Bifurcation Analysis of a Plankton Model with Discrete Delay

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Abstract—In this paper, a delayed plankton-nutrient interaction model consisting of phytoplankton, zooplankton and dissolved nutrient is considered. It is assumed that some species of phytoplankton releases toxin (known as toxin producing phytoplankton (TPP)) which is harmful for zooplankton growth and this toxin releasing process follows a discrete time variation. Using delay as bifurcation parameter, the stability of interior equilibrium point is investigated and it is shown that time delay can destabilize the otherwise stable non-zero equilibrium state by inducing Hopf-bifurcation when it crosses a certain threshold value. Explicit results are derived for stability and direction of the bifurcating periodic solution by using normal form theory and center manifold arguments. Finally, outcomes of the system are validated through numerical simulations.

Keywords—Plankton, Time delay, Hopf-bifurcation, Normal form theory, Center manifold theorem.

I. INTRODUCTION

PLANKTON refer to all single-celled, microscopic organism in marine environment that drift with the oceanic currents. Phytoplankton in particular is capable of photo-synthesis in the presence of sunlight and occupies the first trophic level for all aquatic food chains. Hence, they are producers and recyclers of most of the energy that flows through the oceanic ecosystem. Zooplankton, the herbivores prey on phytoplankton for their food and occupies the next trophic level in aquatic food chain. The rapid increase and decrease of phytoplankton population is a common feature in marine ecology and known as "bloom". Generally, highly nutrient and favorable conditions play a key role in rapid or massive growth of algae and low nutrient concentration as well as unfavorable conditions inevitably limits their growth. Although, the sudden appearance and disappearance of blooms is not well understood; many researchers have studied the nutrient-plankton interaction to understand the importance of nutrient concentration on the growth of plankton [1], [2]. The persistence and co-existence in nutrient-plankton interaction have also been discussed by Ruan [3], [4].

The understanding of the dynamic of plankton-nutrient system becomes complex when additional effects of toxicity (caused due to the release of toxic substances by some phytoplankton species known as harmful phytoplankton) are considered. The role of toxin and nutrient on the plankton system has been discussed in [5]-[8]. Sarkar et al. in [9] and [10] studied the interaction of toxin producing phytoplankton-zooplankton system and concluded that harmful phytoplankton may be used as bio-control agent in the termination of harmful planktonic blooms. It is well known that time delay in biological systems is a reality and it can have complex impact on the dynamic of the system namely loss of stabilty, induced oscillations and periodic solutons [11]-[13]. The interaction of plankton-nutrient model with delay due to gestation and nutrient recycling has also been studied in [14] and [15]. Chattopadhyay et al. in [16] proposed and analyzed a mathematical model of toxic phytoplankton (*Noctilucca Scintillans* belonging to the group Dinoflagellates of the division Dinophyta)-zooplankton (*Paracalanus* belonging to the group Copepoda) interaction and assumed that the liberation of toxic substances by the phytoplankton species in not an instantaneous process but is mediated by some time lag required for maturity of species. Extending the work of [16], Bandyopadhyay et al. [17] and Rehim et al. [18] have studied the global stability of the toxin producing phytoplankton-zooplankton system. Sufficient efforts have already been made to understand the interaction of phytoplankton-zooplankton system with delay in toxin liberation, but the study of nutrient-plankton interaction with delay in toxin liberation by the phytoplankton species is not done so far. In this paper, an open system with three interacting components consisting of phytoplankton (P), zooplankton (Z) and dissolved nutrient (N) is considered. Here, it is assumed that the functional form of biomass conversion by the herbivore is of holling-II type and the predator is obligate that is they does not take nutrient directly. The toxic substance term which causes extra mortality in zooplankton is expressed in holling-I type functional form [19]. It is also taken into account that the liberation of toxic substances by the phytoplankton species follows discrete time variation. The main aim of the present study is to see the effect of this discrete time delay on the nutrient-plankton system and the organisation of our paper is as follows: In subsection A, the mathematical model is presented using simultaneous differential equations and we analyze the stability of the co-existence equilibrium in the absence of delay in subsection B. After that we have considered the delayed plankton model and considering delay as bifurcation parameter the dynamical behavior of the system around coexisting equilibrium is discussed. In subsection C, we have investigated the direction and stability of the bifurcating solution using a technique based upon normal form theory and center manifold theorem. Some

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supports our analytical findings through numerical simulations are given in subsection D. Finally, the basic outcomes of our mathematical findings and their ecological significance are mentioned in Section II.

A. The Mathematical Model

Let $N(t)$ denotes the concentration of nutrient at time 't'. Let x(t) and y(t) be the concentration of phytoplankton and zooplankton population respectively at time 't'. Let N_0 is the constant input of nutrient concentration and a is their absorption rate. Let b and α_1 be the nutrient uptake rate for the phytoplankton population and conversion rate of nutrient for the growth of phytoplankton population, respectively ($b \le \alpha_1$). Let β be the maximal zooplankton ingestion rate and β_1 ($\beta_1 \le \beta$) be the maximal zooplankton conversion rate. Let b_1 be the mortality rate of the phytoplankton population, α_2 be the mortality rate of the zooplankton population and let nutrient are recycled at the rate k after the dead of phytoplankton population. It is assumed zooplankton population decay at the rate of ρ due to toxin producing phytoplankton. The grazing phenomenon is described by the holling-II type functional form with γ as the half saturation constant. Let τ is the time delay which is incorporated with the assumption that the liberation of toxin is not instantaneous it is mediated by some time lag. The biological significance of this time lag lies in the fact this time may be considered the time required for the maturity of toxic-phytoplankton to reduce the grazing impact of zooplankton.

With these assumptions our model system is

$$\begin{cases} \dfrac{dN}{dt} = N_0 - aN - bNx + k_1 b_1 x \\ \dfrac{dx}{dt} = \alpha_1 Nx - b_1 x - \dfrac{\beta xy}{(\gamma + x)} \\ \dfrac{dy}{dt} = \dfrac{\beta_1 xy}{(\gamma + x)} - \alpha_2 y - \rho x(t-\tau) y \end{cases} \tag{1}$$

The initial conditions of the system (1) has the form $N(\theta) = \phi_1(\theta)$, $x(\theta) = \phi_2(\theta)$, $y(\theta) = \phi_3(\theta)$, $\phi_1(\theta) \ge 0$, $\phi_2(\theta) \ge 0$, $\phi_3(\theta) \ge 0$, $\theta \in [-\tau, 0]$, $\phi_1(0) \ge 0$, $\phi_2(0) \ge 0$, $\phi_3(0) \ge 0$, where $\phi_1(\theta), \phi_2(\theta), \phi_3(\theta) \in C([-\tau, 0], R_+^3)$, the banach space of continuous functions mapping the interval $[-\tau, 0]$ into R_+^3 where $R_+^3 = \{(x_1, x_2, x_3) : x_i \ge 0, i = 1,2,3\}$.

