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Exponential Stability Analysis for Uncertain Neural Networks with Discrete and Distributed Time-Varying Delays

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Abstract—This paper studies the problem of exponential stability analysis for uncertain neural networks with discrete and distributed time-varying delays. Together with a suitable augmented Lyapunov Krasovskii function, zero equalities, reciprocally convex approach and a novel sufficient condition to guarantee the exponential stability of the considered system. The several exponential stability criterion proposed in this paper is simpler and effective. Finally,numerical examples are provided to demonstrate the feasibility and effectiveness of our results.

Keywords—Exponential stability, Uncertain Neural networks, LMI approach, Lyapunov-Krasovskii function, Time-varying.

I. Introduction

RECURRENT years neural networks have been studied extensively and have been widely applied within a kind of engineering fields such as associative memories, neuro biology, population dynamics, and computing technology[1-9]. Existing stability criteria can be classified into two categories, that is, delay-independent ones and delay-dependent ones. It is well known that delay-independent ones are usually more conservative than the delay-dependent ones, so much attention has been paid in recent years to the study of delay-dependent stability conditions.

Although neural networks can be implemented by very large scale integrated circuits, there inevitably exist some delays in neural networks due to the limitation of the speed of transmission and switching of signals. It is well known that time-delay is usually a cause of instability and oscillations of recurrent neural networks. Therefore, the problem of stability of recurrent neural networks with time-delay is of importance in both theory and practice.

The problem of exponential stability analysis for uncertain neural networks with discrete and distributed time-varying delays has been studied by many investigators in the past years.mang known results,the time-varying delays varies from 0 to an upper bound,but in practice the delay of lower bound is not restricted to be 0, In this paper, we considered the relationship between the time-varying delay and its lower and upper bound, by means of the Lyapunov-Krasovskii function and the linear matrix inequality(LMI) approach, Note that LMIs can be easily solved by using the Matlab LMI toolbox, Finally numerical examples given to illustrate the effectiveness of the proposed methods.

Notation: Throughout this paper, the superscripts$' - 1'$, $'T'$ stand for the inverse and transpose of a matrix, respectively;$\Re^n$ denotes an n-dimensional Euclidean space;$\Re^{m\times n}$ is the set of all $m \times n$ real matrices; $P > 0$ means that the matrix P is symmetric positive definite; I is an appropriately dimensional identity matrix.

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II. PROBLEM STATEMENT

Consider the following neural networks with time-varying delays:

$$\begin{cases} \dot{z}(t) = -(C + \Delta C(t))z(t) + (A + \Delta A(t))g(z(t)) \\ \qquad + (B + \Delta B(t))g(z(t - \tau(t))) + (D + \Delta D(t)) \\ \qquad \times \int_{t-d}^{t} g(z(s))ds + \mu \\ z(t) = \phi(t), t \in [-\tau_2, 0] \end{cases} \tag{1}$$

where $z(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T \in \Re^n$ is the neuron vector,$g(z(t))) = [g(z_1(t)), g(z_2(t)), \ldots, g(z_n(t))]^T \in \Re^n$ is neuron activation function,$C = diag\{c_1, c_2, \ldots, c_n\} > 0$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times n}$are the connection weight matrices,and and the delayed connection weight matrices, respectively, $\mu = [\mu_1, \mu_2, \ldots, \mu_n]^T$is constant input vector, $\Delta C(t)$,$\Delta A(t)$,$\Delta B(t)$, $\Delta D(t)$ are the parametric uncertainties of system matrices of the form

$$\Delta C(t) = WF(t)E_c, \Delta A(t) = WF(t)E_a, \Delta B(t) = WF(t)E_b, \Delta D(t) = WF(t)E_d \tag{2}$$

with

$$F^T(t)F(t) \leq I, \forall t \geq 0. \tag{3}$$

and $\tau(t)$ is a continuous time-varying function which satisfies

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \dot{\tau}(t) \leq u \tag{4}$$

where τ and u are constants.

The following assumption is made in this paper.

Assumption1. The neuron activation functions $g_i(\cdot)$ in (1) are bounded and satisfy

$$\gamma_i^- \leq \frac{g_i(x)-g_i(y)}{x-y} \leq \gamma_i^+, x, y \in \Re, x \neq y, i = 1, 2, \ldots, n \tag{5}$$

Where $\gamma_i^-, \gamma_i^+ (i = 1, 2, \cdots, n)$ are positive constants.

Assumption1 guarantees the existence of an equilibrium point of system(16).Denote that$z^* = [z_1^*, z_2^*, \ldots, z_n^*]$ is the equilibrium point. Using the transformation$x(\cdot) = z(\cdot) - z^*$ system (1) can be converted to the following error system:

$$\begin{aligned}\dot{x}(t) = &-(C + \Delta C(t))x(t) + (A + \Delta A(t))f(x(t)) \\ &+ (B + \Delta B(t))f(x(t - \tau(t))) + (D + \Delta D(t)) \\ &\times \int_{t-d}^{t} f(x(s))ds\end{aligned} \tag{6}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \Re^n$ is the neuron vector,$f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^T \in \Re^n$ denotes the neuron activation function.$f_i(x(\cdot)) = g_i(z_i(\cdot)) - g_i(z_i^*), i = 1, 2, \ldots, n.$

$$\gamma_i^- \leq \frac{f_i(x_i(t))}{x_i(t)} \leq \gamma_i^+, f_i(0) = 0, i = 1, 2, \ldots, n \tag{7}$$

System(6) can be written as

$$\begin{cases}\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) + D \\ \qquad \times \int_{t-d}^{t} f(x(s))ds + Wp(t) \\ p(t) = F(t)(-E_c x(t) + E_a f(x(t)) + E_b f(x(t - \tau(t))) \\ \qquad + E_d \int_{t-d}^{t} f(x(s))ds\end{cases} \tag{8}$$

By translating d to function $d(t)$, we have

$$\begin{cases}\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) + D \\ \qquad \times \int_{t-d}^{t} f(x(s))ds + Wp(t) \\ p(t) = F(t)(-E_c x(t) + E_a f(x(t)) + E_b f(x(t - \tau(t))) \\ \qquad + E_d \int_{t-d(t)}^{t} f(x(s))ds\end{cases} \tag{9}$$

where $0 \leq d(t) \leq d$.

