World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:8, No:1, 2014

On the Parameter of the Burr Type X under Bayesian Principles

T. N. Sindhu, M. Aslam

***Abstract*—**A comprehensive Bayesian analysis has been carried out in the context of informative and non-informative priors for the shape parameter of the Burr type X distribution under different symmetric and asymmetric loss functions. Elicitation of hyperparameter through prior predictive approach is also discussed. Also we derive the expression for posterior predictive distributions, predictive intervals and the credible Intervals. As an illustration, comparisons of these estimators are made through simulation study.

***Keywords*—**Credible Intervals, Loss Functions, Posterior Predictive Distributions, Predictive Intervals.

I. INTRODUCTION

BURR [1] introduced twelve different forms of cumulative distribution functions for modeling data. Among those twelve distribution functions, Burr-Type X and Burr-Type XII received the maximum attention. Surles and Padgett [2] observed that the Burr-Type X distribution can be used quite effectively in modelling strength data and also modelling general lifetime data. Several aspects of the one-parameter (Scale=1) Burr-Type X distribution were studied by Sartawi and Abu-Salih [3], Jaheen [4]), Ahmad et al. [5], and Raqab [6]. The cumulative distribution function (cdf), and the probability density function (pdf) of the Burr-Type X distribution with shape parameter $\beta > 0$ are respectively as follows,

$$F(x \mid \beta) = \left\{1-\exp\left(-x^2\right)\right\}^{\beta}, \qquad \beta, x > 0. \tag{1}$$

$$f(x \mid \beta) = 2\beta x \exp\left(-x^2\right)\left\{1\text{-}\exp\left(-x^2\right)\right\}^{\beta\text{-}1}, \quad \beta, x > 0. \tag{2}$$

The use of a Bayesian approach allows both sample and prior information to be incorporated into the statistical analysis, which will improve the quality of the inferences and permit a reduction in sample size. The decision-theoretic viewpoint takes into account additional information concerning the possible consequences of our decisions (quantified by a loss function). The main aim of this is to consider the statistical analysis of the unknown parameters under different priors and loss functions.

T. N. Sindhu and M. Aslam are with Department of Statistics, Quad-i-Azam University, Islamabad, CO 44000 Pakistan (e-mail: sindhuqau@gmail.com, aslamsdqu@yahoo.com).

II. LIKELIHOOD FUNCTION AND POSTERIOR DISTRIBUTION

Let $X_1,...,X_n$ be a random sample from Burr-Type X distribution then the likelihood function can be written as:

$$f\left(x_1,...,x_n;\,\beta\right) \propto \beta^n \exp\left[-\beta\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{\,2}\right)\right\}\right] \tag{3}$$

A. Prior and Posterior Distributions

Uniform prior reflects the lack of prior information and the Bayesian methodology can still work. Uniform prior may be proper or improper. Even if Uniform prior is improper, we can still have a proper posterior. Equation (4) presents an improper prior while the posterior given in (5) is proper one having total area uner the curve equals to unity. The uniform prior for β is defined as:

$$p(\beta) \propto k, \quad \beta>0. \tag{4}$$

The posterior distribution under the uniform prior is:

$$p(\beta \mid \mathbf{x}) = \frac{\left\{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{\,2}\right)\right\}\right\}^{n+1}}{\Gamma(n+1)} \times \beta^n \exp\left[-\beta\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{\,2}\right)\right\}\right], \quad \beta > 0. \tag{5}$$

Jeffreys prior is perhaps the most widely used non-informative prior in Bayesian analysis. The only requirement is a likelihood function from which the prior is then derived using Jeffreys' rule, which is to take the prior distribution to be the determinant of the square root of the Fisher information matrix.

$$p(\beta) \propto \frac{1}{\beta}, \quad \beta>0. \tag{6}$$

The posterior distribution under the Jeffreys prior is:

$$p(\beta \mid \mathbf{x}) = \frac{\left\{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{\,2}\right)\right\}\right\}^{n}}{\Gamma(n)} \times \beta^{n-1} \exp\left[-\beta\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{\,2}\right)\right\}\right], \quad \beta > 0. \tag{7}$$

The informative prior for the parameter β is assumed to be exponential distribution:

$$p(\beta) = \tau e^{-\beta\tau}, \qquad \tau > 0,\ \ \beta > 0. \tag{8}$$

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:8, No:1, 2014

The posterior distribution under the assumption of exponential prior is:

$$p(\beta\,|\,\mathbf{x})=\frac{\left\{\tau+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}\right\}^{n+1}}{\Gamma(n+1)}\times\beta^{n}\exp\left[-\beta\left(\tau+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}\right)\right],\quad \beta>0. \tag{9}$$

The informative prior for the parameter β is assumed to be gamma distribution:

$$p(\beta)=\frac{b^{a}}{\Gamma(a)}\beta^{a-1}e^{-b\theta},\qquad a,b,\beta>0. \tag{10}$$