B. Stability Analysis of the Mathematical Model

The given system has three equilibria namely:

(i) The boundary equilibrium $E_1 = (\frac{N_0}{a}, 0, 0)$

(ii) A planar equilibrium $E_2 = (\frac{b_1}{\alpha_1}, \frac{N_0\alpha_1 - ab_1}{b_1(b - \alpha_1 k_1)}, 0)$ exist if $N_0 > \frac{ab_1}{\alpha_1}$ and $\alpha_1 < \frac{b}{k_1}$.

(iii) A positive interior equilibrium $E_* = (N_*, x_*, y_*)$

where $N_* = \frac{N_0 + k_1 b_1 x_*}{a + bx_*}$,

$x_* = \frac{(\beta_1 - \alpha_2 - \rho\gamma) - \sqrt{((\beta_1 - \alpha_2 - \rho\gamma)^2 - 4\rho\alpha_2\gamma)}}{2\rho}$ and

$y_* = \frac{(\alpha_1 N_* - b_1)(\gamma + x_*)}{\beta}$

which exist if $\beta_1 > \alpha_2 + \rho\gamma$ and $N_* > \frac{b_1}{\alpha_1}$.

Proposition: The plankton free equilibria $E_1 = (\frac{N_0}{a}, 0, 0)$ always exist and stable so long the constant input rate of nutrient is less than certain threshold value i.e. $N_0 < \frac{ab_1}{\alpha_1}$. Moreover zooplankton free equilibria i.e. $E_2 = (\frac{b_1}{\alpha_1}, \frac{N_0\alpha_1 - ab_1}{b_1(b - \alpha_1 k_1)}, 0)$ exist and unstable if the growth rate of phytoplankton biomass α_1 satisfies the inequality, $\alpha_1 < min(\frac{ab_1}{N_0}, \frac{b}{k_1})$.

Definition: The Equilibrium E_* is called asymptotically stable (AS) if there exist a $K > 0$ such that

$$sup_{-\tau \le \theta \le 0}[|\emptyset_1(\theta) - N| + |\emptyset_2(\theta) - x_*| + |\emptyset_3(\theta) - y_*|] < \delta$$

which implies that $lim_{t\to\infty}(N(t), x(t), y(t)) = (N_*, x_*, y_*)$, where $(N(t), x(t), y(t))$ is the solution of the system (1) with given initial conditions.

Definition: The equilibrium E_* is absolutely stable if it is AS for all delays $\tau \ge 0$ and is conditionally stable if it is AS for τ in some finite interval.

The characteristic equation of the system at E_* has the following form

$$\lambda^3 + A\lambda^2 + B\lambda + C + (D + E\lambda)e^{-\lambda\tau} = 0 \tag{2}$$

In the absence of delay $(\tau = 0)$, (2) reduces to,

$$\lambda^3 + A\lambda^2 + (B + E)\lambda + (C + D) = 0, \tag{3}$$

where

$$A = a + bx_* - \alpha_1 N_* + b_1 + \frac{\beta\gamma y_*}{(\gamma + x_*)^2} > 0,$$

$$B + E = (a + bx_*)(-\alpha_1 N_* + b_1 + \frac{\beta\gamma y_*}{(\gamma + x_*)^2}) + \frac{\beta\beta_1\gamma x_* y_*}{(\gamma + x_*)^3} - \alpha_1 x_*(-bN_* + k_1 b_1) - \frac{\beta\rho x_* y_*}{(\gamma + x_*)} > 0$$

$$C+D=\frac{\beta x_* y_*(a+bx_*)}{(\gamma+x_*)}\{\frac{\beta_1\gamma}{(\gamma+x_*)^2}-\rho\}>0$$

by using Routh-Hurwitz criterion we know that all the roots of (3) have negative real parts ,i.e. the positive equilibrium E_* is Locally Asymptotically Stable provided that the condition

$$(H_1):\ A(B+E)-(C+D)>0$$

hold.

Remark: The detailed analysis of the model system in the absence of delay is discussed in [7].

Now, we will be interested to determine how delay effects the stability of the positive equilibrium by taking τ as the bifurcation parameter. Before this we shall introduce the following lemma's.

Lemma1.[19] For the polynomial equation $z^3+pz^2+qz+r=0$,

(i). If $r<0$, the equation has at least one positive root;

(ii). If $r\geq 0$ and $\Delta=p^2-3q\leq 0$, then equation has no positive root;

(iii).If $r\geq 0$ and $\Delta=p^2-3q>0$, then equation has positive roots if $z_1^*=\frac{-p+\sqrt{\Delta}}{3}$ and $h(z_1^*)\leq 0$, where $h(z)=z^3+pz^2+qz+r$.

Lemma2.

(i). The positive equilibrium E_* of the system (1) is absolutely stable if and only if the equilibrium E_* of the corresponding ODE system is asymptotically stable and (2) has no purely imaginary roots for any $\tau>0$

(ii). The positive equilibrium E_* of the system (1) is conditionally stable if and only if all the roots of (2) have negative real parts at $\tau=0$ and there exist some positive values τ such that (2) has pair of purely imaginary roots $\pm i\omega$.

Theorem1.:Since $\sum\hat{i}\times\frac{\partial X}{\partial N}=\hat{i}(\frac{\gamma\beta y_*}{(\gamma+x_*)^2}-\rho y_*+\frac{\beta x_*}{(\gamma+x_*)})+\hat{k}(\alpha_1 x_*+bN_*-k_1b_1)\neq 0$, system (1) have one or more periodic solutions, where $X=(N,x,y)$.

Theorem2. The interior equilibrium E_* is conditionally stable if the condition (H_1) holds for the system (1).

Proof. Let $\lambda(\tau)=\xi(\tau)+i\omega(\tau)$ be the eigen value of the system at E_* and for finding the change of stability we assume that for some $\tau>0$, $\iota\omega(\omega>0)$ is a root of the characteristic equation (2), we then have

$$-\iota\omega^3-A\omega^2+B\iota\omega+C+(D+\iota E\omega)e^{-\iota\omega\tau}=0$$

separating the real and imaginary parts, we have

$$A\omega^2-C=D\cos\omega\tau+\omega E\sin\omega\tau \quad (4)$$

$$B\omega-\omega^3=-\omega E\cos\omega\tau+D\sin\omega\tau \quad (3)$$

eliminating ω from above equations and setting $\omega^2=z$, it can be obtained that

$$h(z)=z^3+pz^2+qz+r=0 \quad (6)$$

where $p=A^2-2B$, $q=B^2-2AC-E^2$, $r=C^2-D^2$.

by lemma 1 there exist at least one positive root $\omega=\omega_0$ of (2) satisfying (4) and (5) which implies (2) has a pair of purely imaginary roots of the form $\pm\iota\omega_0$.

