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Weighted Composition Operators Acting between Kind of Weighted Bergman-Type Spaces and the Bers-Type Space

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Abstract—In this paper, we study the boundedness and compactness of the weighted composition operator $W_{u,\phi}$, which is induced by an holomorphic function u and holomorphic self-map ϕ, acting between the $\mathcal{N}_K$-space and the Bers-type space H_α^∞ on the unit disk.

Keywords—Weighted composition operators, $\mathcal{N}_K$-space, Bers-type space.

I. Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane, ∂D it's boundary. $\mathcal{H}(D)$ denotes the class of all analytic functions on D, while $dA(z)$ denotes the Lebesgue measure on the plane, normalized so that $A(D) = 1$. For each $a \in D$, the Green's function with logarithmic singularity at $a \in D$ is denoted by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformations of D. The pseudo-hyperbolic disk $D(a,r)$ is defined by

$$D(a,r) = \{z \in D : |\varphi_a(z)| < r\}.$$

We will frequently use the following easily verified equality:

$$(1 - |\varphi_a(z)|^2) = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

For $p \in (0,\infty)$ and $-1 < \alpha < \infty$, the Bers-type spaces H_α^∞ consists of all $f \in \mathcal{H}(D)$ such that

$$\|f\|_\alpha = \sup_{z\in D} |f(z)|(1-|z|^2)^\alpha < \infty,$$

and $H_{\alpha,0}^\infty$ consists of all $f \in \mathcal{H}(D)$ such that

$$\|f\|_{\alpha,0} = \lim_{|z|\to 1} |f(z)|(1-|z|^2)^\alpha = 0.$$

For more information about several studied on Bers-type spaces we refer to [3], [12].

For $0 < \alpha < \infty$ the α-Bloch space $\mathcal{B}^\alpha$ consists of all $f \in \mathcal{H}(D)$ such that

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in D} (1-|z|^2)^\alpha |f'(z)| < \infty.$$

Moreover, $f \in \mathcal{B}_0^\alpha$ if

$$\|f\|_{\mathcal{B}_0^\alpha} = \lim_{|z|\to 1} |f'(z)|(1-|z|^2)^\alpha = 0.$$

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The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$ (see [11]). For each $\alpha > 0$, we know that $H_\alpha^\infty = \mathcal{B}^{\alpha+1}$ and $H_{\alpha,0}^\infty = \mathcal{B}_0^{\alpha+1}$ (see [13], Proposition 7).

El-Sayed Ahmed and Bakhit in [4] introduced the $\mathcal{N}_K$ spaces (with the right continuous and nondecreasing function $K : [0,\infty) \to [0,\infty)$) consists of $f \in \mathcal{H}(D)$ such that

$$\|f\|_{\mathcal{N}_K}^2 = \sup_{a\in D} \int_D |f(z)|^2 K(g(z,a))dA(z) < \infty.$$

If

$$\lim_{|a|\to 1} \int_D |f(z)|^2 K(g(z,a))dA(z) = 0,$$

then f is said to belong to $\mathcal{N}_{K,0}$. For $K(t) = 1$ it gives the Bergman space. If $\mathcal{N}_K$ consists of just the constant functions, we say that it is trivial. Clearly, if $K(t) = t^p$, then $\mathcal{N}_K = \mathcal{N}_p$; since $g(z,a) \approx (1 - |\varphi_a(z)|^2)$. The $\mathcal{N}_p$-space was introduced by Palmberg in [8]. Finally, when $K(t) = t$, $\mathcal{N}_K$ coincides $\mathcal{N}_1$, the $\mathcal{N}_1$-space was introduced in [7].