Definition1.The equilibrium point of system (16) is said to be globally exponentially stable,if there exist scalars $k \geq 0$ and $\beta > 0$ such that

$$\|x(t)\| \leq \beta e^{-kt} \sup_{-\tau_2 \leq s \leq 0} \|\phi(s) - z^*\|, \forall t > 0. \tag{10}$$

Lemma 1.[9].For any constant positive matrix $Z = Z^T > 0$, $Z \in \Re^{n \times n}$, scalars $h_1 > h_2 > 0$, such that the following integrations are well defined, then

$$\begin{aligned}&-(h_2 - h_1)\int_{h_2}^{h_1} x^T(s)Zx(s)ds \\ &\leq -\int_{h_2}^{h_1} x^T(s)ds Z \int_{h_2}^{h_1} x(s)ds\end{aligned} \tag{11}$$

Lemma 2.[10]The following inequalities are true:

$$0 \leq \int_0^{x_i(t)} [f_i(s) - \gamma_i^- s]ds \leq [f_i(x_i(t)) - \gamma_i^- x_i(t)]x_i(t) \tag{12}$$

$$0 \leq \int_0^{x_i(t)} [\gamma_i^+ s - f_i(s)]ds \leq [\gamma_i^- x_i(t) + f_i(x_i(t))]x_i(t) \tag{13}$$

Lemma 3.[11] For all real vectors a, b and all matrix $Q > 0$ with appropriate dimensions, if follows that:

$$2a^T b \leq a^T Q a + b^T Q^{-1} b \tag{14}$$

Lemma 4.[12] Given symmetric matrices Ω and D, E, of appropriate dimensions,

$$\Omega + DF(t)E + E^T F(t) D^T < 0$$

for all $F(t)$ satisfying $F^T(t)F(t) \leq I$,if and only if there exists some $\varepsilon > 0$ such that

$$\Omega + \varepsilon DD^T + \varepsilon^{-1} E^T E < 0. \tag{15}$$

III. MAIN RESULTS

In this section, we propose a new exponential criterion for the uncertain neural networks with time-varying delays system(9).First,we let $\Delta C(t) = 0, \Delta A(t) = 0, \Delta B(t) = 0$ and $\Delta D(t) = 0$,the system as following:

$$\begin{aligned}\dot{x}(t) = &-Cx(t) + Ag(x(t)) + Bg(x(t - \tau(t))) \\ &+ D\int_{t-d(t)}^{t} g(z(s))ds\end{aligned} \tag{16}$$

Now, we have the following main results.

Theorem1.For given scalars $\Gamma_1 = diag(\gamma_1^-, \gamma_2^-, \ldots, \gamma_n^-)$ $\Gamma_2 = diag(\gamma_1^+, \gamma_2^+, \ldots, \gamma_n^+)$, $d \geq 0$, $u \leq 1$, $\tau_{12} = \tau_2 - \tau_1$, the system(16) is globally exponential stable with the exponential convergence rate index k if there exist symmetric positive definite matrices, Q_i $(i = 1, 2, \ldots, 6)$, R_i $(i = 1, 2, 3, 4)$, P, $S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ * & S_{22} & S_{23} \\ * & * & S_{33} \end{bmatrix}$, positive diagonal matrices M_1, M_2, $\Lambda_1 = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\Lambda_2 = diag(\delta_1, \delta_2, \ldots, \delta_n)$ such that the following LMIs hold:

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0 & 0 & 0 & e_{19} \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 & 0 & e_{29} \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 & 0 & e_{39} \\ * & * & * & e_{44} & e_{45} & e_{46} & 0 & 0 & 0 \\ * & * & * & * & e_{55} & e_{56} & e_{57} & 0 & 0 \\ * & * & * & * & * & e_{66} & e_{67} & e_{68} & 0 \\ * & * & * & * & * & * & e_{77} & e_{78} & 0 \\ * & * & * & * & * & * & * & e_{88} & 0 \\ * & * & * & * & * & * & * & * & e_{99} \end{bmatrix} \leq 0 \tag{17}$$

$$F=\begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0 & 0 & 0 & e_{19} \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 & 0 & e_{29} \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 & 0 & e_{39} \\ * & * & * & f_{44} & 0 & f_{46} & f_{47} & 0 & 0 \\ * & * & * & * & e_{55} & f_{56} & e_{57} & 0 & 0 \\ * & * & * & * & * & e_{66} & f_{67} & e_{68} & 0 \\ * & * & * & * & * & * & f_{77} & f_{78} & 0 \\ * & * & * & * & * & * & * & e_{88} & 0 \\ * & * & * & * & * & * & * & * & e_{99} \end{bmatrix} \leq 0 \quad (18)$$

$$G=\begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0 & 0 & 0 & e_{19} \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 & 0 & e_{29} \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 & 0 & e_{39} \\ * & * & * & g_{44} & 0 & 0 & g_{47} & g_{48} & 0 \\ * & * & * & * & e_{55} & f_{56} & e_{57} & 0 & 0 \\ * & * & * & * & * & g_{66} & f_{67} & e_{68} & 0 \\ * & * & * & * & * & * & f_{77} & e_{78} & 0 \\ * & * & * & * & * & * & * & e_{88} & 0 \\ * & * & * & * & * & * & * & * & e_{99} \end{bmatrix} \leq 0 \quad (19)$$

$$\begin{aligned}
e_{11} &= 2kP - PC - CP - 4k\Gamma_1\Lambda_1 - C(\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1) \\
&\quad + (\frac{\tau_{12}}{3})^2 C(R_1+R_2+R_3)C - (\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1)C \\
&\quad + \sum_{i=1}^{5} Q_i + 4k\Gamma_2\Lambda_2 - 2\Gamma_1 M_1\Gamma_2 \\
e_{12} &= PA + 2k\Lambda_1 - (\Lambda_1 - \Lambda_2)C + (\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1)A \\
&\quad - 2k\Lambda_2 - (\frac{\tau_{12}}{3})^2 C(R_1+R_2+R_3)A + M_1(\Gamma_1+\Gamma_2) \\
e_{13} &= PB + (\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1)B - (\frac{\tau_{12}}{3})^2 C(R_1+R_2+R_3)B \\
e_{19} &= PD + (\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1)D - (\frac{\tau_{12}}{3})^2 C(R_1+R_2+R_3)D \\
e_{22} &= (\Lambda_1 - \Lambda_2)A - 2M_1 + (\frac{\tau_{12}}{3})^2 A^T(R_1+R_2+R_3)A \\
&\quad + A^T(\Lambda_1 - \Lambda_2) \\
e_{23} &= (\Lambda_1 - \Lambda_2)B + (\frac{\tau_{12}}{3})^2 A^T(R_1+R_2+R_3)B \\
e_{29} &= (\Lambda_1 - \Lambda_2)D + (\frac{\tau_{12}}{3})^2 A^T(R_1+R_2+R_3)D \\
e_{33} &= (\frac{\tau_{12}}{3})^2 B^T(R_1+R_2+R_3)B - 2M_2 \\
&\quad - e^{-2k\tau_2}(1-u)Q_6, e_{34} = M_2(\Gamma_1+\Gamma_2) \\
e_{39} &= (\frac{\tau_{12}}{3})^2 B^T(R_1+R_2+R_3)D \\
e_{44} &= -e^{-2k\tau_2}(1-u)Q_1 - 2e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3 \\
&\quad - 2\Gamma_1 M_2\Gamma_2 \\
e_{45} &= e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3, e_{46} = e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3 \\
e_{55} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3 - e^{-2k\tau_1}Q_2 + e^{-2k\tau_1}S_{11} \\
e_{56} &= e^{-2k\tau_1}S_{12}, e_{57} = e^{-2k\tau_1}S_{13} \\
e_{66} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}(Q_3 + S_{11} + R_3 + R_2) \\
&\quad + e^{-2k\tau_1}(S_{22} - R_1) \\
e_{67} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{12} + e^{-2k\frac{2\tau_2+\tau_1}{3}}R_2 + e^{-2k\tau_1}S_{23} \\
e_{77} &= e^{-2k\tau_1}S_{33} - e^{-2k\frac{2\tau_2+\tau_1}{3}}(R_2+Q_4) \\
&\quad - e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{22}, e_{78} = -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{23} \\
e_{88} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{33} - e^{-2k\tau_2}(Q_5+R_1) \\
e_{99} &= (\frac{\tau_{12}}{3})^2 D^T(R_1+R_2+R_3)D - e^{-2kd}R_4 \\
f_{44} &= -e^{-2k\tau_2}(1-u)Q_1 - 2e^{-2k\frac{2\tau_1+\tau_2}{3}}R_2 \\
&\quad - 2\Gamma_1 M_2\Gamma_2 \\
f_{46} &= e^{-2k\frac{2\tau_1+\tau_2}{3}}R_2, f_{47} = e^{-2k\frac{2\tau_1+\tau_2}{3}}R_2, \\
f_{56} &= e^{-2k\tau_1}S_{12} + e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3 \\
f_{67} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{12} + e^{-2k\tau_1}S_{23} \\
f_{78} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{23} + e^{-2k\tau_2}R_1 \\
f_{77} &= e^{-2k\tau_1}S_{33} - e^{-2k\frac{2\tau_2+\tau_1}{3}}(R_2+Q_4) - e^{-2k\tau_2}R_1 \\
g_{44} &= -e^{-2k\tau_2}(1-u)Q_1 - 2e^{-2k\tau_2}R_1 - 2\Gamma_1 M_2\Gamma_2 \\
g_{47} &= 2e^{-2k\tau_2}R_1, g_{48} = 2e^{-2k\tau_2}R_1 \\
g_{66} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}(Q_3 + S_{11} + R_2) + e^{-2k\tau_1}S_{22}
\end{aligned}$$