Further (4) and (5) gives the corresponding $\tau_k>0$ such that (2) has a pair of purely imaginary roots,

$$\tau_k=\frac{1}{\omega_0}\arccos\frac{D(A\omega_0^{\,2}-C)+\omega_0^{\,2}E(\omega_0^{\,2}-B)}{D^2+\omega_0^{\,2}E}+\frac{2k\pi}{\omega_0} \quad (4)$$

Under the condition of (H_1), all the roots of (2) have negative real parts when $\tau=0$. Therefore by lemma 2 the positive equilibrium E_* of system (1) is conditionally stable. This completes the proof.

Next to obtain the transversality condition for the Hopf-bifurcation, we will find the value of $\frac{d\xi}{d\tau}$ at $\xi=0$.

Taking $\lambda(\tau)=\xi(\tau)+i\omega(\tau)$ in (2) and differentiating with respect to τ , we can obtain

$$\begin{cases} m_1\frac{d\xi}{d\tau}-n_1\frac{d\omega}{d\tau}=p_1 \\ n_1\frac{d\xi}{d\tau}+m_1\frac{d\omega}{d\tau}=p_2 \end{cases} \quad (8)$$

where

$$\begin{cases} m_1=-3\omega^2+B+Ecos(\omega\tau)-\tau Dcos(\omega\tau)-\tau\omega Esin(\omega\tau) \\ n_1=2\omega A-Esin(\omega\tau)+\tau Dsin(\omega\tau)-\tau\omega Ecos(\omega\tau) \\ p_1=\omega(Dsin(\omega\tau)-\omega Ecos(\omega\tau)) \\ p_2=\omega(Dcos(\omega\tau)+\omega Esin(\omega\tau)) \end{cases} \quad (9)$$

Solving (8), we get

$$\frac{d\xi}{d\tau}|_{\xi=0}=\frac{m_1p_1+n_1p_2}{m_1^2+n_1^2}$$

or it can be obtained that

$$\frac{d\xi}{d\tau}|_{\xi=0}=\frac{\omega^2}{m_1^2+n_1^2}\{\frac{dh(z)}{dz}\}_{z=\omega^2}\neq 0$$

thus root of (2) crosses the imaginary axis as τ continuously varies from a number less than τ_k to one greater than τ_k. Therefore, the transversality condition holds and the conditions for Hopf bifurcation [20] are then satisfied at $\tau = \tau_0$ which is the least positive value of τ_k given by (7). Based on the above analysis we have the following theorem.

Theorem 3. Suppose that E_* exist and the condition H_1 satisfied for the model system (1), then

(i) if $\tau \in [0, \tau_0]$,the positive equilibrium point E_* is Locally Asymptotically stable;

(ii). if $\tau > \tau_0$,the positive equilibrium point E_* is unstable;

(iii).system undergoes Hopf-bifurcation at $\tau = \tau_0$ around E_*.

C. Direction and Stability of the Hobf-Bifurcation

In this subsection we will determine the stability, direction and period of the periodic solutions bifurcating form E_* and following along the lines of Hassard et al. [20] we will derive the explicit formulae for determining the properties of the Hopf-bifurcation at the critical value of τ_k by using the normal form theory and the center manifold theorem.

Let $x_1 = N - N_*$, $x_2 = x - x_*$ and $x_3 = y - y_*$, rewriting the system (1) by Taylor series expansion about $E_*(N_*, x_*, y_*)$ we have the following system of equations:

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{100}x_1(t) + a_{010}x_2(t) + \sum_{i+j+k\geq 2} a_{ijk} x^i{}_1(t)x^j{}_2(t)x^k{}_3(t) = F^1(x_1,x_2,x_3)\\
\frac{dx_2}{dt} &= b_{100}x_1(t) + b_{010}x_2(t) + b_{001}x_3(t) + \sum_{i+j+k\geq 2} b_{ijk}x^i{}_1(t)x^j{}_2(t)x^k{}_3(t) = F^2(x_1,x_2,x_3)\\
\frac{dx_3}{dt} &= c_{100}x_2(t) + c_{001}x_2(t-\tau) + \sum_{i+j+k\geq 2} c_{ijk} x^i{}_2(t)x^j{}_3(t)x^k{}_2(t-\tau) = F^3(x_1,x_2,x_3)
\end{aligned} \tag{10}
$$

where

$$a_{ijk} = \frac{1}{!i!\,j!k}\frac{\partial^{i+j+k}F^1}{\partial N^i \partial x^j \partial y^k},\ b_{ijk} = \frac{1}{!i!\,j!k}\frac{\partial^{i+j+k}F^2}{\partial N^i \partial x^j \partial y^k},$$

$$c_{ijk} = \frac{1}{!i!\,j!k}\frac{\partial^{i+j+k}F^3}{\partial x^i \partial y^j \partial x^k(t-\tau)}$$

$$a_{100} = -(a + bx_*) \text{ and } a_{010} = (-bN_* + k_1 b_1),\ b_{100} = \alpha_1 x_*,$$

$$b_{010} = \alpha_1 N_* - b_1 - \frac{\beta\gamma y_*}{(\gamma + x_*)^2},\ b_{001} = -\frac{\beta x_*}{(\gamma + x_*)},\ c_{100} = \frac{\gamma\beta y_*}{(\gamma + x_*)^2},$$

$$c_{001} = -\rho y_*$$

and the coefficients of non-linear terms are given by,

$$a_{110} = -b,\ b_{110} = \alpha_1,\ b_{011} = -\frac{\gamma\beta}{(\gamma + x_*)^2},\ b_{020} = \frac{\gamma\beta y_*}{(\gamma + x_*)^3},$$

$$c_{110} = \frac{\gamma\beta_1}{(\gamma + x_*)^2},\ c_{011} = -\rho,\ c_{200} = -\frac{\gamma\beta_1 y_*}{(\gamma + x_*)^3}$$

let $\tau = \tau_k + \mu$, $\bar{u}_i(t) = u_i(\tau t)$ and dropping the bars for simplification of notations, system (10) becomes a functional differential equation in $C = C([-1,0], \Re^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t) \tag{11}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in \Re^3$ and $L_\mu : C \to \Re^3, f : \Re \times C \to \Re^3$ are given respectively, by

$$L_\mu(\varphi) = (\tau_k + \mu)[A_1\varphi(0) + A_2\varphi(-1)] \tag{12}$$

and

$$f(\mu,\varphi) = (\tau_k + \mu)\begin{bmatrix} a_{110}\varphi_1(0)\varphi_2(0) \\ b_{110}\varphi_1(0)\varphi_2(0) + \\ b_{011}\varphi_2(0)\varphi_3(0) + b_{020}\varphi_2^2(0) \\ c_{011}\varphi_2(-1)\varphi_3(0) + \\ c_{110}\varphi_2(0)\varphi_3(0) + c_{200}\varphi_2^2(0) \end{bmatrix} \tag{13}$$

$$A_1 = \begin{bmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & b_{001} \\ 0 & c_{100} & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{001} & 0 \end{bmatrix}$$