The posterior distribution under the assumption of gamma prior is:

$$p(\beta\,|\,\mathbf{x})=\frac{\left\{b+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}\right\}^{n+a}}{\Gamma(n+a)}\times\beta^{n+a-1}\exp\left[-\beta\left(b+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}\right)\right],\quad \beta>0. \tag{11}$$

The informative prior for the parameter β is assumed to be Inverse Levy distribution:

$$p(\beta)=\sqrt{\frac{h}{2\pi}}\beta^{-\frac{1}{2}}e^{-\left(\frac{h\beta}{2}\right)},\qquad h,\ \beta>0. \tag{12}$$

The posterior distribution under the Inverse Levy prior is:

$$p(\beta\,|\,\mathbf{x})=\frac{\left\{\frac{h}{2}+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}\right\}^{n+\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)}\times\beta^{n+\frac{1}{2}-1}\exp\left[-\beta\left(\frac{h}{2}+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}\right)\right],\ \ \beta>0. \tag{13}$$

III. Bayes Estimators and Posterior Risks Under Different Loss Functions

This section enlightens the derivation of the Bayes Estimator (BE) and corresponding Posterior Risks (PR) under different loss functions. The Bayes estimators are evaluated under Squared Error Loss Function (SELF), Precautionary Loss Function (PLF), Weighted Squared Error Loss Function (WSELF), Quasi-Quadratic Loss Function (QQLF), and Squared-Log Error Loss Function (SLELF). The Bayes Estimator (BE) and corresponding Posterior Risks (PR) under different loss functions are given in the following table.

TABLE I
BAYES ESTIMATORS AND POSTERIOR RISKS UNDER DIFFERENT LOSS FUNCTIONS

Loss Function= $L(\beta,\ \hat\beta)$	Bayes Estimator	Posterior Risk
SELF: $\left(\beta-\hat\beta\right)^2$	$E(\beta\,|\,\mathbf{x})$	$Var(\beta\,|\,\mathbf{x})$
PLF: $\frac{\left(\beta-\hat\beta\right)^2}{\hat\beta}$	$\sqrt{E\left(\beta^2\,|\,\mathbf{x}\right)}$	$2\left\{\sqrt{E\left(\beta^2\,|\,\mathbf{x}\right)}-E(\beta\,|\,\mathbf{x})\right\}$
WSELF: $\frac{\left(\beta-\hat\beta\right)^2}{\beta}$	$\left\{E\left(\beta^{-1}\,|\,\mathbf{x}\right)\right\}^{-1}$	$E(\beta\,|\,\mathbf{x})-\left\{E\left(\beta^{-1}\,|\,\mathbf{x}\right)\right\}^{-1}$
QQLF: $\left(e^{-c\hat\beta}-e^{-c\beta}\right)^2$	$\frac{-1}{c}\ln\left\{E\left(e^{-c\beta}\,|\,\mathbf{x}\right)\right\}$	$E\left(e^{-2c\beta}\right)-\left\{E\left(e^{-c\beta}\right)\right\}^2$
SLELF: $\left(\ln\hat\beta-\ln\beta\right)^2$	$\exp\left\{\mathrm{E}(\ln\beta\,|\,\mathbf{x})\right\}$	$\mathrm{E}\left\{(\ln\beta\,|\,\mathbf{x})\right\}^2-\left\{\mathrm{E}(\ln\beta\,|\,\mathbf{x})\right\}^2$

The Bayes Estimators and Posterior Risks under uniform prior are:

$$\hat\beta_{SELF}=\frac{n+1}{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}},\quad \rho\left(\hat\beta_{SELF}\right)=\frac{n+1}{\left\{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}\right\}^{2}}.$$

$$\hat\beta_{PLF}=\frac{\sqrt{(n+1)(n+2)}}{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}},$$

$$\rho\left(\hat\beta_{PLF}\right)=2\left\{\frac{\sqrt{(n+1)(n+2)}}{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}-\frac{(n+1)}{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}\right\}.$$

$$\hat\beta_{WSELF}=\frac{(n)}{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}},\quad \rho\left(\hat\beta_{WSELF}\right)=\frac{1}{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}.$$

$$\hat\beta_{QQLF}=\ln\left(\frac{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}{1+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}\right)^{-(n+1)},$$

$$\rho\left(\hat\beta_{QQLF}\right)=\left(\frac{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}{2+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}\right)^{(n+1)}-\left(\frac{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}{1+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}}\right)^{2(n+1)},\ c=1.$$

$$\hat\beta_{SLELF}=\frac{\exp\left(\psi(n+1)\right)}{\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^{2}\right)\right\}},\qquad \rho\left(\hat\beta_{SLELF}\right)=\left\{\psi'(n+1)\right\}.$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and $\psi'(x)=\frac{d^2}{dx^2}\{\log\Gamma(x)\}=\frac{d}{dx}\left\{\frac{\Gamma'(x)}{\Gamma(x)}\right\}$ is the tri-gamma function.