Proof: Construct a Lyapunov function as follows:

$$V(x_t) = \sum_{i=1}^{6} V_i(x_t)$$

where

$$V_1(x_t) = e^{2kt}x^T(t)Px(t)$$

$$\begin{aligned}
V_2(x_t) &= 2e^{2kt}\sum_{i=1}^{n}[\int_0^{x_i(t)} \lambda_i(f_i(s) - \gamma_i^- s)ds \\
&\quad + \int_0^{x_i(t)} \delta_i(\gamma_i^+ s - f_i(s))ds]
\end{aligned}$$

$$\begin{aligned}
V_3(x_t) &= \int_{t-\tau(t)}^{t} e^{2ks}x^T(s)Q_1x(s)ds \\
&\quad + \int_{t-\tau_1}^{t} e^{2ks}x^T(s)Q_2x(s)ds \\
&\quad + \int_{t-\frac{2\tau_1+\tau_2}{3}}^{t} e^{2ks}x^T(s)Q_3x(s)ds \\
&\quad + \int_{t-\frac{\tau_1+2\tau_2}{3}}^{t} e^{2ks}x^T(s)Q_4x(s)ds \\
&\quad + \int_{t-\tau_2}^{t} e^{2ks}x^T(s)Q_5x(s) \\
&\quad + \int_{t-\tau(t)}^{t} e^{2ks}f^T(x(s))Q_6f(x(s))ds
\end{aligned}$$

$$\begin{aligned}
V_4(x_t) &= \int_{t-\frac{\tau_2+2\tau_1}{3}}^{t-\tau_1} e^{2ks}\begin{bmatrix} x(s) \\ x(s-\frac{\tau_2-\tau_1}{3}) \\ x(s-\frac{2(\tau_2-\tau_1)}{3}) \end{bmatrix}^T \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ * & S_{22} & S_{23} \\ * & * & S_{33} \end{bmatrix} \\
&\quad \times \begin{bmatrix} x(s) \\ x(s-\frac{\tau_2-\tau_1}{3}) \\ x(s-\frac{2(\tau_2-\tau_1)}{3}) \end{bmatrix} ds
\end{aligned}$$

$$V_5(x_t) = \frac{\tau_2-\tau_1}{3}\int_{-\tau_2}^{-\frac{2\tau_2+\tau_1}{3}}\int_{t+\theta}^{t} e^{2ks}\dot{x}(s)^T R_1\dot{x}(s)dsd\theta + \frac{\tau_2-\tau_1}{3}\int_{-\frac{2\tau_2+\tau_1}{3}}^{-\frac{\tau_2+2\tau_1}{3}}\int_{t+\theta}^{t} e^{2ks}\dot{x}(s)^T R_2\dot{x}(s)dsd\theta + \frac{\tau_2-\tau_1}{3}\int_{-\frac{\tau_2+2\tau_1}{3}}^{-\tau_1}\int_{t+\theta}^{t} e^{2ks}\dot{x}(s)^T R_3\dot{x}(s)dsd\theta$$

$$V_6(x_t) = d\int_{-d}^{0}\int_{t+\theta}^{t} e^{2ks} f^T(x(s))R_4 f(x(s))$$

The time derivative of $V(x_t)$ along the trajectory of system (16)is given by

$$\dot{V}(x_t) = \sum_{i=1}^{6}\dot{V}_i(x_t)$$

where

$$\dot{V}_1(x_t) = 2ke^{2kt}x^T(t)Px(t) + 2e^{2kt}x^T(t)P\dot{x}(t) \tag{20}$$

$$\dot{V}_2(x_t) = 4ke^{2kt}\sum_{i=1}^{n}[\int_0^{x_i(t)}\lambda_i(f_i(s)-\gamma_i^- s)ds + \int_0^{x_i(t)}\delta_i(\gamma_i^+ s - f_i(s))ds] + 2e^{2kt}[(f^T(x(t)) - x^T(t)\Gamma_1)\Lambda\dot{x}(t) + (x^T(t)\Gamma_2 - f^T(x(t)))\Delta\dot{x}(t)] \tag{21}$$

$$\begin{aligned}\dot{V}_3(x_t) \le{}& e^{2kt}x^T(t)(\sum_{i=1}^{6}Q_i)x(t) - e^{2k(t-\tau_2)}(1-u)\\ &\times x^T(t-\tau(t))Q_1x(t-\tau(t)) + e^{2kt}f(x(t))Q_6f^T(x(t))\\ &- e^{2k(t-\tau_1)}x^T(t-\tau_1)Q_2x(t-\tau_1)\\ &- e^{2k(t-\frac{2\tau_1+\tau_2}{3})}x^T(t-\frac{2\tau_1+\tau_2}{3})Q_3x(t-\frac{2\tau_1+\tau_2}{3})\\ &- e^{2k(t-\frac{2\tau_2+\tau_1}{3})}x^T(t-\frac{2\tau_2+\tau_1}{3})Q_4x(t-\frac{2\tau_1+\tau_2}{3})\\ &- e^{2k(t-\tau_2)}x^T(t-\tau_2)Q_5x(t-\tau_2)\\ &- e^{2k(t-\tau_2)}(1-u)f(x(t-\tau(t)))Q_6f^T(x(t-\tau(t)))\end{aligned} \tag{22}$$