By Riesz representation theorem, there exists a function $\varsigma(\theta, \mu)$ of bounded variation for $\theta \in [-1,0]$ such that

$$L_\mu(\varphi) = \int_{-1}^{0} d\varsigma(\theta,\mu)\varphi(\theta) \quad for \quad \varphi \in C \tag{14}$$

In fact we can choose

$$\varsigma(\theta,\mu) = (\tau_k + \mu)[A_1\delta(\theta) - A_2\delta(\theta + 1)] \tag{15}$$

where δ denote the Dirac delta function. For $\varphi \in C[(-1,0), \Re^3]$, define

$$A(\mu)\varphi = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta} & \theta \in [-1,0) \\ \int_{-1}^{0} d\varsigma(s,\mu)\varphi(s) & \theta = 0 \end{cases}$$

and

$$R(\mu)(\varphi) = \begin{cases} 0 & \theta \in [-1,0) \\ f(\mu,\varphi) & \theta = 0 \end{cases}$$

then system (11) is equivalent to

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t \qquad (16)$$

where $u_t(\theta) = u(t+\theta)$ for $\theta \in [-1,0]$

for $\psi \in C^1([0,1], \Re^{*3})$, define

$$A^*\psi = \begin{cases} -\dfrac{d\psi(s)}{ds} & s \in (0,1] \\ \int_{-1}^{0}\psi(-t)d\varsigma(t,0) & s = 0 \end{cases}$$

and a bilinear inner product

$$<\psi(s), \varphi(\theta)> = \overline{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)d\varsigma(\theta)\varphi(\xi)d\xi \quad (17)$$

where $\varsigma(\theta) = \varsigma(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. From the results of last section, we know that $\pm \iota\omega_0\tau_k$ are eigen values of $A(0)$, it implies that they are also eigen values of A^*.So the corresponding eigen vectors computation follows the theorem:

Theorem 4. Let $q(\theta) = (1, q_1, q_2)^T e^{\iota\omega_0\tau_k}$ be the eigenvector of $A(0)$ corresponding to $\iota\omega_0\tau_k$ and $q^* = D(1, q_1^*, q_2^*)e^{\iota\omega_0\tau_k}$ be the eigenvector of A^* corresponding to $-\iota\omega_0\tau_k$.

Then $<q^*, q> = 1$ and $<q^*, \overline{q}> = 0$

where

$$q_1 = \frac{\iota\omega_0 - a_{100}}{a_{010}}, q_2 = \frac{(\iota\omega_0 - b_{010})q_1 - b_{100}}{b_{001}}, \; q_1^* = \frac{-\iota\omega_0 - a_{100}}{b_{100}},$$

$$q_2^* = \frac{(\iota\omega_0 + a_{100})b_{001}}{\iota\omega_0 b_{100}}, \text{ and}$$

$$D = \frac{1}{1 + \overline{q}_1 q_1^* + \overline{q}_2 q_2^* + \tau_k e^{\iota\omega_0\tau_k}(c_{001}\overline{q}_1 q_2^*)}.$$

Proof.: As $q(\theta)$ is the eigenvector of $A(0)$ corresponding to $\iota\omega_0\tau_k$, then we have

$$A(0)q(\theta) = \iota\omega_0\tau_k q(\theta)$$

from (12), (15) and by definition of $A(0)$,

$$\{(A_1 + A_2 e^{-\iota\omega_0\tau_k}) - \iota\omega_0 I\}q(0) = 0.$$

simplification gives,

$$q_1 = \frac{\iota\omega_0 - a_{100}}{a_{010}}, \quad q_2 = \frac{(\iota\omega_0 - b_{010})q_1 - b_{100}}{b_{001}}$$

since $q^*(0)$ is the eigen vector of A^* corresponding to eigen value $-\iota\omega_0\tau_k$ and by the definition of A^*,we can obtain

$$\{(A_1^T + A_2^T e^{\iota\omega_0\tau_k}) + \iota\omega_0 I\}(q^*(0))^T = 0$$

and further simplification leads to

$$q_1^* = \frac{-\iota\omega_0 - a_{100}}{b_{100}} \text{ and } q_2^* = \frac{(\iota\omega_0 + a_{100})b_{001}}{\iota\omega_0 b_{100}}.$$

now in order to assure $<q^*(s), q(\theta)> = 1$, we will find D and using (17),

$$\begin{aligned} &<q^*, q> \\ &= \overline{D}\{(1, \overline{q}_1^*, \overline{q}_2^*)(1, q_1, q_2)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta}(1, \overline{q}_1^*, \overline{q}_2^*). \\ &= \overline{D}\{(1 + q_1\overline{q}_1^* + q_2\overline{q}_2^*) - \int_{-1}^{0}(1, \overline{q}_1^*, \overline{q}_2^*)\theta e^{\iota\omega_0\tau_k\theta}d\varsigma(1, q_1, q_2)^T\} \end{aligned}$$

$$= \overline{D}\{1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* + \tau_k c_{001} q_1 \overline{q}_2^* e^{-\iota\omega_0\tau_k}\}$$

To obtain $<q^*(s), q(\theta)> = 1$, we can choose

$$D = \frac{1}{1 + \overline{q}_1 q_1^* + \overline{q}_2 q_2^* + \tau_k c_{001}\overline{q}_1 q_2^* e^{\iota\omega_0\tau_k}}$$

further, $<\psi, A\varphi> = <A^*\psi, \varphi>$, we then have

$$\begin{aligned} -\iota\omega_0\tau_k <q^*, \overline{q}> &= <q^*, A\overline{q}> \\ &= <A^* q^*, \overline{q}> = <-\iota\omega_0\tau_k q^*, \overline{q}> \\ &= \iota\omega_0\tau_k <q^*, \overline{q}>. \end{aligned}$$

Therefore $<q^*, \overline{q}> = 0$. This completes the proof.

Next we will compute the coordinates to describe the center manifold C_0 at $\mu = 0$.

Let u_t be the solution of (16) when $\mu = 0$.