The Bayes Estimators and posterior Risks under the rest of priors can be obtained in a similar manner.

IV. Bayes Credible Interval for the Left Censored Data

The Bayesian credible intervals under informative and non-informative priors, as discussed by Saleem and Aslam [7] are presented in the following. The credible intervals under all priors are:

$$\frac{\chi^2_{2(n+1)\left(\frac{\alpha}{2}\right)}}{2\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}}<\beta_{Uniform}<\frac{\chi^2_{2(n+1)\left(1-\frac{\alpha}{2}\right)}}{2\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}},$$

$$\frac{\chi^2_{2(n)\left(\frac{\alpha}{2}\right)}}{2\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}}<\beta_{Jeffreys}<\frac{\chi^2_{2(n)\left(1-\frac{\alpha}{2}\right)}}{2\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}},$$

$$\frac{\chi^2_{2(n+1)\left(\frac{\alpha}{2}\right)}}{2\left\{\tau+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}}<\beta_{Exponential}<\frac{\chi^2_{2(n+1)\left(1-\frac{\alpha}{2}\right)}}{2\left\{\tau+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}},$$

$$\frac{\chi^2_{2(n+a)\left(\frac{\alpha}{2}\right)}}{2\left\{b+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}}<\beta_{Gamma}<\frac{\chi^2_{2(n+a)\left(1-\frac{\alpha}{2}\right)}}{2\left\{b+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}},$$

and

$$\frac{\chi^2_{2(n+1/2)\left(\frac{\alpha}{2}\right)}}{2\left\{h/2+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}}<\beta_{In-Levy}<\frac{\chi^2_{2(n+1/2)\left(1-\frac{\alpha}{2}\right)}}{2\left\{h/2+\sum_{i=1}^{n}\ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}}.$$

V. Elicitation

Bayesian analysis elicitation of opinion is a crucial step. It helps to make it easy for us to understand what the experts believe in and what their opinions are. In statistical inference the characteristics of a certain predictive distribution proposed by an expert determine the hyperparameters of a prior distribution.

In this article, we focus on a probability elicitation method known as prior predictive elicitation. Predictive elicitation is a method for estimating hyperparameters of prior distributions by inverting corresponding prior predictive distributions. Elicitation of hyperparameter from the prior $p(\beta)$ is conceptually difficult task because we first have to identify prior distribution and then its hyperparameters. The prior predictive distribution is used for the elicitation of the hyperparameters which is compared with the experts' judgment about this distribution and then the hyperparameters are chosen in such a way so as to make the judgment agree closely as possible with the given distribution (reader desires more detail see Grimshaw et al. [8], O'Hagan et al. [9], Jenkinson [10] and Leon et al. [11]. According to Aslam [12], the method of assessment is to compare the predictive distribution with experts' assessment about this distribution and then to choose the hyperparameters that make the assessment agree closely with the member of the family. He discusses three important methods to elicit the hyperparameters: (i) Via the Prior Predictive Probabilities (ii) Via Elicitation of the Confidence Levels (iii) Via the Predictive Mode and Confidence Level. We will use the prior predictive approach by Aslam [12].

A. Prior Predictive Distribution

The prior predictive distribution is:

$$p(y)=\int_0^\infty p(y\mid\beta)p(\beta)\,d\beta\,. \tag{14}$$

The predictive distribution under exponential prior is:

$$p(y)=\int_0^\infty 2\beta y\exp\left(-y^2\right)\left\{1\text{-}\exp\left(-y^2\right)\right\}^{\beta\text{-}1}\tau\exp\{-\beta\tau\}d\beta \tag{15}$$

After some simplification it reduces as

$$p(y)=\frac{2\tau y\exp\left(-y^2\right)\left\{1-\exp\left(-y^2\right)\right\}^{-1}}{\left[\tau+\ln\left\{1-\exp\left(-y^2\right)\right\}^{-1}\right]^2},\qquad y>0. \tag{16}$$

The predictive distribution under gamma prior is:

$$p(y)=\frac{2yab^a\exp\left(-y^2\right)\left\{1-\exp\left(-y^2\right)\right\}^{-1}}{\left[b+\ln\left\{1-\exp\left(-y^2\right)\right\}^{-1}\right]^{(a+1)}},\qquad y>0. \tag{17}$$

$$p(y)=\frac{y\sqrt{h/2}\exp\left(-y^2\right)\left\{1-\exp\left(-y^2\right)\right\}^{-1}}{\left[\frac{h}{2}+\ln\left\{1-\exp\left(-y^2\right)\right\}^{-1}\right]^{3/2}},\qquad y>0. \tag{18}$$

By using the method of elicitation defined by Aslam (2003) [12], we obtain the following hyper-parameters,

$\tau=0.0025844$, $a=3.89562$, $b=0.006953$ and $h=0.00012475$.