$$\begin{aligned}\dot{V}_4(x_t) \le{}& e^{2k(t-\tau_1)}\begin{bmatrix}x(t-\tau_1)\\ x(t-\frac{2\tau_1+\tau_2}{3})\\ x(t-\frac{2\tau_2+\tau_1}{3})\end{bmatrix}^T\begin{bmatrix}S_{11}&S_{12}&S_{13}\\ *&S_{22}&S_{23}\\ *&*&S_{33}\end{bmatrix}\\ &\times\begin{bmatrix}x(t-\tau_1)\\ x(t-\frac{2\tau_1+\tau_2}{3})\\ x(t-\frac{2\tau_2+\tau_1}{3})\end{bmatrix}\\ &- e^{2k(t-\frac{2\tau_1+\tau_2}{3})}\begin{bmatrix}x(t-\frac{2\tau_1+\tau_2}{3})\\ x(t-\frac{2\tau_2+\tau_1}{3})\\ x(t-\tau_2)\end{bmatrix}^T\begin{bmatrix}S_{11}&S_{12}&S_{13}\\ *&S_{22}&S_{23}\\ *&*&S_{33}\end{bmatrix}\\ &\times\begin{bmatrix}x(t-\frac{2\tau_1+\tau_2}{3})\\ x(t-\frac{2\tau_2+\tau_1}{3})\\ x(t-\tau_2)\end{bmatrix}\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{V}_5(x_t) = {}&(\frac{\tau_2-\tau_1}{3})^2e^{2kt}\dot{x}^T(t)(R_1+R_2+R_3)\dot{x}(t)\\ &- \frac{\tau_2-\tau_1}{3}\int_{t-\tau_2}^{t-\frac{2\tau_2+\tau_1}{3}}e^{2ks}\dot{x}^T(s)R_1\dot{x}(s)ds\\ &- \frac{\tau_2-\tau_1}{3}\int_{t-\frac{2\tau_2+\tau_1}{3}}^{t-\frac{2\tau_1+\tau_2}{3}}e^{2ks}\dot{x}^T(s)R_2\dot{x}(s)ds\\ &- \frac{\tau_2-\tau_1}{3}\int_{t-\frac{2\tau_1+\tau_2}{3}}^{t-\tau_1}e^{2ks}\dot{x}^T(s)R_3\dot{x}(s)ds\end{aligned} \tag{24}$$

(1)When $\tau_1 \le \tau(t) \le \frac{2\tau_1+\tau_2}{3}$. Based on the bounds lemma of[16], we have

$$\begin{aligned}-\frac{\tau_2-\tau_1}{3}\int_{t-\frac{2\tau_1+\tau_2}{3}}^{t-\tau_1}e^{2ks}\dot{x}^T(s)R_3\dot{x}(s)ds \le{}& e^{2k(t-\frac{2\tau_1+\tau_2}{3})}\\ &\times\begin{bmatrix}x(t-\tau_1)\\ x(t-\tau(t))\\ x(t-\frac{2\tau_1+\tau_2}{3})\end{bmatrix}^T\begin{bmatrix}-R_3&R_3&0\\ *&-2R_3&R_3\\ *&*&-R_3\end{bmatrix}\\ &\times\begin{bmatrix}x(t-\tau_1)\\ x(t-\tau(t))\\ x(t-\frac{2\tau_1+\tau_2}{3})\end{bmatrix}\end{aligned} \tag{25}$$

$$\begin{aligned}&-\frac{\tau_2-\tau_1}{3}\int_{t-\tau_2}^{t-\frac{2\tau_2+\tau_1}{3}}e^{2ks}\dot{x}^T(s)R_1\dot{x}(s)ds \le -e^{2k(t-\tau_2)}\\ &\times[x(t-\frac{2\tau_2+\tau_1}{3})-x(t-\tau_2)]^TR_1[x(t-\frac{2\tau_2+\tau_1}{3})\\ &-x(t-\tau_2)]\end{aligned} \tag{26}$$

$$\begin{aligned}&-\frac{\tau_2-\tau_1}{3}\int_{t-\frac{2\tau_2+\tau_1}{3}}^{t-\frac{2\tau_1+\tau_2}{3}}e^{2ks}\dot{x}^T(s)R_2\dot{x}(s)ds \le -e^{2k(t-\frac{2\tau_2+\tau_1}{3})}\\ &\times[x(t-\frac{2\tau_1+\tau_2}{3})-x(\frac{2\tau_2+\tau_1}{3})]^TR_2[x(t-\frac{2\tau_1+\tau_2}{3})\\ &-x(t-\frac{2\tau_2+\tau_1}{3})]\end{aligned} \tag{27}$$

(2)When $\frac{2\tau_1+\tau_2}{3} \le \tau(t) \le \frac{2\tau_2+\tau_1}{3}$. we have

$$\begin{aligned}-\frac{\tau_2-\tau_1}{3}\int_{t-\frac{2\tau_1+\tau_2}{3}}^{t-\frac{2\tau_2+\tau_1}{3}}e^{2ks}\dot{x}^T(s)R_2\dot{x}(s)ds \le{}& e^{2k(t-\frac{2\tau_2+\tau_1}{3})}\\ &\times\begin{bmatrix}x(t-\frac{2\tau_1+\tau_2}{3})\\ x(t-\tau(t))\\ x(t-\frac{2\tau_2+\tau_1}{3})\end{bmatrix}^T\begin{bmatrix}-R_2&R_2&0\\ *&-2R_2&R_2\\ *&*&-R_2\end{bmatrix}\\ &\times\begin{bmatrix}x(t-\frac{2\tau_1+\tau_2}{3})\\ x(t-\tau(t))\\ x(t-\frac{2\tau_2+\tau_1}{3})\end{bmatrix}\end{aligned} \tag{28}$$

$$\begin{aligned}&-\frac{\tau_2-\tau_1}{3}\int_{t-\tau_2}^{t-\frac{2\tau_2+\tau_1}{3}}e^{2ks}\dot{x}^T(s)R_1\dot{x}(s)ds \le -e^{2k(t-\tau_2)}\\ &\times[x(t-\frac{2\tau_2+\tau_1}{3})-x(t-\tau_2)]^TR_1[x(t-\frac{2\tau_2+\tau_1}{3})\\ &-x(t-\tau_2)]\end{aligned} \tag{29}$$

$$-\frac{\tau_2-\tau_1}{3}\int_{t-\frac{2\tau_1+\tau_2}{3}}^{t-\tau_1} e^{2ks}\dot{x}^T(s)R_3\dot{x}(s) \le -e^{2k(t-\frac{2\tau_1+\tau_2}{3})} \times [x(t-\tau_1)-x(t-\frac{2\tau_1+\tau_2}{3})]^T R_3[x(t-\tau_1) - x(t-\frac{2\tau_1+\tau_2}{3})] \tag{30}$$