Define

$$z(t) = <q^*, u_t>, \quad W(t,\theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\} \quad (18)$$

on the center manifold C_0, we have

$$W(t,\theta) = W(z(t), \overline{z}(t), \theta),$$

where

$$W(z(t),\overline{z}(t),\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}+........ \quad (19)$$

z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. We will consider only real solutions as W is real if u_t is real. so for solution $u_t \in C_0$ of (16),

$$\begin{aligned}\dot{z}(t) &= \iota\omega_0\tau_k z+\overline{q}^*(0)f(0,(W(z,\overline{z},\theta)+2Rezq(\theta)))\\ &= \iota\omega_0\tau_k z+\overline{q}^*(0)f_0(z,\overline{z})\end{aligned}$$

we rewrite this equation as

$$\dot{z}(t)=\iota\omega_0\tau_k z+g(z,\overline{z})$$

where

$$\begin{aligned}g(z,\overline{z}) &= \overline{q}^*(0)f_0(z,\overline{z})\\ &= g_{20}(\theta)\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z}^2}{2}+g_{21}\frac{z^2\overline{z}}{2}+........\end{aligned} \quad (20)$$

thus from (18) and (19), we have

$$\begin{aligned}u_t &= W(t,\theta)+2Re\{z(t)q(\theta)\}\\ &= W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}+(1,q_1,q_2)^T e^{\iota\omega_0\tau_k\theta}z\\ &\quad +(1,\overline{q}_1,\overline{q}_2)^T(\theta)e^{-\iota\omega_0\tau_k\theta}\overline{z}+....\end{aligned}$$

Using (13), (20) can be expressed as,

$$\begin{aligned}g(z,\overline{z}) &= \overline{q}^*(0)f_0(z,\overline{z})=\overline{q}^*(0)f(0,u_t)\\ &= \tau_k\overline{D}(1,\overline{q}_1^*,\overline{q}_2^*)\begin{bmatrix} a_{110}u_{1t}(0)u_{2t}(0)\\ b_{110}u_{1t}(0)u_{2t}(0)+\\ b_{011}u_{2t}(0)u_{3t}(0)+b_{020}u_{2t}^2(0)\\ c_{011}u_{2t}(-1)u_{3t}(0)+\\ c_{110}u_{2t}(0)u_{3t}(0)+c_{200}u_{2t}^2(0)\end{bmatrix}\\ &= \tau_k\overline{D}[\xi_{01}\{W_{20}^1(0)\frac{z^2}{2}+W_{11}^1(0)z\overline{z}+W_{02}^1(0)\frac{\overline{z}^2}{2}+z+\overline{z}\}\\ &\quad \{W_{20}^2(0)\frac{z^2}{2}+W_{11}^2(0)z\overline{z}+W_{02}^2(0)\frac{\overline{z}^2}{2}+q_1z+\overline{q}_1\overline{z}\}\\ &\quad +\xi_{02}\{W_{20}^2(0)\frac{z^2}{2}+W_{11}^2(0)z\overline{z}+W_{02}^2(0)\frac{\overline{z}^2}{2}+q_1z+\overline{q}_1\overline{z}\}\\ &\quad \{W_{20}^3(0)\frac{z^2}{2}+W_{11}^3(0)z\overline{z}+W_{02}^3(0)\frac{\overline{z}^2}{2}+q_2z+\overline{q}_2\overline{z}\}\\ &\quad +\xi_{03}\{W_{20}^2(0)\frac{z^2}{2}+W_{11}^2(0)z\overline{z}+W_{02}^2(0)\frac{\overline{z}^2}{2}+q_1z+\overline{q}_1\overline{z}\}^2\\ &\quad +\xi_{04}\{W_{20}^2(-1)\frac{z^2}{2}+W_{11}^2(-1)z\overline{z}+W_{02}^2(-1)\frac{\overline{z}^2}{2}+\\ &\quad q_1e^{-\iota\omega_0\tau_k}z+\overline{q}_1e^{\iota\omega_0\tau_k}\overline{z}\}\{W_{20}^3(0)\frac{z^2}{2}+W_{11}^3(0)z\overline{z}\\ &\quad +W_{02}^3(0)\frac{\overline{z}^2}{2}+q_2z+\overline{q}_2\overline{z}\}]\end{aligned} \quad (21)$$

where

$$\xi_{01}=a_{110}+b_{110}\overline{q}_1^*,\ \xi_{02}=b_{011}\overline{q}_1^*+c_{110}\overline{q}_2^*,$$

$$\xi_{03}=b_{020}\overline{q}_1^*+c_{200}\overline{q}_2^*,\ \xi_{04}=c_{011}\overline{q}_2^*$$

Comparison of coefficients with (19) gives,

$$\begin{aligned}g_{20} &= \overline{D}\tau_k\{2q_1\xi_{01}+2\xi_{02}q_1q_2+2\xi_{03}q_1^2+2q_1q_2\xi_{04}e^{-\iota\omega_0\tau_k}\}\\ g_{11} &= \overline{D}\tau_k\{2Req_1\xi_{01}+2Re(q_1\overline{q}_2)\xi_{02}+2q_1\overline{q}_1\xi_{03}+\\ &\quad 2Re(q_1\overline{q}_2e^{-\iota\omega_0\tau_k})\xi_{04}\}\\ g_{02} &= \overline{D}\tau_k\{2\overline{q}_1\xi_{01}+2\overline{q}_1\overline{q}_2\xi_{02}+2\overline{q}_1^2\xi_{03}+2\overline{q}_1\overline{q}_2\xi_{04}e^{\iota\omega_0\tau_k}\}\\ g_{21} &= \overline{D}\tau_k\{\xi_{01}(W_{20}^1(0)\overline{q}_1+2q_1W_{11}^1(0)+2W_{11}^2(0)+W_{20}^2(0))+\\ &\quad \xi_{02}(W_{20}^2(0)\overline{q}_2+2q_2W_{11}^2(0)+2q_1W_{11}^3(0)+\overline{q}_1W_{20}^3(0))+\xi_{03}(W_{20}^2(0)\overline{q}_1\\ &\quad +2q_1W_{11}^2(0)+2q_1W_{11}^2(0)+\overline{q}_1W_{20}^2(0))+\\ &\quad \xi_{04}(\overline{q}_2W_{20}^2(-1)+2q_2W_{11}^2(-1)+2q_1e^{-\iota\omega_0\tau_k}W_{11}^3(0)+\\ &\quad \overline{q}_1e^{\iota\omega_0\tau_k}W_{20}^3(0))\end{aligned} \quad (22)$$

since $W_{20}(\theta)$ and $W_{11}(\theta)$ are in g_{21}, we still to compute it. From (16) and (18), we have

$$\begin{aligned}\dot{W} &= \dot{u}(t)-\dot{z}q-\dot{\overline{z}}\overline{q}\\ &= \begin{cases} A(0)W-2Re\{\overline{q}^*(0)f_0q(\theta)\} & \theta\in[-1,0)\\ A(0)W-2Re\{\overline{q}^*(0)f_0q(\theta)\} & s=0\end{cases}\\ &\cong A(0)W+H(z,\overline{z},\theta)\end{aligned} \quad (23)$$

where

$$H(z,\overline{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\overline{z}+H_{02}(\theta)\frac{\overline{z}^2}{2}+........ \quad (24)$$