VI. Posterior Predictive Distribution

The predictive distribution contains the information about the independent future random observation given preceding observations. The reader desire more details can see Bansal [13].

The posterior predictive distribution of the future observation $y=x_{n+1}$ is

$$p(y \mid \mathbf{x}) = \int_0^\infty p(\beta \mid \mathbf{x})\, p(y \mid \beta)\, d\beta \qquad (19)$$

where $p(y) = 2\beta y \exp\left(-y^2\right)\left\{1\text{-}\exp\left(-y^2\right)\right\}^{\beta\text{-}1}$ is the future observation density and $p\left(\beta \mid \mathbf{x}\right)$ is the posterior distribution obtained by incorporating the likelihood with the respective prior distributions. A $\left(1-\alpha\right)100\%$ Bayesian interval (L, U) can be obtained by solving the following two equations simultaneously

$$\int_{-\infty}^{L} p(y \mid \mathbf{x})dy = \frac{k}{2} = \int_U^\infty p(y \mid \mathbf{x})dy$$

The posterior predictive distribution of the future observation $y = x_{n+1}$ under uniform prior is

$$p(y \mid \mathbf{x}) = \frac{2y(n+1)\left\{\sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}^{(n+1)} \exp\left(-y^2\right)\left\{1-\exp\left(-y^2\right)\right\}^{-1}}{\left[\sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\} + \ln\left\{1-\exp\left(-y^2\right)\right\}^{-1}\right]^{(n+2)}}, \quad y > 0. \qquad (20)$$

The posterior predictive distribution of the future observation $y = x_{n+1}$ under Jeffreys prior is

$$p(y \mid \mathbf{x}) = \frac{2y(n)\left\{\sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}^{(n)} \exp\left(-y^2\right)\left\{1-\exp\left(-y^2\right)\right\}^{-1}}{\left[\sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\} + \ln\left\{1-\exp\left(-y^2\right)\right\}^{-1}\right]^{(n+1)}}, \quad y > 0. \qquad (21)$$

The posterior predictive distribution of the future observation $y = x_{n+1}$ under exponential prior is

$$p(y \mid \mathbf{x}) = \frac{2y(n+1)\left\{\tau + \sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}^{(n+1)} \exp\left(-y^2\right)\left\{1-\exp\left(-y^2\right)\right\}^{-1}}{\left[\tau + \sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\} + \ln\left\{1-\exp\left(-y^2\right)\right\}^{-1}\right]^{(n+2)}}, \quad y > 0. \qquad (22)$$

The posterior predictive distribution of the future observation $y = x_{n+1}$ under gamma prior is

$$p(y \mid \mathbf{x}) = \frac{2y(n+a)\left\{b + \sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}^{(n+a)} \exp\left(-y^2\right)\left\{1-\exp\left(-y^2\right)\right\}^{-1}}{\left[b + \sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\} + \ln\left\{1-\exp\left(-y^2\right)\right\}^{-1}\right]^{(n+a+1)}}, \quad y > 0. \qquad (23)$$

The posterior predictive distribution of the future observation $y = x_{n+1}$ under In-Levy prior is

$$p(y \mid \mathbf{x}) = \frac{2y(n+1/2)\left\{h/2 + \sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}^{(n+1/2)} \exp\left(-y^2\right)\left\{1-\exp\left(-y^2\right)\right\}^{-1}}{\left[h/2 + \sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\} + \ln\left\{1-\exp\left(-y^2\right)\right\}^{-1}\right]^{(n+3/2)}}, \quad y > 0. \qquad (24)$$

and predictive interval under uniform prior is:

$$\frac{\left\{\sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}^{(n+1)}}{\left[\sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\} + \ln\left\{1-\exp\left(-L^2\right)\right\}^{-1}\right]^{(n+1)}} = \frac{k}{2},$$

$$\frac{\left\{\sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\}\right\}^{(n+1)}}{\left[\sum_{i=1}^n \ln\left\{1-\exp\left(-x_i^2\right)\right\} + \ln\left\{1-\exp\left(-U^2\right)\right\}^{-1}\right]^{(n+1)}} = 1-\frac{k}{2}.$$

The predictive intervals under the rest of priors can be obtained in a similar manner.

VII. Simulation Study

This section shows how simulation can be helpful and illuminating way to approach problems in Bayesian analysis. Bayesian problems of updating estimates can be handled easily and straight forwardly with simulation. Since we can express the distribution function of the Burr Type X as well as its inverse in closed form, the inversion method of simulation is straightforward to implement. The study has been carried out for different value of n using $\beta \in 8$ and $12.$ Sample size is varied to observe the effect of small and large samples on the estimators. Changes in the estimators and their risks have been determined when changing the loss function and the prior distribution of β while keeping the sample size fixed. All these results are based on 5,000 repetitions. Tables II – XVI, give the estimated value of the parameter, posterior risks (PR) and 95% confidence limits (LCL & UCL) for the parameter. The results are summarized in the following Tables. The first entry is the simulated Bayes Estimator. The second entry is the simulate Posterior Risk.