(3)When $\frac{2\tau_2+\tau_1}{3} \le \tau(t) \le \tau_2$. Based on the bounds lemma of[16], we have

$$-\frac{\tau_2-\tau_1}{3}\int_{t-\tau_2}^{t-\frac{2\tau_2+\tau_1}{3}} e^{2ks}\dot{x}^T(s)R_1\dot{x}(s)ds \le e^{2k(t-\tau_2)} \times \begin{bmatrix} x(t-\frac{2\tau_2+\tau_1}{3}) \\ x(t-\tau(t)) \\ x(t-\tau_2) \end{bmatrix}^T \begin{bmatrix} -R_1 & R_1 & 0 \\ * & -2R_1 & R_1 \\ * & * & -R_1 \end{bmatrix} \times \begin{bmatrix} x(t-\frac{2\tau_2+\tau_1}{3}) \\ x(t-\tau(t)) \\ x(t-\tau_2) \end{bmatrix} \tag{31}$$

$$-\frac{\tau_2-\tau_1}{3}\int_{t-\frac{2\tau_2+\tau_1}{3}}^{t-\frac{2\tau_1+\tau_2}{3}} e^{2ks}\dot{x}^T(s)R_2\dot{x}(s)ds \le -e^{2k(t-\frac{2\tau_2+\tau_1}{3})} \times [x(t-\frac{2\tau_1+\tau_2}{3})-x(\frac{2\tau_2+\tau_1}{3})]^T R_2[x(t-\frac{2\tau_1+\tau_2}{3}) - x(t-\frac{2\tau_2+\tau_1}{3})] \tag{32}$$

$$-\frac{\tau_2-\tau_1}{3}\int_{t-\frac{2\tau_1+\tau_2}{3}}^{t-\tau_1} e^{2ks}\dot{x}^T(s)R_3\dot{x}(s) \le -e^{2k(t-\frac{2\tau_1+\tau_2}{3})} \times [x(t-\tau_1)-x(t-\frac{2\tau_1+\tau_2}{3})]^T R_3[x(t-\tau_1) - x(t-\frac{2\tau_1+\tau_2}{3})] \tag{33}$$

$$\begin{aligned}\dot{V}_6(x_t) &= d^2e^{2kt}f^T(x(s))R_4f(x(s)) \\ &\quad - d\int_{t-d}^{t} e^{2ks}f^T(x(s))R_4f(x(s))ds \\ &\le d^2e^{2kt}f^T(x(t))R_4f(x(t)) \\ &\quad - e^{2k(t-d)}\int_{t-d(t)}^{t} f^T(x(s))ds R_4 \int_{t-d(t)}^{t} f(x(s))ds\end{aligned} \tag{34}$$

In order to derive less conservative results,we add the following inequalities with positive diagonal matrices M_1, M_2

$$e^{2kt}[-2f^T(x(t))M_1f(x(t)) + 2x^T(t)M_1(\Gamma_1+\Gamma_2)f(x(t)) - 2x^T(t)\Gamma_1M_1\Gamma_2x(t)] \ge 0 \tag{35}$$

$$e^{2kt}[-2f^T(x(t-\tau(t)))M_2f(x(t-\tau(t))) + 2x^T(t-\tau(t))M_2 \times (\Gamma_1+\Gamma_2)f(x(t-\tau(t)) - 2x^T(t-\tau(t))\Gamma_1M_2\Gamma_2 \times x(t-\tau(t))] \ge 0 \tag{36}$$

(1)When $\tau_1 \le \tau(t) \le \frac{2\tau_1+\tau_2}{3}$. According to (17),from $(20)-(27),(34)-(36)$.Then one can obtain

$$\dot{V}(x_t) \le e^{2kt}[\xi^T(t)E\xi(t)] \le 0 \tag{37}$$

where

$$\begin{aligned}\xi^T(t) =&[x(t), f(x(t)), f(x(t-\tau(t))), x(t-\tau(t)), x(t-\tau_1), \\ &x(t-\frac{2\tau_1+\tau_2}{3}), x(t-\frac{2\tau_2+\tau_1}{3}), x(t-\tau_2), \\ &\int_{t-d(t)}^{t} f(x(s))ds]\end{aligned}$$

(2)When $\frac{2\tau_1+\tau_2}{3} \le \tau(t) \le \frac{2\tau_2+\tau_1}{3}$. According to (18), from $(20)-(24),(28)-(30),(34)-(36)$. Then one can obtain

$$\dot{V}(x_t) \le e^{2kt}[\xi^T(t)F\xi(t)] \le 0 \tag{38}$$

(3)When $\frac{2\tau_2+\tau_1}{3} \le \tau(t) \le \tau_2$ According to (19), from $(20)-(24),(31)-(36)$.Then one can obtain

$$\dot{V}(x_t) \le e^{2kt}[\xi^T(t)G\xi(t)] \le 0 \tag{39}$$

Therefore, if $(17)-(19)$, are satisfied, the system is globally exponentially stable.

On the other hand,

$$V_1(x_0) \le \lambda_{\max}(P)\|x(0)\|^2 \le \lambda_{\max}(P)\sup_{-\tau_2\le s\le 0}\|x(s)\|^2 \tag{40}$$

$$\begin{aligned}V_2(x_0) &\le 2[f(x(0)) - \Gamma_1x(0)]^T\Lambda x(0) + 2[\Gamma_2x(0) \\ &\quad - f(x(0))]^T\Delta x(0) \\ &\le 2\lambda_{\max}(\Gamma_2-\Gamma_1)(\lambda_{\max}(\Lambda)+\lambda_{\max}(\Delta)) \\ &\quad \times \sup_{-\tau_2\le s\le 0}\|x(s)\|^2\end{aligned} \tag{41}$$

$$\begin{aligned}V_3(x_0) &\le (\tau_2\lambda_{\max}(Q_1) + \tau_1\lambda_{\max}(Q_2) + \frac{2\tau_1+\tau_2}{3}\lambda_{\max}(Q_3) \\ &\quad + \frac{\tau_1+2\tau_2}{3}\lambda_{\max}(Q_4) + \tau_2\lambda_{\max}(Q_5) + \tau_2\gamma\lambda_{\max}(Q_6)) \\ &\quad \times \sup_{-\tau_2\le s\le 0}\|x(s)\|^2\end{aligned} \tag{42}$$

$$\begin{aligned}V_4(x_0) &\le \int_{-\frac{\tau_2+2\tau_1}{3}}^{\tau_1} e^{2ks}\begin{bmatrix} x(s) \\ x(s-\frac{\tau_{12}}{3}) \\ x(s-\frac{2\tau_{12}}{3}) \end{bmatrix}^T S \begin{bmatrix} x(s) \\ x(s-\frac{\tau_{12}}{3}) \\ x(s-\frac{2\tau_{12}}{3}) \end{bmatrix} ds \\ &\le \int_{-\frac{\tau_2+2\tau_1}{3}}^{\tau_1} [x^T(s)S_{11}x(s) + 2x^T(s)S_{12}x(s-\frac{\tau_{12}}{3}) \\ &\quad + 2x^T(s)S_{13}x(s-\frac{2\tau_{12}}{3}) + x^T(s-\frac{\tau_{12}}{3})S_{22}x(s-\frac{\tau_{12}}{3}) \\ &\quad + 2x^T(s-\frac{\tau_{12}}{3})S_{23}x(s-\frac{2\tau_{12}}{3}) + x^T(s-\frac{2\tau_{12}}{3})S_{33} \\ &\quad \times x(s-\frac{2\tau_{12}}{3})]ds\end{aligned} \tag{43}$$

