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# Dynamic Analysis by a Family of Time Marching Procedures Based On Numerically Computed Green's Functions

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***Abstract*—**In this work, a new family of time marching procedures based on Green's function matrices is presented. The formulation is based on the development of new recurrence relationships, which employ time integral terms to treat initial condition values. These integral terms are numerically evaluated taking into account Newton-Cotes formulas. The Green's matrices of the model are also numerically computed, taking into account the generalized-α method and subcycling techniques. As it is discussed and illustrated along the text, the proposed procedure is efficient and accurate, providing a very attractive time marching technique.

***Keywords*—**Dynamics, Time-Marching, Green's Function, Generalized-α Method, Subcycling.

## I. Introduction

TIME dependent hyperbolic equations have numerous applications in various branches of science and in practical engineering design. Since it is usually very difficult to obtain analytical transient responses for these equations, numerical techniques must be applied to find approximate solutions, and step-by-step time integration algorithms are routinely employed when dynamic problems are focused, because of their various inherent advantages to solve a great deal of initial value problems.

In this work, time marching algorithms based on numerically computed Green's function matrices are focused. Recently, Tamma et al. [1] and Zhou and Tamma [2] derived a new family of unconditionally explicit/implicit algorithms based on analytical solution of first-order ordinary differential equations in which the concept of Green's functions is implicitly present. Other similar approaches have been the subject of some papers discussion, and are usually referred to as ''precise time-step integration methods'' [3]-[5]. In Soares [6], [7] and Soares and Mansur [8], a formulation that implicitly computes Green's function matrices taking into account standard time domain numerical procedures was presented, in order to obtain a new family of time marching schemes. Later on, step response and Green's function matrices were computed explicitly by Mansur et al. [9], taking into account explicit time marching routines. The methodology was extended to analyze dynamic models discretized by different numerical techniques [10], [11] and to efficiently analyze coupled problems [12]-[15]. The analysis of a variety of coupled problems by the proposed procedure could be carried out very effectively since it allows coupled sub-domains to be treated independently, at each time step (similarly, it also allows a quite efficient analysis of some non-linear models [6]-[8]). The direct computation of Green's function matrices in order to develop time marching routines to analyze hyperbolic models has been carried out in transformed domains as well, being the frequency [16]-[18] and the Laplace [19] domain mostly focused. Recently, the methodology has been generalized [20] and also applied to analyze heat conduction models [21]-[23], extrapolating the initially focused context of wave propagation problems and structural dynamic analyses.

In the present work a family of time marching algorithms based on Green's function matrices is discussed. Here, dynamic models are studied and time step relationships based on Green's function matrices are presented, allowing more efficient and accurate time marching techniques to be developed.

## II. Time Marching Procedure Based On Numerically Computed Green's Functions

The governing system of equations describing a linear dynamic model is given by [24]:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{F}(t) \tag{1}$$

where **M**, **C** and **K** are mass, damping and stiffness matrices, respectively, $\mathbf{F}(t)$ stands for the force vector and $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$ and $\ddot{\mathbf{U}}(t)$ are displacement, velocity and acceleration vectors, respectively.

Supposing that the analysis begins at a time instant $t$ and that a $\Delta t$ period of analysis is focused, the following analytical recurrence relationships can be obtained to compute displacements and velocities [20]:

$$\mathbf{U}(t+\Delta t) = \mathbf{U}(t) - \int_0^{\Delta t} \mathbf{G}(\Delta t - \tau)\, d\tau\, \mathbf{K}\mathbf{U}(t) + \mathbf{G}(\Delta t)\mathbf{M}\dot{\mathbf{U}}(t) + \int_t^{t+\Delta t} \mathbf{G}(t+\Delta t-\tau)\mathbf{F}(\tau)\, d\tau \tag{2a}$$

$$\dot{\mathbf{U}}(t+\Delta t) = -\int_0^{\Delta t} \dot{\mathbf{G}}(\Delta t - \tau)\, d\tau\, \mathbf{K}\mathbf{U}(t) + \dot{\mathbf{G}}(\Delta t)\mathbf{M}\dot{\mathbf{U}}(t) + \int_t^{t+\Delta t} \dot{\mathbf{G}}(t+\Delta t-\tau)\mathbf{F}(\tau)\, d\tau \tag{2b}$$

where the Green's function matrices that appear in (2) can be computed through the solution of the following initial