From a change of variable we see that the coordinate function z belongs to $\mathcal{N}_K$ space if and only if

$$\sup_{a\in D} \int_D \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} K\left(\log\frac{1}{|z|}\right) dA(z) < \infty.$$

Simplifying the above integral in polar coordinates, we conclude that $\mathcal{N}_K$ space is nontrivial if and only if

$$\sup_{t\in(0,1)} \int_0^1 \frac{(1-t)^2}{(1-tr^2)^3} K\left(\log\frac{1}{r}\right) rdr < \infty. \quad (1)$$

We assume from now that all $K : [0,\infty) \to [0,\infty)$ to appear in this paper are right-continuous and nondecreasing function. Moreover, we always assume that condition (1) is satisfied, so that the $\mathcal{N}_K$ space we study is not trivial.

Given $u \in \mathcal{H}(D)$ and ϕ a holomorphic self-map of D. The weighted composition operator $W_{u,\phi} : \mathcal{H}(D) \to \mathcal{H}(D)$ is defined by

$$W_{u,\phi}(f)(z) = u(z)(f \circ \phi)(z), \quad z \in D.$$

It is obvious that $W_{u,\phi}$ can be regarded as a generalization of the multiplication operator $M_u f = u \cdot f$ and composition operator $C_\phi f = f \circ \phi$. The behavior of those operators is studied extensively on various spaces of holomorphic functions (see for example [3], [4], [6], [7], [8]). El-Sayed Ahmed and Bakhit in [4] considered the composition operator $C_\phi f = f\circ\phi$ on the space $\mathcal{N}_K$. They gave complete characterizations for the boundedness and compactness of $C_\phi : \mathcal{N}_K \to H_\alpha^\infty$. However the boundedness and compactness of the case $C_\phi : H^\infty_\alpha \to \mathcal{N}_K$ remain to be studied.

In this paper, we will characterize the boundedness and compactness of the case $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ and $W_{u,\phi} : \mathcal{N}_K \to H^\infty_\alpha$. Our situations have not been covered by a recent progress of studies of weighted composition operators. Of course, the results in this paper will also give the characterizations of the case $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ and the case $W_{u,\phi} : \mathcal{N}_K \to H^\infty_\alpha$ is a generalization of the results in [4], [8] and [10]. Furthermore, by the derivative operator $f \mapsto f', Q_K$-spaces (see [9]) are closely related to $\mathcal{N}_K$-spaces and Bloch-type spaces $\mathcal{B}^\alpha$ related to H^∞_α.

For a subarc $I \subset \partial D$, let

$$S(I) = \{r\zeta \in D : 1 - |I| < r < 1, \zeta \in I\}.$$

If $|I| \geq 1$ then we set $S(I) = D$. For $0 < p < \infty$, we say that a positive measure $d\mu$ is a p-Carleson measure on D if

$$\sup_{I \subset \partial D} \frac{\mu(S(I))}{|I|^p} < \infty.$$

Here and henceforth $\sup_{I \subset \partial D}$ indicates the supremum taken over all subarcs I of ∂D. Note that $p = 1$ gives the classical Carleson measure (see [1], [2]). A positive measure $d\mu$ is said to be a K-Carleson measure on D if

$$\sup_{I \subset \partial D} \int_{S(I)} K\left(\frac{1 - |z|}{|I|}\right) d\mu(z) < \infty.$$

Clearly, if $K(t) = t^p$, then μ is a K-Carleson measure on D if and only if $(1 - |z|^2)d\mu$ is a p-Carleson measure on D.

Pau in [9] proved the following results:

Lemma 1. Let K satisfy (1) and μ be a positive measure. Then

(i) μ is a $K-$Carleson measure if and only if

$$\sup_{a \in D} \int_D K(1 - |\varphi_a(z)|^2) dA(z) < \infty. \qquad (2)$$

(ii) μ is a compact $K-$Carleson measure if and only if (2) holds and

$$\lim_{|a| \to 1} \int_D K(1 - |\varphi_a(z)|^2) dA(z) = 0.$$

Lemma 2. Let K satisfy (1) and let $f \in \mathcal{H}(D)$. Then the following are equivalent.

(i) $f \in \mathcal{N}_K$.

(ii) $\sup_{a \in D} \int_D |(f \circ \varphi_a)(z)|^2 \, K(1 - |z|^2) dA(z) < \infty$.