According to Lemma 3

$$\begin{aligned}V_4(x_0) &\le \frac{\tau_{12}}{3}[\lambda_{\max}(S_{11}) + \lambda_{\max}(S_{12}) + \lambda_{\max}(S_{13})]\|x(s)\|^2 \\ &\quad + \frac{\tau_{12}}{3}[\lambda_{\max}(S_{12}) + \lambda_{\max}(S_{13}) + \lambda_{\max}(S_{22})] \\ &\quad \times \|x(s-\frac{\tau}{3})\|^2 + \frac{\tau}{3}[\lambda_{\max}(S_{22}) + \lambda_{\max}(S_{23})\end{aligned}$$

$$
\begin{aligned}
&+\lambda_{\max}(S_{33})]\|x(s)\|^2\\
&\le \frac{\tau_{12}}{3}[\lambda_{\max}(S_{11})+2\lambda_{\max}(S_{12})+2\lambda_{\max}(S_{13})\\
&+\lambda_{\max}(S_{22})+2\lambda_{\max}(S_{23})+\lambda_{\max}(S_{33})]\\
&\times \sup_{-\tau_2\le s\le 0}\|x(s)\|^2
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\dot{x}^T(t)\dot{x}(t) \le& 4[\lambda_{\max}(C^TC)+\gamma^2[\lambda_{\max}(A^TA)+\lambda_{\max}(B^TB)]\\
&+d^2\gamma^2\lambda_{\max}(D^TD)]\sup_{-\tau_2\le s\le 0}\|x(s)\|^2\\
\gamma =& \max_{1\le i\le n}\{|\gamma_i^-|,|\gamma_i^+|\}
\end{aligned}
$$

$$
\begin{aligned}
V_5(x_0) \le& \frac{\tau_{12}}{3}\lambda_{\max}(R_1)\int_{-\tau_2}^{-\frac{2\tau_2+\tau_1}{3}}\int_{\theta}^{0}\dot{x}^T(s)\dot{x}(s)dsd\theta\\
&+\frac{\tau_{12}}{3}\lambda_{\max}(R_2)\int_{-\frac{2\tau_2+\tau_1}{3}}^{-\frac{\tau_1+2\tau_2}{3}}\int_{\theta}^{0}\dot{x}^T(s)\dot{x}(s)dsd\theta\\
&+\frac{\tau_{12}}{3}\lambda_{\max}(R_3)\int_{-\frac{\tau_1+1\tau_2}{3}}^{\tau_1}\int_{\theta}^{0}\dot{x}^T(s)\dot{x}(s)dsd\theta\\
\le& \frac{\tau_{12}}{3}[\frac{\tau_{12}(\tau_1+5\tau_2)}{9}\lambda_{\max}(R_1)\\
&+\frac{\tau_{12}(\tau_1+\tau_2)}{3}\lambda_{\max}(R_2)\\
&+\frac{\tau_{12}}{3}[\frac{\tau_{12}(5\tau_1+\tau_2)}{9}\lambda_{\max}(R_1)][\lambda_{\max}(C^TC)\\
&+\gamma^2\lambda_{\max}(A^TA)+\gamma^2\lambda_{\max}(B^TB)]\sup_{-\tau_2\le s\le 0}\|x(s)\|^2
\end{aligned}
\tag{45}
$$

$$
V_6(x_0) \le \frac{d^3\gamma^2}{2}\lambda_{\max}(R_4)\sup_{-\tau_2\le s\le 0}\|x(s)\|^2 \tag{46}
$$

According to $(33)-(38)$, we can get the following inequalities:

$$
V(x(0)) \le \omega \sup_{-\tau_2\le s\le 0}\|x(s)\|^2 \tag{47}
$$

where

$$
\begin{aligned}
\omega =& \lambda_{\max}(P)+2\lambda_{\max}(\Gamma_2-\Gamma_1)(\lambda_{\max}(\Lambda)+\lambda_{\max}(\Delta))\\
&+(\tau_2\lambda_{\max}(Q_1)+\tau_1\lambda_{\max}(Q_2)+\frac{2\tau_1+\tau_2}{3}\lambda_{\max}(Q_3)\\
&+\frac{\tau_1+2\tau_2}{3}\lambda_{\max}(Q_4)+\tau_2\lambda_{\max}(Q_5)+\tau_2\gamma\lambda_{\max}(Q_6)\\
&+\frac{\tau_{12}}{3}[\lambda_{\max}(S_{11})+\lambda_{\max}(S_{12})+\lambda_{\max}(S_{13})]\|x(s)\|^2\\
&+\frac{\tau_{12}}{3}[\lambda_{\max}(S_{12})+\lambda_{\max}(S_{13})+\lambda_{\max}(S_{22})]\\
&\times\|x(s-\frac{\tau}{3})\|^2+\frac{\tau}{3}[\lambda_{\max}(S_{22})+\lambda_{\max}(S_{23})+\lambda_{\max}(S_{33})]\\
&\times\|x(s)\|^2+\frac{\tau_{12}}{3}[\lambda_{\max}(S_{11})+2\lambda_{\max}(S_{12})+2\lambda_{\max}(S_{13})\\
&+\lambda_{\max}(S_{22})+2\lambda_{\max}(S_{23})+\lambda_{\max}(S_{33})]\\
&+\frac{\tau_{12}}{3}[\frac{\tau_{12}(\tau_1+5\tau_2)}{9}\lambda_{\max}(R_1)+\frac{\tau_{12}(\tau_1+\tau_2)}{3}\lambda_{\max}(R_2)\\
&+\frac{\tau_{12}}{3}[\frac{\tau_{12}(5\tau_1+\tau_2)}{9}\lambda_{\max}(R_1)][\lambda_{\max}(C^TC)\\
&+\gamma^2\lambda_{\max}(A^TA)+\gamma^2\lambda_{\max}(B^TB)]+\frac{d^3\gamma^2}{2}\lambda_{\max}(R_4)
\end{aligned}
\tag{48}
$$

On the other hand, we have

$$
V(x(t)) \ge e^{2kt}\lambda_{\min}(P)\|x(t)\|^2 \tag{49}
$$

Therefore

$$
\|x(t)\| \le \sqrt{\frac{\omega}{\lambda_{\min}(P)}}e^{-kt}\sup_{-\tau_2\le s\le 0}\|x(s)\| \tag{50}
$$

Thus,according to definition1 the system (16) is exponentially stable, the proof is completed. ■