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condition problem, within the time period $\Delta t$:

$$\mathbf{M}\ddot{\mathbf{G}}(t)+\mathbf{C}\dot{\mathbf{G}}(t)+\mathbf{K}\mathbf{G}(t)=\mathbf{0}\ ;\ \mathbf{G}(0)=\mathbf{0}\ \text{ and }\ \dot{\mathbf{G}}(0)=\mathbf{M}^{-1} \quad (3)$$

The solution of the dynamic problem described by (3) can be carried out numerically, providing the numerical computation of the Green's functions of the model. The accuracy, stability and efficiency of the recurrence relations (2) are intimately related to the computation of these Green's matrices. Once the Green's function matrices and the time convolution integrals expressed in (2) are properly computed, this time marching procedure may become extremely accurate, providing a very attractive methodology (one should observe that recurrence relations (2) are the analytical solutions of the dynamic model, hence, a very accurate procedure is expected, once the terms in relations (2) are properly computed).

In this work, the generalized-α method is employed to compute the Green's matrices of the model (one should observe that several other numerical procedures could be employed, which would result in different time marching techniques). In the generalized-α method, the time discrete equation of motion employed to compute the Green's matrices of the model is written as:

$$\mathbf{M}[\alpha_m\ddot{\mathbf{G}}^{\eta}+(1-\alpha_m)\ddot{\mathbf{G}}^{\eta+1}]+\mathbf{C}[\alpha_k\dot{\mathbf{G}}^{\eta}+(1-\alpha_k)\dot{\mathbf{G}}^{\eta+1}]+\mathbf{K}[\alpha_k\mathbf{G}^{\eta}+(1-\alpha_k)\mathbf{G}^{\eta+1}]=\mathbf{0} \quad (4)$$

and the finite difference formulas of the Newmark method are retained, as described below:

$$\mathbf{G}^{\eta+1}=\mathbf{G}^{\eta}+\Delta t\dot{\mathbf{G}}^{\eta}+\Delta t^2(1/2-\mu_1)\ddot{\mathbf{G}}^{\eta}+\Delta t^2\mu_1\ddot{\mathbf{G}}^{\eta+1} \quad (5a)$$

$$\dot{\mathbf{G}}^{\eta+1}=\dot{\mathbf{G}}^{\eta}+\Delta t(1-\mu_2)\ddot{\mathbf{G}}^{\eta}+\Delta t\mu_2\ddot{\mathbf{G}}^{\eta+1} \quad (5b)$$

where $\mu_1$ and $\mu_2$ stand for the Newmark's parameters and $\alpha_m$ and $\alpha_k$ describe the generalized-α method. By substituting approximations (5) into (4), the following time-marching procedure arises, which enables the computation of the Green's function matrix of accelerations:

$$\begin{aligned}&[(1-\alpha_k)\mu_1\Delta t^2\mathbf{K}+(1-\alpha_k)\mu_2\Delta t\mathbf{C}+(1-\alpha_m)\mathbf{M}]\ddot{\mathbf{G}}^{\eta+1}=\\&=-\mathbf{K}\mathbf{G}^{\eta}+-[\mathbf{C}+(1-\alpha_k)\Delta t\mathbf{K}]\dot{\mathbf{G}}^{\eta}+\\&-[\alpha_m\mathbf{M}+(1-\alpha_k)(1-\mu_2)\Delta t\mathbf{C}+(1-\alpha_k)(1/2-\mu_1)\Delta t^2\mathbf{K}]\ddot{\mathbf{G}}^{\eta}\end{aligned} \quad (6)$$

In the generalized-α method, second-order accuracy and maximal high-frequency dissipation are achieved if $\mu_1=(1+\alpha_k-\alpha_m)^2/4$ and $\mu_2=1/2+\alpha_k-\alpha_m$ are adopted. For $\alpha_k=\alpha_m=0$, the method reduces to the Newmark method; for $\alpha_m=0$, the method reduces to the HHT method; and for $\alpha_k=0$, the method reduces to the Bossak method.