TABLE II
BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER UNIFORM PRIOR $\beta = 8$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	8.46759 (2.4381)	8.68941 (0.2757)	8.29601 (0.2765)	7.55051 ($3.9 * 10^{-6}$)	8.40028 (0.0328)	8.27339 (0.0166)
50	8.32283 (1.3859)	8.38431 (0.1620)	8.12765 (0.1626)	7.68619 ($1.1 * 10^{-6}$)	8.28254 (0.0198)	8.19250 (0.0099)
100	8.17035 (0.6680)	8.21998 (0.8079)	8.09848 (0.0810)	7.84072 ($2.4 * 10^{-7}$)	8.10474 (0.0099)	8.06961 (0.0050)
500	8.03248 (0.1290)	8.04653 (0.0160)	8.03510 (0.0161)	7.95991 ($1.9 * 10^{-8}$)	8.01117 (0.0020)	8.01857 (0.0009)
1000	8.00758 (0.0641)	8.02194 (0.0080)	8.01460 (0.0080)	7.99279 ($8.1 * 10^{-9}$)	8.00708 (0.0009)	8.00649 (0.0005)

TABLE III

BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER JEFFREYS PRIOR $\beta = 8$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	8.27099 (2.3633)	8.53576 (0.2731)	8.09421 (0.2774)	7.30027 ($5.4*10^{-6}$)	8.09909 (0.0339)	7.97947 (0.0171)
50	8.14480 (1.3553)	8.25066 (0.1614)	8.07769 (0.1649)	7.60696 ($1.3*10^{-6}$)	8.08484 (0.0202)	7.98562 (0.0102)
100	8.08115 (0.65950)	8.10073 (0.0804)	8.05988 (0.0814)	7.76675 ($2.6*10^{-7}$)	8.03342 (0.0101)	7.99320 (0.0050)
500	8.01424 (0.1287)	8.01093 (0.0160)	8.01807 (0.0161)	7.95808 ($1.9*10^{-8}$)	8.01699 (0.0020)	7.99851 (0.0010)
1000	8.00255 (0.0641)	8.00835 (0.0080)	8.00555 (0.0080)	7.98234 ($8.3*10^{-9}$)	8.01094 (0.0015)	8.00535 (0.0005)

TABLE IV

BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER EXPONENTIAL PRIOR $\beta = 8$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	8.46918 (2.3929)	8.72654 (0.2749)	8.30877 (0.2770)	7.58197 ($3.8*10^{-6}$)	8.41362 (0.0328)	8.11734 (0.0166)
50	8.33205 (1.3892)	8.32308 (0.1608)	8.16838 (0.1634)	7.68309 ($1.0*10^{-6}$)	8.18345 (0.019801)	8.08847 (0.0099)
100	8.14593 (0.6640)	8.23147 (0.0809)	8.06133 (0.0806)	7.84383 ($2.3*10^{-7}$)	8.10193 (0.009950)	8.08400 (0.0050)
500	8.03297 (0.1291)	8.02778 (0.0160)	8.01874 (0.0161)	7.96369 ($1.9*10^{-8}$)	8.02124 (0.001998)	8.02976 (0.0010)
1000	8.01062 (0.0642)	8.01887 (0.0080)	8.01556 (0.0081)	7.98184 ($8.4*10^{-9}$)	8.00470 (0.000999)	8.00699 (0.00050)

TABLE V

BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER GAMMA PRIOR $\beta = 8$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	9.29426 (2.6300)	9.54997 (0.2757)	9.02036 (0.2742)	8.21571 ($1.4*10^{-6}$)	9.19908 (0.0299)	9.04031 (0.01512)
50	8.88501 (1.4953)	8.83332 (0.1617)	8.70335 (0.1646)	8.15623 ($5.3*10^{-7}$)	8.65049 (0.018728)	8.59468 (0.0094)
100	8.41808 (0.6896)	8.46774 (0.0809)	8.30063 (0.8067)	8.09926 ($1.5*10^{-7}$)	8.35970 (0.009672)	8.30013 (0.0049)
500	8.05681 (0.1291)	8.10657 (0.0161)	8.07138 (0.0161)	8.01384 ($1.8*10^{-8}$)	8.05571 (0.001987)	8.05503 (0.0010)
1000	8.04445 (0.0645)	8.04327 (0.0081)	8.03973 (0.0080)	8.01176 ($7.9*10^{-6}$)	8.04296 (0.000997)	8.03320 (0.0005)