(iii) $|f(z)|^2 dA(z)$ is a K-Carleson measure on D.

Lemma 3. (Test function in $\mathcal{N}_K$ see [5], Lemma 2.2) Let K satisfy (1). For $w \in D$ we define

$$h_w(z) = \frac{1 - |w|^2}{(1 - \overline{w}z)^2}.$$

Then $h_w \in \mathcal{N}_K$ and $\|h_w\|_{\mathcal{N}_K} \leq 1$.

The following lemma proved by Ueki (see [10], Lemma 2):

Lemma 4. (Test function in H^∞_α) For each $\alpha \in (0, \infty)$, $\theta \in [0, 2\pi)$, $r \in (0, 1]$ and $w \in D$, we put

$$h_{\theta,r}(w) := \sum_{k=0}^{\infty} 2^{k\alpha} (re^{i\theta})^{2^k} w^{2^k}.$$

Then $h_{\theta,r} \in H^\infty_\alpha$ and $\|h_{\theta,r}\|_{H^\infty_\alpha} \leq 1$.
In particular, $h_{\theta,r} \in H^\infty_{\alpha,0}$ if $r \in (0, 1)$.

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \leq C\|f\|_X$ for all maps $f \in X$. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y of $H(\Delta)$, T is compact from X to Y if and only if for each bounded sequence $\{x_n\} \in X$, the sequence $\{Tx_n\} \in Y$ contains a subsequence converging to some limit in Y.

Two quantities A_f and B_f, both depending on an $f \in \mathcal{H}(D)$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on D we have: $\frac{1}{C} B_f \leq A_f \leq C B_f$. If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$. As usual, the letter C will denote a positive constant, possibly different on each occurrence.

II. Weighted composition operators from H^∞_α into $\mathcal{N}_K$ spaces

In this section, we characterize the boundedness and compactness of weighted composition operators $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$. First, in the following result, we describe the boundedness of $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$.

Theorem 1. Let $K : [0, \infty) \to [0, \infty)$ be a nondecreasing function and ϕ be a holomorphic self-map of D. For $\alpha \in (0, \infty)$ and $u \in \mathcal{H}(D)$, then the following are equivalent

(i) $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ is a bounded operator.

(ii) u and ϕ satisfy:

$$\sup_{a \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(g(z, a)) dA(z) < \infty. \qquad (3)$$

(iii) u and ϕ satisfy:

$$\sup_{I \subset \partial D} \int_{S(I)} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(1 - |z|) dA(z) < \infty. \qquad (4)$$

Proof. $(\mathbf{ii}) \Rightarrow (\mathbf{i})$. We assume that condition (3) holds and let

$$\sup_{a \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(g(z, a)) dA(z) < C,$$

where C is a positive constant. If $f \in H^\infty_\alpha$, then for all $a \in D$, we have

$$\begin{aligned} & \|W_{u,\phi}(f)\|_{\mathcal{N}_K} \\ = \;& \sup_{z \in D} \int_D |u(z)|^2 |f(\phi(z))|^2 K(g(z, a)) dA(z) \\ \leq \;& \|f\|^2_{H^\infty_\alpha} \sup_{z \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(g(z, a)) dA(z) \\ \leq \;& C\|f\|^2_{H^\infty_\alpha}. \end{aligned}$$

(i) $\Rightarrow$ **(ii)**. Suppose that $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is bounded, then

$$\|W_{u,\phi}(f)\|_{\mathcal{N}_K} \le \|f\|_{H_\alpha^\infty}.$$

For each $\alpha \in (0,\infty), \theta \in [0,2\pi)$ we set the test function $h_\theta = h_{\theta,1}$ which is defined in Lemma 4 with $w = \phi(z_0)$. Fix $w \in D$, by Fubini's theorem we have

$$\begin{aligned} 1 &\ge \int_0^{2\pi} \|W_{u,\phi}(h_\theta)\|_{\mathcal{N}_K} \frac{d\theta}{2\pi} \\ &\ge \int_D |u(z)|^2 K(g(z,a)) \left(\int_0^{2\pi} |h_\theta(\phi(z))|^2 \frac{d\theta}{2\pi} \right) dA(z). \end{aligned}$$