In order to achieve a more accurate numerical procedure, the Green's matrices of the model can be computed within a time step $\Delta t$, taking into account a subcycling technique, i.e.: the system of equations (6) is solved $n$ times (as well as the actualizations described by (5)), considering a time sub-step of $\Delta t/n$. As a consequence, the following matrices are computed considering the subcycling technique: $\mathbf{G}^1$, $\mathbf{G}^2$, …, $\mathbf{G}^n$ etc; which are the numerical approximations to the following Green's matrices: $\mathbf{G}(\Delta t/n)$, $\mathbf{G}(2\Delta t/n)$, …, $\mathbf{G}(\Delta t)$ etc. To numerically evaluate the time integrals described in (2), the trapezoidal rule is considered here, taking into account the $n$ sub-steps employed to evaluate the Green's function matrices, allowing to obtain the following recurrence relationships:

$$\mathbf{U}^{t+\Delta t}=\mathbf{U}^{t}-\mathbf{J}^{n}\mathbf{K}\mathbf{U}^{t}+\mathbf{G}^{n}\mathbf{M}\dot{\mathbf{U}}^{t}+\mathbf{L}_1^{n}\mathbf{F}^{t}+\mathbf{L}_2^{n}\mathbf{F}^{t+\Delta t} \quad (7a)$$

$$\dot{\mathbf{U}}^{t+\Delta t}=-\dot{\mathbf{J}}^{n}\mathbf{K}\mathbf{U}^{t}+\dot{\mathbf{G}}^{n}\mathbf{M}\dot{\mathbf{U}}^{t}+\dot{\mathbf{L}}_1^{n}\mathbf{F}^{t}+\dot{\mathbf{L}}_2^{n}\mathbf{F}^{t+\Delta t} \quad (7b)$$

where

$$\mathbf{L}_1^{n}=\tfrac{1}{2}(\tfrac{\Delta t}{n})\mathbf{G}^{n}+(\tfrac{\Delta t}{n})\sum_{i=1}^{n-1}(\tfrac{i}{n})\mathbf{G}^{i} \quad (8a)$$

$$\mathbf{L}_2^{n}=\tfrac{1}{2}(\tfrac{\Delta t}{n})\mathbf{G}^{0}+(\tfrac{\Delta t}{n})\sum_{i=1}^{n-1}(1-\tfrac{i}{n})\mathbf{G}^{i} \quad (8b)$$

$$\mathbf{J}^{n}=\mathbf{L}_1^{n}+\mathbf{L}_2^{n} \quad (8c)$$

and $\dot{\mathbf{L}}_1^{n}$, $\dot{\mathbf{L}}_2^{n}$ and $\dot{\mathbf{J}}^{n}$ can be analogously computed. In (7), the $\mathbf{L}$ terms are evaluated considering a linear behavior of the acting forces within the time interval $[t,t+\Delta t]$, i.e.: $\mathbf{F}(\tau)=\mathbf{F}^{t}+(\mathbf{F}^{t+\Delta t}-\mathbf{F}^{t})(\tau-t)/\Delta t$, for $t<\tau<t+\Delta t$.

The time marching procedure (7) becomes extremely accurate once high values of $n$ are selected, allowing considerably large values of $\Delta t$ to be considered, without damaging the accuracy and the stability of the time marching technique. Once considerably larger time steps $\Delta t$ are allowed, the time marching procedure (7) may become highly efficient to analyse long period responses. In Fig. 1, the spectral radius of the amplification matrix of the new procedure are depicted taking into account several $\mu_1$, $\mu_2$, $\alpha_m$ and $\alpha_k$ values, as well as $n$ = 1, 4 and 10. Results of the standard generalized-α method are also depicted in the figure, for reference.

## III. Numerical Applications

In this section, two numerical applications are presented to further illustrate some aspects of the proposed methodology. In the first application, a shear-building model is analyzed and results are computed taking into account the Newmark's trapezoidal rule and linear acceleration methodologies. In the second application, a more complex physical model is focused. In this case, a clamped beam discretized by finite elements is considered and the Central Difference Method is employed to compute the related results.