Substituting (24) above and comparing the coefficients, we have

$$\begin{aligned}(A(0)-2\iota\omega_0\tau_kI)W_{20}(\theta) &= -H_{20}(\theta)\\ A(0)W_{11}(0) &= -H_{11}(\theta)\end{aligned} \quad (25)$$

from (23) and for $\theta\in[-1,0)$

$$\begin{aligned}H(z,\overline{z},\theta) &= -\overline{q}^*(0)f_0q(\theta)-q^*(0)\overline{f_0}\overline{q}(\theta)\\ &= -g(z,\overline{z})q(\theta)-\overline{g}(z,\overline{z})\overline{q}(\theta)\end{aligned} \quad (26)$$

Comparing the coefficients with (24) we get

$$H_{20}(\theta)=-g_{20}q(\theta)-\overline{g}_{02}\overline{q}(\theta) \quad (27)$$

and

$$H_{11}(\theta)=-g_{11}q(\theta)-\overline{g}_{11}\overline{q}(\theta) \quad (28)$$

now from the definition of $A(0)$, (25) and (27), we obtain

$$\dot{W}_{20}(\theta) = 2\iota\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta)$$

Solving it and for $q(\theta) = (1, q_1, q_2)^T e^{\iota\omega_0\tau_k\theta}$, we have

$$W_{20}(\theta) = \frac{\iota g_{20}}{\omega_0\tau_k} q(0)e^{\iota\omega_0\tau_k\theta} + \frac{\iota\overline{g}_{02}}{3\omega_0\tau_k}\overline{q}(0)e^{-\iota\omega_0\tau_k\theta} + E_1 e^{2\iota\omega_0\tau_k\theta} \quad (29)$$

where $E_1 = (E_1^1, E_1^2, E_1^3)$ is a constant vector.

Similarly, (25) and (28) gives,

$$W_{11}(\theta) = -\frac{\iota g_{11}}{\omega_0\tau_k} q(0)e^{\iota\omega_0\tau_k\theta} + \frac{\iota\overline{g}_{11}}{3\omega_0\tau_k}\overline{q}(0)e^{-\iota\omega_0\tau_k\theta} + E_2 \quad (30)$$

finally, we will seek the values of E_1 and E_2.

from the definition of $A(0)$ and (25) , we have

$$\int_{-1}^{0} d\varsigma(\theta)W_{20}(\theta) = 2\iota\omega_0\tau_k W_{20}(0) - H_{20}(0) \quad (31)$$

and

$$\int_{-1}^{0} d\varsigma(\theta)W_{11}(\theta) = H_{11}(0) \quad (32)$$

where $\varsigma(\theta) = \varsigma(0,\theta)$.

For $\theta = 0$ and using (20), (23) and (26)

$$\begin{aligned} H(z,\overline{z},0) &= -2Re(\overline{q}^*(0)f_0 q(0)) + f_0 \\ &= -\overline{q}^*(0)f_0 q(0) - q^*(0)\overline{f_0}\overline{q}(0) + f_0 \end{aligned}$$

or

$$\begin{aligned} & H_{20}(\theta)\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \ldots\ldots. \\ &= -q(0)\{g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \ldots\ldots\} \\ & -\overline{q}(0)\{\overline{g}_{20}\frac{z^2}{2} + \overline{g}_{11}z\overline{z} + \overline{g}_{02}\frac{\overline{z}^2}{2} + \ldots.\} + f_0 \end{aligned} \quad (33)$$

Now from (13)

$$f_0 = \tau_k \begin{bmatrix} a_{110}u_{1t}(0)u_{2t}(0) \\ b_{110}u_{1t}(0)u_{2t}(0) + \\ b_{011}u_{2t}(0)u_{3t}(0) + b_{020}u_{2t}^2(0) \\ c_{011}u_{2t}(-1)u_{3t}(0) + \\ c_{110}u_{2t}(0)u_{3t}(0) + c_{200}u_{2t}^2(0) \end{bmatrix}$$

From (18), we have

$$\begin{aligned} u_t(\theta) &= W(t,\theta) + 2Rez(t)q(\theta) \\ &= W(t,\theta) + z(t)q(\theta) + \overline{z}(t)\overline{q}(\theta) \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + z(t)q(\theta) \\ &\quad + \overline{z}(t)\overline{q}(\theta)\ldots\ldots\ldots \end{aligned}$$

Thus we can obtain

$$f_0 = 2\tau_k \begin{bmatrix} \Gamma_{11} \\ \Gamma_{12} \\ \Gamma_{13} \end{bmatrix} \frac{z^2}{2} + \tau_k \begin{bmatrix} \Gamma_{21} \\ \Gamma_{22} \\ \Gamma_{23} \end{bmatrix} z\overline{z} + \ldots\ldots\ldots \quad (34)$$

where

$$\Gamma_{11} = a_{110}q_1,$$

$$\Gamma_{12} = b_{110}q_1 + b_{011}q_1q_2 + b_{020}q_1^2,$$

$$\Gamma_{13} = c_{011}q_1q_2e^{-\iota\omega_0\tau_k} + c_{110}q_1q_2 + c_{200}q_1^2,$$

$$\Gamma_{21} = 2Re\{q_1\}a_{110},$$

$$\Gamma_{22} = 2Re(q_1)b_{110} + 2Re(q_1\overline{q}_2)b_{011} + 2b_{020}q_1\overline{q}_1,$$

$$\Gamma_{23} = 2Re(q_1\overline{q}_2e^{-\iota\omega_0\tau_k})c_{011} + 2Re(q_1\overline{q}_2)c_{110} + 2c_{200}q_1\overline{q}_1$$

By (33) and (34), we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau_k \begin{bmatrix} \Gamma_{11} \\ \Gamma_{12} \\ \Gamma_{13} \end{bmatrix} \quad (35)$$

and

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + \tau_k \begin{bmatrix} \Gamma_{21} \\ \Gamma_{22} \\ \Gamma_{23} \end{bmatrix} \quad (36)$$