TABLE VI

BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER INVERSE LEVY PRIOR $\beta = 8$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	8.41141 (2.4001)	8.52824 (0.2729)	8.14856 (0.2762)	7.41366 ($4.4*10^{-6}$)	8.25164 (0.0333)	8.1528 (0.0169)
50	8.27273 (1.3823)	8.38402 (0.1636)	8.07998 (0.1632)	7.61606 ($1.3*10^{-6}$)	8.20454 (0.0199)	8.06762 (0.0101)
100	8.12617 (0.6636)	8.15631 (0.0806)	8.02695 (0.0807)	7.79372 ($2.6*10^{-7}$)	8.03997 (0.0099)	8.06203 (0.0050)
500	8.01546 (0.1286)	8.01787 (0.0160)	8.00163 (0.01607)	7.96359 ($1.97*10^{-8}$)	8.00926 (0.0020)	8.02628 (0.0010)
1000	8.00044 (0.0640)	8.01690 (0.0080)	8.00141 (0.0081)	7.98520 ($8.3*10^{-9}$)	8.00383 (0.0010)	8.01357 (0.0005)

TABLE VII

BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER UNIFORM PRIOR $\beta = 12$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	12.8802 (5.5469)	13.13750 (0.4138)	12.34340 (0.4114)	10.74490 $(8.4 * 10^{-8})$	12.68612 (0.0328)	12.40400 (0.0166)
50	12.47360 (3.1050)	12.61290 (0.2437)	12.2079 (0.2442)	11.21090 $(8.3 * 10^{-9})$	12.27030 (0.0198)	12.28260 (0.0099)
100	12.34550 (1.5234)	12.27900 (0.1207)	12.16310 (0.1216)	11.56070 $(8.7 * 10^{-10})$	12.24060 (0.0099)	12.14290 (0.0050)
500	12.04930 (0.2904)	12.04690 (0.0240)	12.02830 (0.0241)	11.89040 $(2.4 * 10^{-11})$	12.04523 (0.0020)	12.03630 (0.0009)
1000	12.01940 (0.1445)	12.02820 (0.0120)	11.99840 (0.01199)	11.95330 (7.8×10^{-12})	12.01031 (0.0009)	12.01040 (0.0005)

TABLE VIII

BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER JEFFREYS PRIOR $\beta = 12$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	12.33840 (5.2453)	12.55170 (0.4082)	12.12580 (0.4181)	10.34420 $(1.1 * 10^{-7})$	12.26450 (0.0339)	11.85501 (0.0171)
50	12.26160 (3.0754)	12.24680 (0.2413)	12.03910 (0.2457)	10.98850 $(1.4 * 10^{-8})$	12.05512 (0.0202)	11.95540 (0.0102)
100	12.12190 (1.4837)	2.14580 (0.1206)	12.02940 (0.1215)	11.40990 $(1.4 * 10^{-9})$	12.054110 (0.0101)	11.99490 (0.0050)
500	12.01771 (0.2895)	12.02120 (0.0241)	12.0205 (0.0241)	11.89910 $(2.4 * 10^{-11})$	12.01230 (0.0020)	11.99770 (0.0010)
1000	12.00940 (0.1444)	12.01240 (0.0120)	12.0023 (0.0120)	11.94720 $(7.9 * 10^{-12})$	12.00410 (0.0015)	12.00260 (0.0005)

TABLE IX

BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER EXPONENTIAL PRIOR $\beta = 12$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	12.88930 (5.5567)	12.94980 (0.4079)	12.36370 (0.4121)	10.77990 $(6.7 * 10^{-8})$	12.64230 (0.0328)	12.49420 (0.0166)
50	12.49460 (3.1301)	12.57380 (0.2430)	12.12630 (0.2425)	11.14130 $(9.1 * 10^{-9})$	12.32650 (0.0198)	12.23650 (0.0099)
100	12.18630 (1.4855)	12.22510 (0.1202)	12.05650 (0.1206)	11.58901 $(9.1 * 10^{-10})$	12.17011 (0.0099)	12.13341 (0.0050)
500	12.04570 (0.2902)	12.07340 (0.0241)	12.06500 (0.0241)	11.89680 $(2.3 * 10^{-11})$	12.03480 (0.0020)	12.01950 (0.0009)
1000	12.02650 (0.1447)	10.02400 (0.0120)	11.99250 (0.0120)	11.96610 $(7.6 * 10^{-12})$	11.99940 (0.0009)	12.01840 (0.0005)