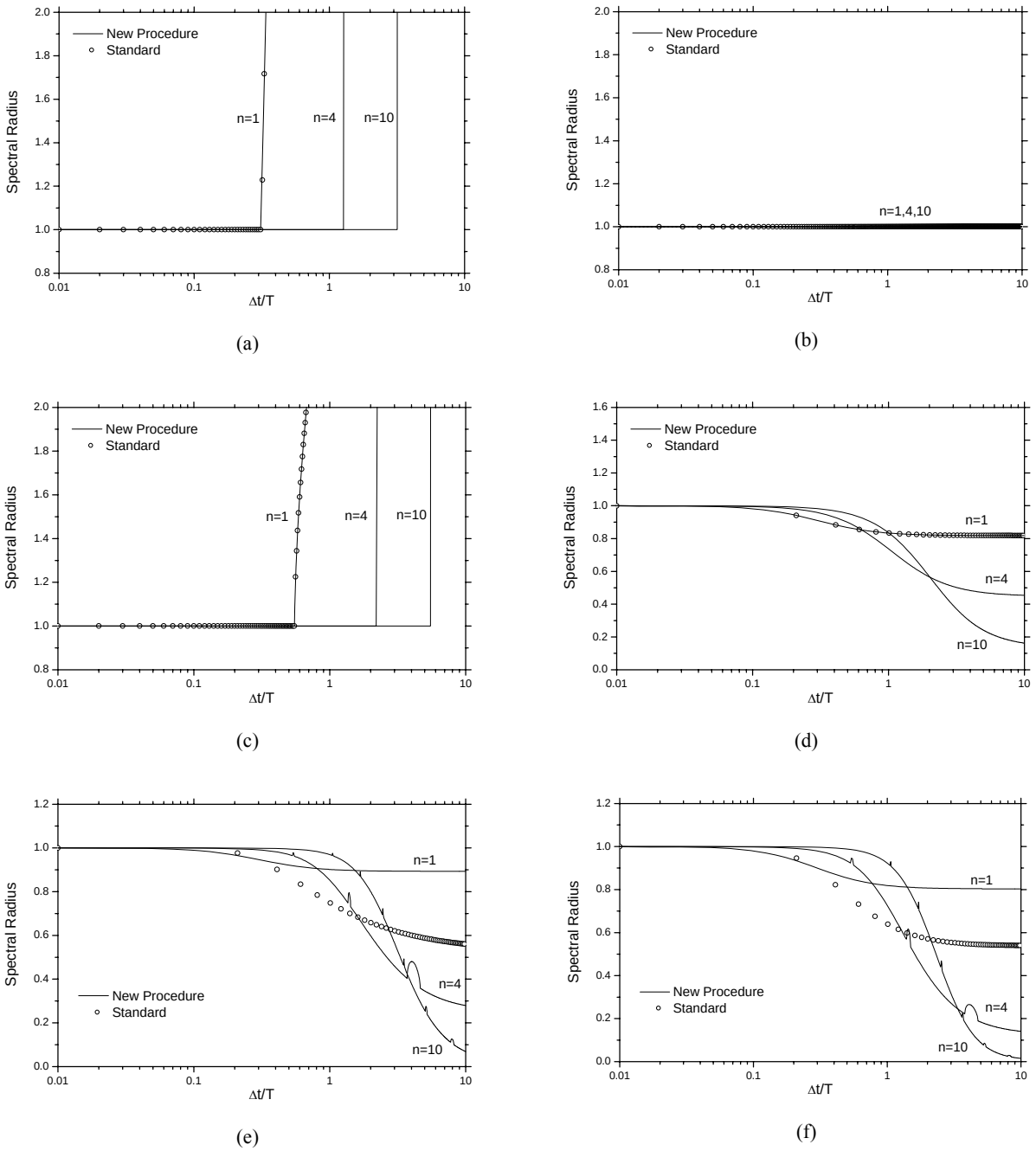



Fig. 1 Spectral radius: (a) Central Difference Method ( $\mu_1 = 0$ , $\mu_2 = 1/2$ , $\alpha_k = \alpha_m = 0$ ); (b) Trapezoidal Rule ( $\mu_1 = 1/4$ , $\mu_2 = 1/2$ , $\alpha_k = \alpha_m = 0$ ); (c) Linear Acceleration Method ( $\mu_1 = 1/6$ , $\mu_2 = 1/2$ , $\alpha_k = \alpha_m = 0$ ); (d) Damped Newmark ( $\mu_1 = (\mu_2 + 1/2)^2/4$ , $\mu_2 = 0.6$ , $\alpha_k = \alpha_m = 0$ ); (e) HHT Method ( $\mu_1 = (1 + \alpha_k - \alpha_m)^2/4$ , $\mu_2 = 1/2 + \alpha_k - \alpha_m$ , $\alpha_k = 0.3$ , $\alpha_m = 0$ ); (f) Bossak Method ( $\mu_1 = (1 + \alpha_k - \alpha_m)^2/4$ , $\mu_2 = 1/2 + \alpha_k - \alpha_m$ , $\alpha_k = 0$ , $\alpha_m = -0.3$ )