Substituting (29) and (35) into (31), we find,

$$\{2\iota\omega_0\tau_k I - \int_{-1}^{0} e^{2\iota\omega_0\tau_k\theta} d\varsigma(\theta)\}E_1 = 2\tau_k \begin{bmatrix} \Gamma_{11} \\ \Gamma_{12} \\ \Gamma_{13} \end{bmatrix}$$

which leads to

$$\begin{bmatrix} 2\iota\omega_0 - a_{100} & a_{010} & 0 \\ -b_{100} & 2\iota\omega_0 - b_{010} & -b_{001} \\ 0 & -c_{100} - c_{001}e^{-2\iota\omega_0\tau_k} & 2\iota\omega_0 \end{bmatrix} E_1 = 2\begin{bmatrix} \Gamma_{11} \\ \Gamma_{12} \\ \Gamma_{13} \end{bmatrix}$$

and we can easily obtain the following

$$E_1^{(1)} = \frac{1}{M}\{-2(4\omega_0^2 + 2\iota\omega_0 b_{010} + b_{001}(c_{100} + e^{-2\iota\omega_0\tau_k}))\Gamma_{11} - 4\iota\omega_0 d_2\Gamma_{12} - 2a_{010}b_{001}\Gamma_{13}\}$$

$$E_1^{(2)} = \frac{1}{M}\{(2\iota\omega_0 - a_{100})(4\iota\omega_0\Gamma_{12} + 2b_{001}\Gamma_{13}) + 4\iota\omega_0 b_{100}\Gamma_{11}\}$$

$$E_1^{(3)} = \frac{1}{M}\{(2\iota\omega_0 - a_{100})[2(2\iota\omega_0 - b_{010})\Gamma_{13} + 2(c_{100} + c_{001}e^{-2\iota\omega_0\tau_k})\Gamma_{12}] + 2a_{010}b_{100}\Gamma_{13} + 2b_{100}(c_{100} + c_{001}e^{-2\iota\omega_0\tau_k})\Gamma_{11}\}\ and$$

$$M = (2\iota\omega_o - a_{100})\{(-4\omega_0^2 - 2\iota\omega_0 b_{010}) - b_{001}c_{100} - b_{001}c_{001}e^{-2\iota\omega_0\tau_k}\} + 2\iota\omega_0 b_{100}a_{010}$$

Similarly substituting (30) and (36) into (32), we get

$$\begin{bmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & b_{001} \\ 0 & c_{100} + c_{001} & 0 \end{bmatrix} E_2 = \begin{bmatrix} \Gamma_{21} \\ \Gamma_{22} \\ \Gamma_{23} \end{bmatrix}$$

therefore we have

$$E_2^{(1)} = \frac{1}{N}(a_{010}b_{001}\Gamma_{23} - b_{001}(c_{100} + c_{001})\Gamma_{21})$$

$$E_2^{(2)} = -\frac{1}{N}(a_{100}b_{001}\Gamma_{23})$$

$$E_2^{(3)} = \frac{1}{N}(a_{100}b_{010} - b_{100}a_{010})\Gamma_{23} + (c_{100} + c_{001})(b_{100}\Gamma_{21} - a_{100}\Gamma_{22}))$$

and $N = -a_{100}b_{010}(c_{100} + c_{001})$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (29) and (30) and further g_{21} can be computed from (22).

Thus we can compute the following values:

$$c_1(0) = \frac{\iota}{2\omega_0\tau_k}\{g_{20}g_{11} - 2|g_{11}|^2 - \frac{(|g_{02}|)^2}{3}\} + \frac{g_{21}}{2}$$

$$\mu_2 = -\frac{Re\{c_1(0)\}}{Re\{\frac{d\lambda(\tau_k)}{d\tau}\}},$$

$$\beta_2 = 2Re\{c_1(0)\},$$

$$T_2 = -\frac{Im\{c_1(0)\} + \mu_2 Im\{\frac{d\lambda(\tau_k)}{d\tau}\}}{\omega_0\tau_k},\ k = 0,1,2,.....$$

where g_{ij} are given by (22).

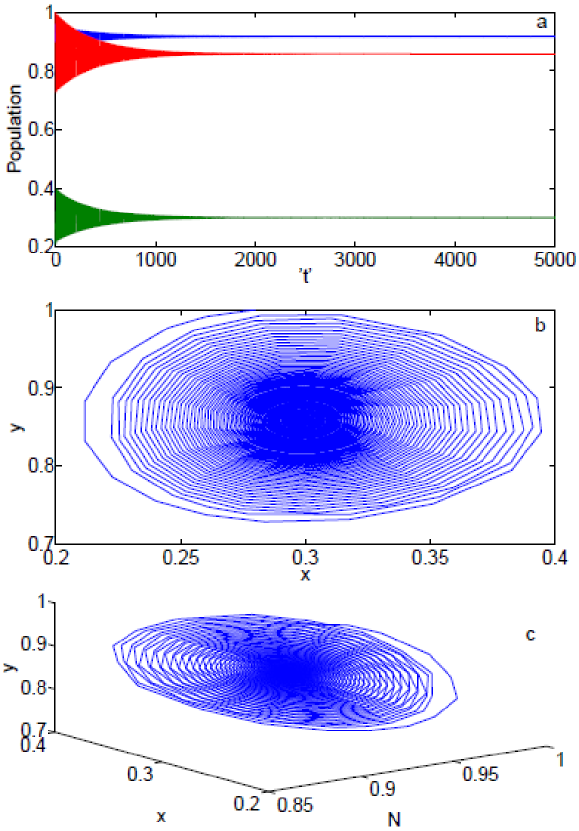

Fig. 1 (a), (b) and (c) convergence of trajectories towards E_*at $\tau = 6.3$

Theorem5. Due to Hassard et al. [20], we gives the properties of the Hopf bifurcation at the critical value of $\tau = \tau_0$ as follows:

(i) If $\mu_2 > 0(< 0)$; Hopf bifurcation is supercritical (subcritical).

(ii) If $\beta_2 < 0(> 0)$; the bifurcating periodic solutions are stable (unstable.)

(iii) If $T_2 > 0(< 0)$; period of the bifurcating periodic solution increases(decreases).

D. Numerical Simulation

In this subsection, we will provide a numerical example to dignify our theoretical findings. We have considered the given system by choosing a set of parameters $N_0 = 3$, $a = 2.8$, $b = 1.75$, $k_1 = 0.3$, $b_1 = 0.55$, $\alpha_1 = 1.50$, $\beta = 1.25$, $\beta_1 = 1.21$, $\gamma = 1$, $\alpha_2 = 0.25$, $\rho = 0.095$. i.e.

$$\begin{cases} \frac{dN}{dt} = 3 - 2.8N - 1.75Nx + 0.3(0.55)x \\ \frac{dx}{dt} = 1.50Nx - 0.55x - \frac{1.25xy}{(1+x)} \\ \frac{dy}{dt} = \frac{1.21xy}{(1+x)} - 0.25y - 0.095x(t-\tau)y \end{cases} \quad (37)$$

using the package of DDE 23 in Matlab, we have integrated the system (37) with initial data $N(t)=1.0$, $x(t)=0.28$, $y(t)=0.90$ and observe that the local asymptotic stability condition (H_1) in the absence of time delay is evidently satisfied. The system trajectories approaches to positive interior equilibrium at $E_*(0.9193,0.2953,0.8586)$ in the form of a stable focus as shown in Fig. 1. Further we find a purely imaginary root $\iota\omega_0$ of (2) with $\omega_0=0.4613$ and after some algebraic calculations one can find the minimum value of the delay parameter $'\tau'$ for the model system (1) for which the stability behavior changes and the this critical value is given by $\tau_0=6.714$ such that, the co-existence equilibrium E_* remain stable for $0\le\tau\le 6.714$ (see Figs. 1 (a)-(c)) and is unstable for $\tau\ge 6.714$ (see Figs. 2 (a)-(c)). Finally the stability determining quantities for Hopf-bifurcating periodic solutions are given by

$$c_1(0)=3.1737e+002+7.7626e+001i\,,$$

$\mu_2=-2.6060e+004$, $\beta_2=634.7315$ and $T_2=-39.8943$ Using Theorem 5, we can conclude that the Hopf bifurcation is subcritical in nature as well as the bifurcating periodic solutions are unstable and decreases as τ increases through its critical value τ_0.

