1,jj)));
    end
end
disp(p)
%Taking output in EXCELL format
xlswrite('simulation',p);
```

Fig. 5 The program text of solving equation in Matlab

TABLE I
PRESSURE DISTRIBUTION IN ALL SECTIONS IN ONE SECOND BY ΔT=0.1 S, ΔX=1 M, LENGTH 10 M WITH PRESSURE AT ENTRANCE (FIRST BLOCK) 100 MPA AND AT EXIT (LAST BLOK) 20 MPA

Time (s)	Pressure (MPa)									
	First block	Second block	Third block	4th block	5th block	6th block	7th block	8th block	9th block	10th block
0.1	100	20	20	20	20	20	20	20	20	20
0.1	100	20	20	20	20	20	20	20	20	20
0.2	100	28	20	20	20	20	20	20	20	20
0.3	100	34.4	20.8	20	20	20	20	20	20	20
0.4	100	39.6	22.08	20.08	20	20	20	20	20	20
0.5	100	43.89	23.63	20.27	20.01	20	20	20	20	20
0.6	100	47.47	25.32	20.58	20.03	20	20	20	20	20
0.7	100	50.51	27.06	21	20.09	20	20	20	20	20
0.8	100	53.12	28.8	21.52	20.17	20.01	20	20	20	20
0.9	100	55.37	30.5	22.11	20.29	20.03	20.01	20	20	20
1	100	57.35	32.15	22.77	20.44	20.05	20.01	20	20	20

TABLE II
PRESSURE DISTRIBUTION IN ALL SECTIONS IN 5 SECONDS BY ΔT=0.1 S, ΔX=1 M, LENGTH 10 M WITH PRESSURE AT ENTRANCE (FIRST BLOCK) 100 MPA AND AT EXIT (LAST BLOK) 20 MPA

Time (s)	Pressure (MPa)									
	First block	Second block	Third block	4th block	5th block	6th block	7th block	8th block	9th block	10th block
0.1	100	20	20	20	20	20	20	20	20	20
0.2	100	28	20	20	20	20	20	20	20	20
0.3	100	34.4	20.8	20	20	20	20	20	20	20
0.4	100	39.6	22.08	20.08	20	20	20	20	20	20
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
4.6	100	79.19	60.55	45.6	34.84	27.88	23.82	21.68	20.61	20
4.7	100	79.41	60.92	46.02	35.22	28.17	24.02	21.78	20.65	20
4.8	100	79.62	61.28	46.43	35.6	28.46	24.21	21.89	20.7	20
4.9	100	79.82	61.63	46.83	35.97	28.75	24.4	22.01	20.75	20
5	100	80.02	61.97	47.22	36.33	29.04	24.6	22.12	20.8	20

TABLE III
PRESSURE DISTRIBUTION IN ALL SECTIONS IN 10 SECONDS BY ΔT=0.1 S, ΔX=1 M, LENGTH 10 M WITH PRESSURE AT ENTRANCE (FIRST BLOCK) 100 MPA AND AT EXIT (LAST BLOK) 20 MPA

Time (s)	Pressure (MPa)									
	First block	Second block	Third block	4th block	5thblock	6th block	7th block	8th block	9th block	10th block
0.01	100	20	20	20	20	20	20	20	20	20
0.02	100	20.8	20	20	20	20	20	20	20	20
0.03	100	21.58	20.01	20	20	20	20	20	20	20
0.04	100	22.35	20.02	20	20	20	20	20	20	20
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
9.96	100	85.77	72.24	60	49.43	40.69	33.73	28.25	23.85	20
9.97	100	85.78	72.26	60.01	49.44	40.71	33.74	28.26	23.85	20
9.98	100	85.79	72.27	60.03	49.46	40.73	33.76	28.27	23.86	20
9.99	100	85.79	72.28	60.05	49.48	40.75	33.77	28.28	23.86	20
10	100	85.8	72.29	60.06	49.5	40.76	33.79	28.29	23.87	20

TABLE IV
PRESSURE DISTRIBUTION IN ALL SECTIONS IN 100 SECONDS BY ΔT=0.1 S, ΔX=1 M, LENGTH 10 M WITH PRESSURE AT ENTRANCE (FIRST BLOCK) 100 MPA AND AT EXIT (LAST BLOK) 20 MPA

Time (s)	Pressure (MPa)									
	First block	Second block	Third block	4^{th} block	5^{th}block	6^{th}block	7^{th}block	8^{th}block	9^{th}block	10^{th} block
0.1	100	20	20	20	20	20	20	20	20	20
0.2	100	28	20	20	20	20	20	20	20	20
0.3	100	34.4	20.8	20	20	20	20	20	20	20
0.4	100	39.6	22.08	20.08	20	20	20	20	20	20
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
99.6	100	91.11	82.22	73.33	64.44	55.56	46.67	37.78	28.89	20
99.7	100	91.11	82.22	73.33	64.44	55.56	46.67	37.78	28.89	20
99.8	100	91.11	82.22	73.33	64.44	55.56	46.67	37.78	28.89	20
99.9	100	91.11	82.22	73.33	64.44	55.56	46.67	37.78	28.89	20
100	100	91.11	82.22	73.33	64.44	55.56	46.67	37.78	28.89	20

REFERENCES

[1] B. Guo and X. Wang, "Testing of New High-Order Finite Difference Methods for Solving the Convection-Diffusion Equation," *E-Journal of Reservoir Engineering,* pp. 1-14, 2005.

[2] D. K. E. De Carvalho, P. R. M. Lyra and R. B. Willmersdorf, "A First Step Towards a Petroleum Reservoir Simulator Using an Edge-Based Unstructured Finite Volume Formulation," *The 2nd Brazilian Congress on R&D in Petroleum and Gas,* 2003.

[3] T. Pal Singh, "An Improved Finite Difference Method for Well Models in Numerical Reservoir Simulation," *2nd International Conference on Current Trends in Technology,* 08–10 Dec. 2011.

[4] K. Hvistendahl Karlsen, K. A. Lieb and N. H. Risebro, "A Fast Marching Method for Reservoir Simulation," *Computational Geosciences,* vol. 4, pp. 185–206, 2000.

[5] G. J. Moridis, D. A. Mc Vay, D. L. Reddell and T. A. Blasingame, "The Laplace Transform Finite Difference (LTFD) Numerical Method for the Simulation of Compressible Liquid Flow in Reservoirs," *SPE Advanced Technology Series,* vol. 2, pp. 122–131, Apr. 1994.

[6] Y. Liu, K. Mrinal Sen, A. John and G. Katherine, "Acoustic VTI Modeling by a New Time-Space Domain High-Order Finite-Difference Method," *SEG Annual Meeting,* Houston, Texas, Oct. 2009.

[7] F. Liu and W. Shi, "Numerical Solutions of Two-Dimensional Burgers' Equations by Lattice Boltzmann Method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 150–157, 2011.