By Parseval's formula as in [10], when $|\phi(z)| > \frac{1}{\sqrt{2}}$, we have

$$\int_0^{2\pi} |h_\theta(\phi(z))|^2 \frac{d\theta}{2\pi} \ge \frac{1}{(1-|\phi(z)|^2)^{2\alpha}}.$$

Hence we obtain

$$\int_{D_{\frac{1}{\sqrt{2}}}} \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(g(z,a)) dA(z) \le 1, \qquad (5)$$

for any $a \in D$, where $D_{\frac{1}{\sqrt{2}}} = \{z \in D : |\phi(z)| > \frac{1}{\sqrt{2}}\}$.
By noting that $u \in \mathcal{N}_K$, for any $a \in D$, we have

$$\int_{|\phi(z)| \le \frac{1}{\sqrt{2}}} \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(g(z,a)) dA(z) \le C\|u\|_{\mathcal{N}_K}. \qquad (6)$$

Inequalities (5) and (6) show that the condition (3) is true.
(iii) $\Rightarrow$ **(i)**. For every $f \in H_\alpha^\infty$ it follows that

$$\begin{aligned} &\sup_{I \subset \partial D} \int_{S(I)} |u(z)|^2 |f(\phi(z))|^2 K(1-|z|) dA(z) \\ \le\ & \|f\|^2_{H_\alpha^\infty} \sup_{I \subset \partial D} \int_{S(I)} \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(1-|z|) dA(z). \end{aligned}$$

Combining this with condition (4), we see that

$$d\mu := |u(z)|^2 |f(\phi(z))|^2 K(1-|z|) dA(z)$$

is a K-Carleson measure. Thus Lemma 1 implies that $W_{u,\phi}(f) \in \mathcal{N}_K$ and

$$\begin{aligned} & \|W_{u,\phi}(f)\|_{\mathcal{N}_K} \\ =\ & \sup_{z \in D} \int_D |u(z)|^2 |f(\phi(z))|^2 K(g(z,a)) dA(z) \\ \le\ & \|f\|^2_{H_\alpha^\infty} \sup_{z \in D} \int_D \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(g(z,a)) dA(z) \\ \le\ & C\|f\|^2_{H_\alpha^\infty}, \end{aligned}$$

and so $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is bounded. **(i)** $\Rightarrow$ **(iii)**. Assume that $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is bounded. Fix an arc $I \subset \partial D$, again we consider the test function $h_\theta, \theta \in [0,2\pi)$. By Lemma 1 and Lemma 4, we have

$$\int_{S(I) \cap D_{\frac{1}{\sqrt{2}}}} \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K((1-|z|)/|I|) dA(z) \le 1.$$

Since $u \in \mathcal{N}_K$ by the boundedness of $W_{u,\phi}$, it follows from Lemma 1 that $|u(z)|^2 dA(z)$ is a K-Carleson measure and

$$\sup_{I \subset \partial D} \int_{S(I)} |u(z)|^2 dA(z) \le \|u\|^2_{\mathcal{N}_K}.$$

Then we have

$$\int_{S(I) \cap \{|\phi(z)| \le \frac{1}{\sqrt{2}}\}} \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K\left(\frac{(1-|z|)}{|I|} \right) dA(z) \le \|u\|^2_{\mathcal{N}_K}.$$

Hence, we obtain the condition (4) and we accomplish the proof.
Under the same assumption in Theorem 1 we obtain the following theorem.

Theorem 2. Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and ϕ be a holomorphic self-map of D. For $\alpha \in (0,\infty)$ and $u \in \mathcal{H}(D)$, then the following are equivalent

(i) $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is a compact operator.