*A. Application 1*

A simple four-store shear building is analyzed here. A sketch of the model is depicted in Fig. 2. The mass and the stiffness values are adopted the same for all floors, they are: $m = 5 \cdot 10^4 kg$ and $k = 2.5 \cdot 10^7 N/m$. A force, whose time dependence is shown in Fig. 2 (c), is applied at the fourth floor of the model. Several numerical techniques are considered to analyse the shear building. In the first analyses, the standard Newmark's trapezoidal rule and linear

acceleration method are applied to analyze the model considering a small time step ( $\Delta t = 0.0025s$ ), in order to provide reference results for comparisons. Next, the same standard procedures are applied to analyze the model considering a larger time step ( $\Delta t = 0.25s$ ). Taking into account this large time step, the model is finally analyzed considering the proposed formulation, adopting $n$ = 10. The obtained results are depicted in Fig. 3.

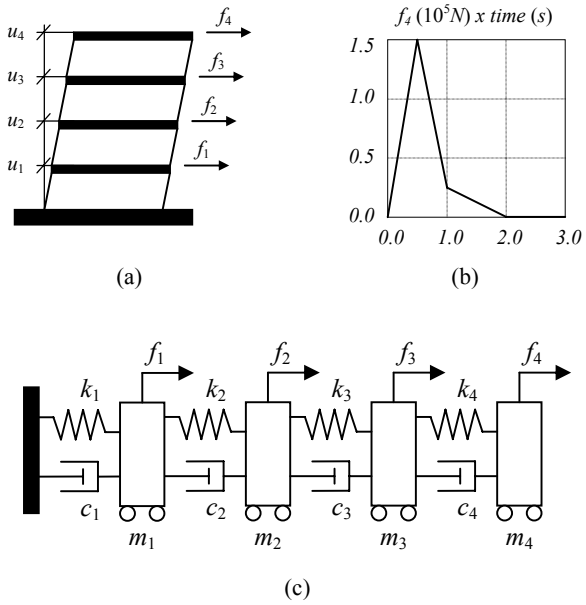
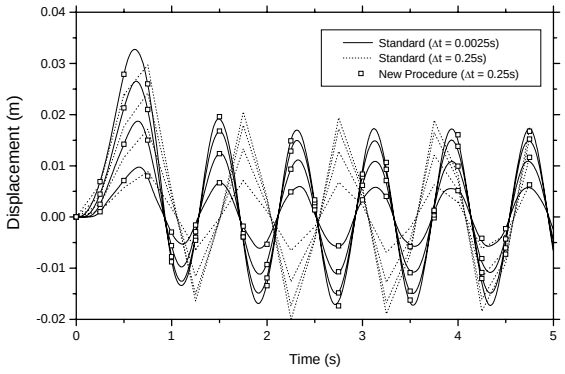
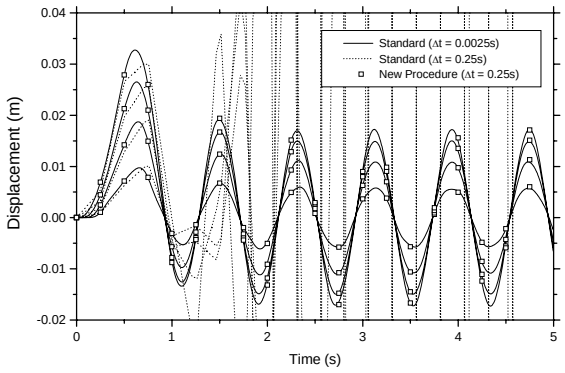



Fig. 2 Shear building: (a) four-store shear building model; (b) load applied at the fourth floor; (c) equivalent spring-dashpot-mass model

As can be observed in Fig. 3, for the larger time step, standard procedures provide very inaccurate results: an expressive period elongation takes place considering the trapezoidal rule, as depicted in Fig. 3 (a); and unstable results arise considering the linear acceleration method, as described in Fig. 3 (b). On the other hand, even for a large time step, stability and accuracy are observed in Fig. 3 taking into account the proposed methodology. As a matter of fact, by adopting an appropriate number of sub-steps $n$, the proposed methodology can become as accurate as one wishes and critical time steps (above which instabilities arise) can be made as high as it may be desired (as illustrated, for instance, in Figs. 1 (a) and (c)).