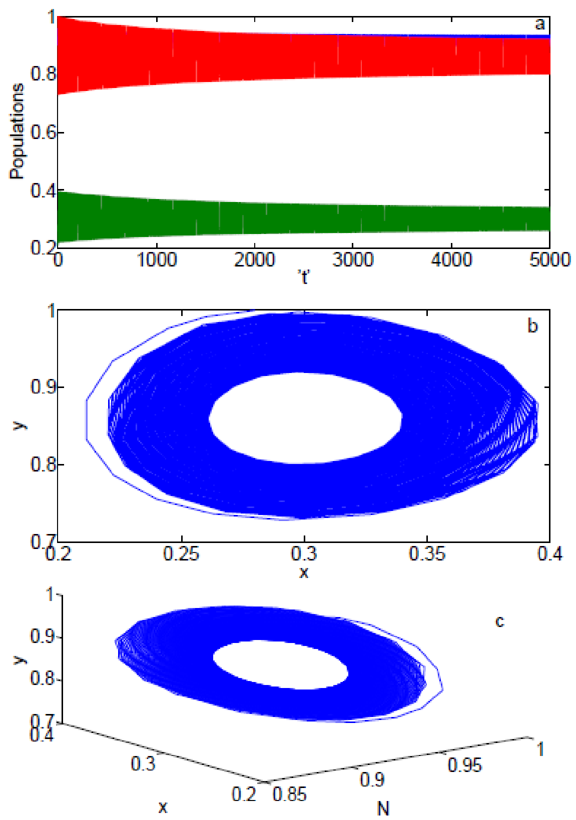

Fig. 2 (a), (b) and (c) Hopf-bifurcation at τ = 6.9.

II. Conclusion

In the present paper, a delayed plankton-nutrient interaction model system is analyzed with the assumption that the toxin liberation by the phytoplankton species follows a discrete time variation. Firstly the stability of the given system in the absence of delay is discussed and it is shown that interior equilibrium remained stable under certain conditions. Next we have considered the plankton-nutrient interaction in the presence of delay and it is observed that the system does not possess any periodic orbit for $\tau\in[0,6.714)$. But when time delay τ crosses a threshold value $\tau_0=6.714$ the system enters into a Hopf-bifurcation and a periodic orbit around equilibrium state E_* appearers.

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References

[1] G.T. Evans, J.S. Parslow, A model of annual plankton cycles, Biol. Oceanogr. 3 (1985) 327-427.
[2] S. Busenberg, K.S. Kishore, P. Austin, G. Wake, The dynamics of a model of a plankton-nutrient interaction, J. Math. Biol. 52 (5) (1990) 677-696.
[3] S. Ruan, Persistence and coexistence in zooplankton-phytoplankton-nutrient models with instantaneous nutrient recycling, J. Math. Biol. 31 (6) (1993) 633-654.
[4] S. Ruan, Oscillations in Plankton models with nutrient recycling, J. Theor. Biol. 208 (2001) 15-26.
[5] S. Chakarborty, S. Roy, J. Chattopadhyay, Nutrient-limiting toxin producing and the dynamics of two phytoplankton in culture media: A mathematical model, J. Ecological Modelling 213(2) (2008) 191-201.
[6] S. Pal, Samrat Chatterjee, J. Chattopadhyay, Role of toxin and nutrient for the occurrence and termination of plankton bloom-results drawn from field observations and a mathematical model, J. Biosystem 90 (2007) 87-100.
[7] S. Khare, J. Dhar, O. P. Misra, Role of toxin producing phytoplankton on a plankton ecosystem, Nonlinear Analysis: Hybrid Systems, 4 (2010) p.496-502.
[8] S. R. J. Jang, J. Baglama, J. Rick, Nutrient-phytoplankton-zooplankton models with a toxin, Math. Comput. Model. 43 (2006) 105-118.
[9] R. R. Sarkar, J. Chattopadhyay, Occurence of planktonic blooms under environmental fluctuations and its possible control mechanism-mathematical models and experimental observations, J. Theor. Biol.224 (2003) 501-516.
[10] R. R. Sarkar, S. Pal, J. Chattopadhyay, Role of two toxin-producing plankton and their effect on phytoplankton-zooplankton system-a mathematical study by experimental findings, J. Biosystem 80 (2005) 11-23.
[11] J. M. Cushing, Integrodifferential Equations and Delay Models in Population Dynamics. Springer-Verlag, Heidelberg (1977).
[12] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics. Kluwer Academic (1992).
[13] Y. Kuang, Delay differential equations with applications in population dynamics. Academic Press, New York, (1993).
[14] Shigui Ruan, The effect of delays on stability and persistence in plankton models, Nonlinear Anal. 24 (1995), 575-585.
[15] K. Das, S. Ray, Effect of delay on nutrient cycling in phytoplankton-zooplankton interactions in estuarine system. Ecol. model.215 (2008) 69-76.
[16] J. Chattopadhyay, R. R. Sarkar, A. Abdllaoui, A delay differential equation model on harmful algal blooms in the presence of toxic substances, IMA J. Math. Appl. Med. Biol. 19 (2002) 137-161.
[17] Tapan Saha, Malay Bandyopadhyay, Dynamical analysis of toxin producing Phytoplankton-Zooplankton interactions, Nonlinear Analysis: Real World Applications 10 (2009) 314-332.

[18] M. Rehim, M. Imran, Dynamical analysis of a delay model of phytoplankton-zooplankton interaction, Applied Mathematical Modelling 36 (2012) 638-647.
[19] Y. Song, M. Han, J. Wei, Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays. Physica D, 200 (2005) 185-204.
[20] B.D. Hassard, N.D. Kazarinoff, Y. H. Wan., Theory and Applications of Hopf bifurcation. Cambridge: Cambridge University Press, 1981.