(ii) u and ϕ satisfy:

$$\lim_{r \to 1} \sup_{a \in D} \int_{D_r} \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(g(z,a)) dA(z) = 0.$$

(iii) u and ϕ satisfy:

$$\lim_{r \to 1} \sup_{I \subset \partial D} \int_{S(I) \cap D_r} \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(1-|z|) dA(z) = 0,$$

where $D_r = \{z \in D : |\phi(z)| > r\}$.

Theorem 3. Suppose $\alpha \in (0,\infty), u \in \mathcal{H}(D)$ and let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and ϕ be a holomorphic self-map of D. Then $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is a bounded operator if and only if $\frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} dA(z)$ is a K-Carleson measure.

Proof. Necessity. By Lemma 1, it suffices to prove that

$$\sup_{a \in D} \int_D \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(1-|\varphi_a(z)|^2) dA(z) < \infty.$$

Since K is nondecreasing and $(1-t^2) \le 2\log\frac{1}{t}$, for $t \in (0,1]$, we have $1-|\varphi_a(z)|^2 \le 2\log\frac{1}{|\varphi_a(z)|} \le 2g(z,a)$, for all $z, a \in D$. Using Theorem 1, we have

$$\begin{aligned} & \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(1-|\varphi_a(z)|^2) dA(z) \\ \le\ & \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(2g(z,a)) dA(z) \\ \le\ & \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(g(z,a)) dA(z) < \infty. \end{aligned}$$

Sufficiency. Assume that $\frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} dA(z)$ is a K-Carleson measure. Then

$$\sup_{a \in D} \int_D \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(1-|\varphi_a(z)|^2) dA(z) < \infty.$$

We obtain that for all $f \in H_\alpha^\infty$,

$$\begin{aligned} & \sup_{a \in D} \int_D |W_{u,\phi}(f)(z)|^2 K(1-|\varphi_a(z)|^2) dA(z) \\ =\ & \sup_{a \in D} \int_D |u(z)|^2 |f(\phi(z))|^2 K(1-|\varphi_a(z)|^2) dA(z) \\ \le\ & \|f\|_{H_\alpha^\infty} \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1-|\phi(z)|^2)^{2\alpha}} K(1-|\varphi_a(z)|^2) dA(z) \\ \le\ & \infty. \end{aligned}$$

By Lemma 1, $W_{u,\phi}(f) \in \mathcal{N}_K$. Thus $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is a bounded operator. The proof is completed.

III. Weighted composition operators from $\mathcal{N}_K$ into H_α^∞

In this section, we will consider the operator $W_{u,\phi} : \mathcal{N}_K \to H_\alpha^\infty$. The case $u \equiv 1$ can be found in the work [4] by El-Sayed Ahmed and Bakhit.

Theorem 4. Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and ϕ be a holomorphic self-map of D. For $\alpha \in (0,\infty)$ and $u \in \mathcal{H}(D)$, then $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is a bounded operator if and only if

$$\sup_{a\in D} \frac{|u(z)|^2(1-|z|^2)^\alpha}{1-|\phi(z)|^2} < \infty. \tag{7}$$

Proof. We know that $\mathcal{N}_K \subset H_1^\infty$, for each nondecreasing function $K : [0,\infty) \to [0,\infty)$ (see [4], Proposition 2.1). First assume that condition (7) holds. Then

$$\begin{aligned} \|W_{u,\phi}(f)\|_{H_\alpha^\infty} &= \sup_{z\in D} |u(z)||f(\phi(z))|(1-|z|^2)^\alpha \\ &\le \|f\|_{H_1^\infty} \sup_{z\in D} \frac{|u(z)|(1-|z|^2)^\alpha}{1-|\phi(z)|^2} \\ &\le C\|f\|_{\mathcal{N}_K}. \end{aligned}$$