(a)




(b)

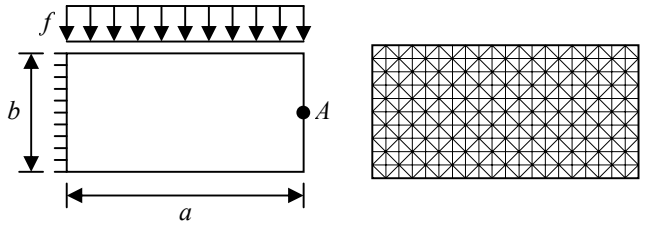
Fig. 3 Displacement time history results considering the new ($n$ = 10) and the standard time marching procedures: (a) Trapezoidal Rule; (b) Linear Acceleration Method

*B. Application 2*

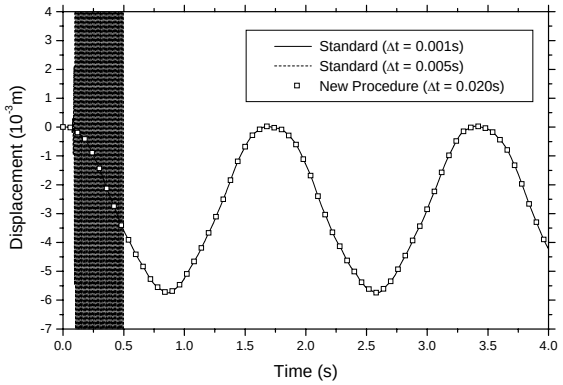
In this application a clamped beam is analyzed. The geometry, boundary conditions and finite element mesh adopted for the model is depicted in Figs. 4 (a) and (b) (400 linear triangular finite elements are employed). The geometry of the beam is defined by $a$ = 1.0$m$ and $b$ = 0.5$m$. The model is submitted to a suddenly applied load, which is kept constant along time. The material properties of the beam are: $\nu$ = 0.0 (Poisson's ratio); $E$ = 100.0 $N/m^2$ (Young modulus); $\rho$ = 1.5 $kg/m^3$ (mass density).

As before, several numerical approaches are employed to analyze the model. First, in order to provide reference results, the standard Central Difference Method is considered, adopting $\Delta t = 0.001s$ and $\Delta t = 0.005s$. Next, the proposed formulation (associated with the Central Difference Method) is employed to analyse the beam, adopting $\Delta t = 0.020s$ and $n$ = 5. The obtained results are depicted in Fig. 4 (c). Fig. 4 (c) shows the vertical displacements obtained at point A (see Fig. 4 (a)). As can be observed, unstable results arise considering the Central Difference Method with $\Delta t = 0.005s$, whereas stable and accurate results are obtained by the proposed

methodology with the much larger time step $\Delta t = 0.020s$. These results illustrate once again the previously highlighted fact that the proposed methodology can become as accurate as one wishes and critical time steps can be made as high as it may be desired.




(a) (b)




(c)

Fig. 4 Clamped beam: (a) sketch of the model; (b) finite element mesh; (c) displacement time history results considering the proposed procedure ($n$ = 5) and the Central Difference Method (results are plotted just for $0s \le t \le 0.5s$ when $\Delta t = 0.005s$)

## IV. Conclusions

In this work, a time marching procedure based on Green's functions matrices is discussed. It considers the numerical evaluation of the Green's matrices, as well as their time integrations. In the present methodology, time integral terms are employed not only to treat external forces, but also to take into account initial displacement contributions. The generalized-α method is considered to compute the Green's matrices of the model and the 2 points Newton-Cotes quadrature rule (trapezoidal rule) is employed to numerically evaluate the required time integrals. Both procedures take into account a sub-step technique, which is applied in order to improve the accuracy and stability of the methodology.

As it is discussed and illustrated along the text, the proposed formulation is efficient and accurate. It only requires the solution of one initial condition problem to compute the fundamental matrices of the recurrence relationships, rendering more efficient procedures than formulations based on step response matrices [5], [9]. At the same time, the procedure maintains the good accuracy and stability aspects that are obtained by adopting step response matrices (briefly, it can be said that the procedure inherits the merits of the step response matrices, without inheriting its drawbacks). Moreover, by adopting a sub-step technique, the accuracy and stability of the methodology can be chosen as good as one wishes.

It is important to highlight that several numerical methods and combinations are possible to compute the discussed terms of the recurrence relationships, rendering different time marching techniques. This vast field of possibilities can be explored in future works, allowing more effective time marching procedures to be developed.

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