Theorem 2. For given scalars $\Gamma_1 = diag(\gamma_1^-,\gamma_2^-,\ldots,\gamma_n^-)$, $\Gamma_2 = diag(\gamma_1^+,\gamma_2^+,\ldots,\gamma_n^+)$, $d\ge 0$, $u\le 1$,$\tau_{12}=\tau_2-\tau_1$, the system(9) is globally exponentially stable with the exponential convergence rate index k if there exist symmetric positive definite matrices P, $Q_i(i=1,2,\ldots,6)$,$R_i(i=1,2,3,4)$, $S=\begin{bmatrix}S_{11}&S_{12}&S_{13}\\ *&S_{22}&S_{23}\\ *&*&S_{33}\end{bmatrix}$ positive diagonal matrices M_1, M_2, $\Lambda_1 = diag(\lambda_1,\lambda_2,\ldots,\lambda_n)$, $\Lambda_2 = diag(\delta_1,\delta_2,\ldots,\delta_n)$, a scalar $\varepsilon>0$,such that the following LMIs hold:

$$
\begin{aligned}
&E+\varepsilon\Upsilon_1\Upsilon_1^T+\varepsilon^{-1}\Upsilon_2^T\Upsilon_2<0.\\
&F+\varepsilon\Upsilon_1\Upsilon_1^T+\varepsilon^{-1}\Upsilon_2^T\Upsilon_2<0.\\
&G+\varepsilon\Upsilon_1\Upsilon_1^T+\varepsilon^{-1}\Upsilon_2^T\Upsilon_2<0.
\end{aligned}
\tag{51}
$$

where

$$
\begin{aligned}
&\Upsilon_1 = [\overline{\Upsilon}_1,(\Lambda_1-\Lambda_2)W+\Xi,\Xi,0,0,0,0,0,\Xi]^T\\
&\Upsilon_1 = [-E_c,E_a,E_b,0,0,0,0,0,E_d]\\
&\overline{\Upsilon}_1 = PW-(\Gamma_2\Lambda_2-\Gamma_1\Lambda_1)W+\Xi\\
&\Xi = \frac{1}{2}(R_1+R_2+R_3)W\times\frac{\tau_{12}}{3}^2
\end{aligned}
$$

Proof: In Theorem 1, we replace C,A,B,D with $C+\Delta C(t), A+\Delta A(t), B+\Delta B(t), D+\Delta D(t)$, then adding to Lemma 4, we can get the results. This completes the proof. ■

Colloary 1.For given scalars $\Gamma_1 = diag(\gamma_1^-,\gamma_2^-,\ldots,\gamma_n^-)$, $\Gamma_2 = diag(\gamma_1^+,\gamma_2^+,\ldots,\gamma_n^+)$, $d\ge 0$, $u\le 1$, $\tau_{12}=\tau_2-\tau_1$, the system(16) is globally exponential stable with the exponential convergence rate index k if there exist symmetric positive definite matrices P, $Q_i(i=2,\ldots,5)$, $R_i(i=1,2,3,4)$, $S=\begin{bmatrix}S_{11}&S_{12}&S_{13}\\ *&S_{22}&S_{23}\\ *&*&S_{33}\end{bmatrix}$ positive diagonal matrices M_1, M_2, $\Lambda_1 = diag(\lambda_1,\lambda_2,\ldots,\lambda_n)$, $\Lambda_2 = diag(\delta_1,\delta_2,\ldots,\delta_n)$ such that the following LMIs hold:

$$
E=\begin{bmatrix}
e_{11}&e_{12}&e_{13}&0&0&0&0&0&e_{19}\\
*&e_{22}&e_{23}&0&0&0&0&0&e_{29}\\
&&e_{33}&e_{34}&0&0&0&0&e_{39}\\
&&*&e_{44}&e_{45}&e_{46}&0&0&0\\
&&*&*&e_{55}&e_{56}&e_{57}&0&0\\
&&*&*&*&e_{66}&e_{67}&e_{68}&0\\
&&*&*&*&*&e_{77}&e_{78}&0\\
&&*&*&*&*&*&e_{88}&0\\
&&*&*&*&*&*&*&e_{99}
\end{bmatrix}\le 0 \tag{52}
$$

$$F=\begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0 & 0 & 0 & e_{19} \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 & 0 & e_{29} \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 & 0 & e_{39} \\ * & * & * & f_{44} & 0 & f_{46} & f_{47} & 0 & 0 \\ * & * & * & * & e_{55} & f_{56} & e_{57} & 0 & 0 \\ * & * & * & * & * & e_{66} & f_{67} & e_{68} & 0 \\ * & * & * & * & * & * & f_{77} & f_{78} & 0 \\ * & * & * & * & * & * & * & e_{88} & 0 \\ * & * & * & * & * & * & * & * & e_{99} \end{bmatrix} \leq 0 \quad (53)$$

$$G=\begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0 & 0 & 0 & e_{19} \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 & 0 & e_{29} \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 & 0 & e_{39} \\ * & * & * & g_{44} & 0 & 0 & g_{47} & g_{48} & 0 \\ * & * & * & * & e_{55} & f_{56} & e_{57} & 0 & 0 \\ * & * & * & * & * & g_{66} & f_{67} & e_{68} & 0 \\ * & * & * & * & * & * & f_{77} & e_{78} & 0 \\ * & * & * & * & * & * & * & e_{88} & 0 \\ * & * & * & * & * & * & * & * & e_{99} \end{bmatrix} \leq 0 \quad (54)$$

$$\begin{aligned}
e_{11} &= 2kP - PC - CP - 4k\Gamma_1\Lambda_1 - C(\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1) \\
&\quad + (\frac{\tau_{12}}{3})^2 C(R_1+R_2+R_3)C - (\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1)C \\
&\quad + \sum_{i=1}^{5} Q_i + 4k\Gamma_2\Lambda_2 - 2\Gamma_1 M_1\Gamma_2 \\
e_{12} &= PA + 2k\Lambda_1 - (\Lambda_1 - \Lambda_2)C + (\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1)A \\
&\quad - (\frac{\tau_{12}}{3})^2 C(R_1+R_2+R_3)A + M_1(\Gamma_1+\Gamma_2) - 2k\Lambda_2 \\
e_{13} &= PB + (\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1)B - (\frac{\tau_{12}}{3})^2 C(R_1+R_2+R_3)B \\
e_{19} &= PD + (\Gamma_2\Lambda_2 - \Gamma_1\Lambda_1)D - (\frac{\tau_{12}}{3})^2 C(R_1+R_2+R_3)D \\
e_{22} &= (\Lambda_1 - \Lambda_2)A - 2M_1 + (\frac{\tau_{12}}{3})^2 A^T(R_1+R_2+R_3)A \\
&\quad + A^T(\Lambda_1 - \Lambda_2) \\
e_{23} &= (\Lambda_1 - \Lambda_2)B + (\frac{\tau_{12}}{3})^2 A^T(R_1+R_2+R_3)B \\
e_{29} &= (\Lambda_1 - \Lambda_2)D + (\frac{\tau_{12}}{3})^2 A^T(R_1+R_2+R_3)D \\
e_{33} &= (\frac{\tau_{12}}{3})^2 B^T(R_1+R_2+R_3)B - 2M_2 \\
e_{34} &= M_2(\Gamma_1+\Gamma_2) \\
e_{39} &= (\frac{\tau_{12}}{3})^2 B^T(R_1+R_2+R_3)D \\
e_{44} &= -2e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3 - 2\Gamma_1 M_2\Gamma_2 \\
e_{45} &= e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3, e_{46} = e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3 \\
e_{55} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3 - e^{-2k\tau_1}Q_2 + e^{-2k\tau_1}S_{11} \\
e_{56} &= e^{-2k\tau_1}S_{12}, e_{57} = e^{-2k\tau_1}S_{13} \\
e_{66} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}(Q_3 + S_{11} + R_3 + R_2) \\
&\quad + e^{-2k\tau_1}(S_{22} - R_1) \\
e_{67} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{12} + e^{-2k\frac{2\tau_2+\tau_1}{3}}R_2 + e^{-2k\tau_1}S_{23} \\
e_{77} &= e^{-2k\tau_1}S_{33} - e^{-2k\frac{2\tau_2+\tau_1}{3}}(R_2+Q_4) \\
&\quad - e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{22}, e_{78} = -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{23} \\
e_{88} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{33} - e^{-2k\tau_2}(Q_5+R_1) \\
e_{99} &= (\frac{\tau_{12}}{3})^2 D^T(R_1+R_2+R_3)D - e^{-2kd}R_4 \\
f_{44} &= -2e^{-2k\frac{2\tau_1+\tau_2}{3}}R_2 - 2\Gamma_1 M_2\Gamma_2 \\
f_{46} &= e^{-2k\frac{2\tau_1+\tau_2}{3}}R_2, f_{47} = e^{-2k\frac{2\tau_1+\tau_2}{3}}R_2, \\
f_{56} &= e^{-2k\tau_1}S_{12} + e^{-2k\frac{2\tau_1+\tau_2}{3}}R_3 \\
f_{67} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{12} + e^{-2k\tau_1}S_{23} \\
f_{78} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}S_{23} + e^{-2k\tau_2}R_1 \\
e_{77} &= e^{-2k\tau_1}S_{33} - e^{-2k\frac{2\tau_2+\tau_1}{3}}(R_2+Q_4) - e^{-2k\tau_2}R_1 \\
g_{44} &= -2e^{-2k\tau_2}R_1 - 2\Gamma_1 M_2\Gamma_2 \\
g_{47} &= 2e^{-2k\tau_2}R_1, g_{48} = 2e^{-2k\tau_2}R_1 \\
g_{66} &= -e^{-2k\frac{2\tau_1+\tau_2}{3}}(Q_3 + S_{11} + R_2) + e^{-2k\tau_1}S_{22}
\end{aligned}$$