TABLE X

BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER GAMMA PRIOR $\beta = 12$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	13.9065 (5.9019)	4.1639 (0.4089)	13.54440 (0.4117)	11.67940 $(2.5 * 10^{-8})$	13.74801 (0.0299)	13.41232 (0.0151)
50	13.2103 (3.3054)	13.3101 (0.2436)	12.8730 (0.2434)	11.83030 $(3.7 * 10^{-9})$	13.05020 (0.0187)	12.91070 (0.0094)
100	12.5664 (1.5355)	12.6614 (0.1210)	12.4293 (0.1208)	11.87040 $(5.7 * 10^{-10})$	12.52310 (0.0097)	12.54610 (0.0049)
500	12.1238 (0.2923)	12.1358 (0.0241)	12.1015 (0.0241)	11.97601 $(1.9 * 10^{-11})$	12.07220 (0.0020)	12.12400 (0.0010)
1000	12.0691 (0.1453)	12.0611 (0.0121)	12.0521 (0.0120)	11.98530 $(7.4 * 10^{-12})$	12.05640 (0.0010)	12.04717 (0.0005)

TABLE XI
BAYES ESTIMATES AND THE POSTERIOR RISKS UNDER INVERSE LEVY PRIOR $\beta = 12$

n	SELF	PLF	WSELF	QQLF	SLELF	ELF
30	12.55712 (5.3667)	12.7211 (0.40717)	12.2087 (0.4139)	10.61152 ($9.7*10^{-8}$)	12.3056 (0.0333)	12.14396 (0.0169)
50	12.3987 (3.1056)	12.3853 (0.2417)	12.0890 (0.2442)	11.00060 ($1.2*10^{-9}$)	12.3020 (0.0199)	12.09972 (0.0101)
100	12.1652 (1.4884)	12.2159 (0.1207)	12.0492 (0.1211)	11.54650 ($8.7*10^{-10}$)	12.1156 (0.0099)	12.05964 (0.0050)
500	12.0263 (0.2896)	12.0199 (0.0240)	12.0155 (0.0241)	11.87160 ($2.4*10^{-11}$)	12.01110 (0.0020)	12.01610 (0.0010)
1000	12.0135 (0.1444)	12.0106 (0.0120)	12.0034 (0.12009)	11.9469 ($8.*10^{-12}$)	12.00880 (0.0010)	12.01308 (0.0005)

TABLE XII
THE LOWER (LL), THE UPPER (UL) AND THE WIDTH OF THE 95% CI UNDER UNIFORM PRIOR

n	$\beta = 8$		Width	$\beta = 12$		Width
	LL	UL		LL	UL	
30	5.60110	11.3886	5.78750	8.45011	17.1814	8.73129
50	6.13768	10.6547	4.51702	9.10803	15.8110	6.70297
100	6.59971	9.75746	3.15775	9.86386	14.58331	4.71944
500	7.32973	8.73340	1.40367	10.9874	13.0915	2.10410
1000	7.52839	8.52164	0.99325	11.2625	12.7484	1.48590

TABLE XIII
THE LOWER (LL), THE UPPER (UL) AND THE WIDTH OF THE 95% CI UNDER JEFFREYS PRIOR

n	$\beta = 8$		Width	$\beta = 12$		Width
	LL	UL		LL	UL	
30	5.38248	11.0753	5.69282	8.12029	16.7088	8.58851
50	5.99837	10.4707	4.47233	8.90130	15.5380	6.63670
100	6.52740	9.66939	3.14199	9.75569	14.4516	4.69591
500	7.31443	8.71669	1.40226	10.9645	13.0665	2.10200
1000	7.52063	8.51338	0.99275	11.2509	12.7361	1.48520

TABLE XIV
THE LOWER (LL), THE UPPER (UL) AND THE WIDTH OF THE 95% CI UNDER EXPONENTIAL PRIOR

n	$\beta = 8$		Width	$\beta = 12$		Width
	LL	UL		LL	UL	
30	5.59725	11.3803	5.78305	8.44136	17.1636	8.72224
50	6.13512	10.6502	4.51508	9.10239	15.8012	6.69881
100	6.59840	9.75544	3.15704	9.86080	14.5788	4.71800
500	7.32943	8.73303	1.40360	10.9867	13.0907	2.10400
1000	7.52823	8.52146	0.99323	11.2622	12.7480	1.48580

TABLE XV
THE LOWER (LL), THE UPPER (UL) AND THE WIDTH OF THE 95% CI UNDER GAMMA PRIOR

n	$\beta = 8$		Width	$\beta = 12$		Width
	LL	UL		LL	UL	
30	6.22679	12.6880	6.46121	9.38525	18.4920	9.10675
50	6.53510	11.1739	4.63880	9.69227	16.5721	6.87983
100	6.80577	10.0067	3.20093	10.1689	14.9516	4.78270
500	7.37322	8.78078	1.40756	11.0520	13.1618	2.10980
1000	7.55044	8.54507	0.99463	11.2952	12.7831	1.48790