This implies that $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is a bounded operator.
Conversely, assume that $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is bounded, then

$$\|W_{u,\phi}(f)\|_{H_\alpha^\infty} \le \|f\|_{\mathcal{N}_K}.$$

Fix a point $z_0 \in D$, and let h_w be the test function in Lemma 3 with $w = \phi(z_0)$. Then,

$$\begin{aligned} 1 \ge \|h_w\|_{\mathcal{N}_K} &\ge C_1\|W_{u,\phi}(h_w)\|_{H_\alpha^\infty} \\ &\ge \frac{|u(z_0)|(1-|w|^2)}{|1-\overline{w}\phi(z_0)|^2}(1-|z_0|^2)^\alpha \\ &= \frac{|u(z_0)|(1-|z_0|^2)^\alpha}{1-|\phi(z_0)|^2}, \end{aligned}$$

where C_1 is a positive constant. This completes the proof of the theorem.

Theorem 5. Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and ϕ be a holomorphic self-map of D. For $\alpha \in (0,\infty)$ and $u \in \mathcal{H}(D)$, then $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is a compact operator if and only if

$$\lim_{r\to 1} \sup_{z\in D_r} \frac{|u(z)|(1-|z|^2)^\alpha}{1-|\phi(z)|^2} = 0. \tag{8}$$

Proof. First assume that $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is compact and suppose that there exists $\varepsilon_0 > 0$ a sequence $\{z_n\} \subset D$ such that

$$\frac{|u(z_n)|(1-|z_n|^2)^\alpha}{1-|\phi(z_n)|^2} \ge \varepsilon_0,$$

whenever $|\phi(z_n)| > 1 - \frac{1}{n}$.
Clearly, we can assume that $w_n = \phi(z_n)$ tends to $w_0 \in \partial D$ as $n \to \infty$. Let $h_{w_n} = \frac{(1-|w_n|^2)}{(1-\overline{w_n}z)^2}$ be the function in Lemma 3. Then $h_{w_n} \to h_{w_0}$ with respect to the compact-open topology. Define $f_n = h_{w_n} - h_{w_0}$. By Lemma 3, we have $\|f_n\|_{\mathcal{N}_K} \le 1$ and $f_n \to 0$ uniformly on compact subsets of D. Thus, $f_n \circ \phi \to 0$ in H_α^∞ by assumption. But, for n big enough,

$$\begin{aligned} &\|W_{u,\phi}(f_n)\|_{H_\alpha^\infty} \\ \ge\ & |u(z_n)||h_{w_n}(\phi(z_n)) - h_{w_0}(\phi(z_n))|(1-|z_n|^2)^\alpha \\ \ge\ & \underbrace{\frac{|u(z_n)|(1-|z_n|^2)^\alpha}{1-|\phi(z_n)|^2}}_{\ge \varepsilon_0} \underbrace{\left|1 - \frac{(1-|w_n|^2)(1-|w_0|^2)}{\left|1-\overline{w_0}\, w_n\right|}\right|}_{=1}, \end{aligned}$$

which is a contradiction.
To prove the necessity of (8), we assume that for all $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that

$$\frac{|u(z)|(1-|z|^2)^\alpha}{1-|\phi(z)|^2} < \varepsilon,$$

whenever $|\phi(z)| > \delta$. Let $\{f_n\}$ be a bounded sequence in $\mathcal{N}_K$ norm which converges to zero on compact subsets of D. Clearly, we may assume that $|\phi(z)| > \delta$. Then

$$\begin{aligned} &\|W_{u,\phi}(f_n)\|_{H_\alpha^\infty} \\ =\ & \sup_{z\in D} |u(z)||f_n(\phi(z))|(1-|z|^2)^\alpha \\ =\ & \sup_{z\in D} \frac{|u(z)|(1-|z|^2)^\alpha}{1-|\phi(z)|^2}\left|f_n(\phi(z))\right|(1-|\phi(z)|^2) \\ \le\ & \varepsilon\, C\|f_n\|_{H_1^\infty} \le \varepsilon\, C\, \|f_n\|_{\mathcal{N}_K} \le \varepsilon. \end{aligned}$$

It follows that $W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K$ is a compact operator. This completes the proof of the theorem.

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