Proof: Choosing $Q_1 = 0, Q_6 = 0$ in Theorem 1,one can easily obtain this result. ■

Remark 1. This paper not only divides the delay interval$[\tau_1, \tau_2]$ into $[\tau_1, \frac{\tau_1+\tau_2}{2}]$ and $[\frac{\tau_1+\tau_2}{2}, \tau_2]$,but also divides $[\tau_1, \tau_2]$ into $[\tau_1, \frac{2\tau_1+\tau_2}{3}]$, $[\frac{2\tau_1+\tau_2}{3}, \frac{\tau_1+2\tau_2}{3}]$,$[\frac{\tau_1+2\tau_2}{3}, \tau_2]$ .Each segments has a different Lyapunov matrix,which have potential to yield less conservative results.

Remark 2. Unlike other papers [17-18],which $0 \leq \tau(t) \leq \tau$, in this paper we let $\tau_1 \leq \tau(t) \leq \tau_2$, consider $\tau_1 \neq 0$. Thus our results can obtain better for exponential stability criteria.

IV. Numerical examples

In this section, we provide the simulation of examples to illustrate the effectiveness of our method.

Example 1. Consider the system (16) with the following parameters:

$$C=\begin{bmatrix} 2.3 & 0 & 0 \\ 0 & 3.4 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}, A=\begin{bmatrix} 0.9 & -1.5 & 0.1 \\ -1.2 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.8 \end{bmatrix},$$
$$B=\begin{bmatrix} 0.8 & 0.6 & 0.2 \\ 0.5 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.5 \end{bmatrix}, D=\begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}$$
$\Gamma_1 = diag(0,0,0)$, $\Gamma_2 = diag(0.2, 0.2, 0.2)$

For the case of $\tau_2 = d$, $k = 0, \tau_1 = 0$, the upper bounds of τ for unknown u is derived by Corollary 1 in our paper and the results are listed in Table I. This example shows that the stability condition in this paper gives much less conservative results.

For the case of $d = 0.2$, $k = 2, \tau_1 = 0.5$, and various u, the maximum τ_2 are shown in Table II.

Example 2. Consider the system (16) with the following

TABLE I
ALLOWABLE UPPER BOUND OF τ FOR EXAMPLE 1

Method	Maximum of allowable τ
[13]	1.833
[14]	3.597
[15]	5.068
[16]	6.938
[17]	9.338
[18]	11.588
corollary 1	13.459

TABLE II
ALLOWABLE UPPER BOUND OF τ_2 FOR EXAMPLE 1

conditions	Theorem 1
$\tau_1 = 0.5, d = 0.2, , k = 2, u = 0.5$	6.3
$\tau_1 = 0.5, d = 0.2, k = 2, u = 0.8$	6.0
$\tau_1 = 0.5, d = 0.2, k = 2, u = 0.9$	5.8

TABLE III
ALLOWABLE UPPER BOUND OF k FOR EXAMPLE 2

conditions	[15]	[16]	[17]	Theorem 1
$\tau_2 = 0.5, d = 0.2, u = 0$	0.46	0.58	0.67	3.60
$\tau_2 = 0.5, d = 0.2, u = 0.5$	0.21	0.35	0.45	3.59

TABLE IV
ALLOWABLE UPPER BOUND OF τ_2 FOR EXAMPLE 3

τ_1	0.3	0.5	0.7	1.0	2.0
Theorem	15.200	15.111	15.000	14.670	14.668

parameters:

$$C = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}, A = \begin{bmatrix} 1.2 & -0.8 & 0.6 \\ 0.5 & -1.5 & 0.7 \\ -0.8 & -1.2 & -1.4 \end{bmatrix},$$

$$B = \begin{bmatrix} -1.4 & 0.9 & 0.5 \\ -0.6 & 1.2 & 0.8 \\ 0.5 & -0.7 & 1.1 \end{bmatrix}, D = \begin{bmatrix} 1.8 & 0.7 & -0.8 \\ 0.6 & 1.4 & 1.0 \\ -0.4 & -0.6 & 1.2 \end{bmatrix}$$

$\Gamma_1 = diag(-1.2, 0, -2.4)$, $\Gamma_2 = diag(0, 1.4, 0)$

For various τ_1, d and u, the maximum of the exponential convergence rate index k calculated by Theorem 1 in this paper are listed in Table III.

Example 3. Consider the system (16) with the following parameters:

$$C = \begin{bmatrix} 3.99 & 0 \\ 0 & 2.99 \end{bmatrix}, A = \begin{bmatrix} 1.188 & 0.09 \\ 0.09 & 1.188 \end{bmatrix}$$

$$,B = \begin{bmatrix} 0.009 & 0.14 \\ 0.05 & 0.09 \end{bmatrix}, D = \begin{bmatrix} 0.45 & -0.2 \\ 0.3 & 0.42 \end{bmatrix}$$

$\Gamma_1 = diag(0, 0)$, $\Gamma_2 = diag(1, 1)$

The corresponding upper of τ_2 for various τ_1 by Theorem 1 (letting $k = 1, u = 0.8, d = 0.3$) in Table IV.

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