TABLE XVI
THE LOWER (LL), THE UPPER (UL) AND THE WIDTH OF THE 95% CI UNDER INVERSE LEVY PRIOR

n	$\beta = 8$		Width	$\beta = 12$		Width
	LL	UL		LL	UL	
30	5.49160	11.2319	5.74030	8.28484	16.9448	8.65996
50	6.06794	10.5626	4.49466	9.00449	15.6744	6.66991
100	6.56355	9.71338	3.14983	9.80969	14.5174	4.70771
500	7.32207	8.72504	1.40297	10.9759	13.0790	2.10310
1000	7.52451	8.51751	0.99300	11.2567	12.7422	1.48550

VIII. CONCLUDING REMARKS

The simulation study has displayed some interesting properties of the Bayes estimates. After an extensive study of results, conclusions are drawn regarding the behavior of the estimators. The risks of the estimates seem to be large in case when the value of the parameter is large and small for relative smaller value of the parameter except under quasi-quadratic loss function. However, the risks under said loss functions are reduced as the sample size increases. Another interesting remark concerning the risks of the estimates is that increasing (decreasing) the value of the parameter reduces (increases) the risks of the estimates under quasi-quadratic loss function. The performance of squared-log error loss function and entropy loss function is independent of choice of parametric value. The above study depicts that the estimated value of the parameter converges to the true value of the parameter by increasing the sample size. In comparison of non-informative priors, the uniform prior, provides the better estimates as the corresponding risks are least under said loss functions except PLF, QQLF and SELF. While the uniform and the exponential priors are equally efficient under ELF and SLELF. Under SELF and PLF In-Levy prior produces more efficient estimates as compared to the other informative priors, and for the rest loss functions, the gamma prior provides the better estimates as the corresponding risks are least under said loss functions with few exceptions.

The Credible intervals are in accordance with the point estimates, that is, the width of credible interval is inversely proportional to sample size. Tables XI-XV reveal that the effect of the prior information in the form of narrower width of interval. The Credible interval assuming Inverse Levy prior is much narrower than the credible intervals assuming informative and non-informative priors. It is the use of prior information that makes a difference in terms of gain in precision. The study can further be extended by considering generalized versions of the distribution under variety of circumstances.

REFERENCES

[1] I. W. Burr, "Cumulative Frequency Distribution," *Annals of Mathematical Statistics*, vol. 13, pp. 215-232, 1942.

[2] J.G. Surles and W.J. Padgett, "Inference for Reliability and Stress-Strength for a Scaled Burr Type X Distribution," *Lifetime Data Analysis*, vol. 7, pp. 187-200, 2001.

[3] H.A. Sartawi and M.S. Abu-Salih, "Bayes Prediction Bounds for the Burr Type X Model," *Communications in Statistics - Theory and Methods*, vol. 20, pp. 2307-2330, 1991.

[4] Z.F. Jaheen, "Empirical Bayes Estimation of the Reliability and Failure Rate Functions of the Burr Type X Failure Model," *Journal of Applied Statistical Sciences*, vol. 3, pp. 281-288, 1996.

[5] K.E. Ahmad, M.E. Fakhry, and Z.F. Jaheen, "Empirical Bayes Estimation of $P(Y < X)$ and Characterization of Burr-Type X Model," *Journal of Statistical Planning and Inference*, vol. 64, pp. 297-308, 1997.

[6] M.Z. Raqab, "Order Statistics from the Burr Type X Model," *Computers Mathematics and Applications*, vol. 36, pp. 111-120, 1998.

[7] M. Saleem, M. Aslam, "On Bayesian Analysis of the Rayleigh Survival Times under Assuming the Rayleigh Censor Time," *Pak. J. of Statistics*, vol. 25, pp. 71-82, 2009.

[8] S. D. Grimshaw, B. J. Collings, W. A. Larsen, and C. R. Hurt, "Eliciting Factor Importance in a Designed Experiment," *Technometrics*, vol. 43, pp. 133-146, 2001.

[9] A. O'Hagan, C. E. Buck, A. Daneshkhah, A. Eiser, P. H. Garthwaite, D. J. Jenkinson, J. E. Oakley, and T. Rakow, "Uncertain Judgements: Eliciting Expert Probabilities," *John Wiley & Sons*, 2006.

[10] D. Jenkinson, "The Elicitation of Probabilities: A Review of the Statistical Literature," *Department of Probability and Statistics, University of Sheffield*, 2005.

[11] J. C. Leon, J. F. Vazquez-Polo, and L.R. Gonzalez, "Environmental and Resource Economics," *Kluwer Academic Publishers*, vol. 26, pp. 199-210, 2003.

[12] M. Aslam, "An Application of Prior Predictive Distribution to Elicit the Prior Density," *Journal of Statistical Theory and Applications*, vol. 2, pp. 70-83, 2003.

[13] A. K. Bansal, "Bayesian Parametric Inference," *Norsa Publishing House*, New *Delhi*.