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Generalized Stokes' Problems for an Incompressible Couple Stress Fluid

M.Devakar, T.K.V.Iyengar

Abstract—In this paper, we investigate the generalized Stokes' problems for an incompressible couple stress fluid. Analytical solution of the governing equations is obtained in Laplace transform domain for each problem. A standard numerical inversion technique is used to invert the Laplace transform of the velocity in each case. The effect of various material parameters on velocity is discussed and the results are presented through graphs. It is observed that, the results are in tune with the observation of V.K.Stokes in connection with the variation of velocity in the flow between two parallel plates when the top one is moving with constant velocity and the bottom one is at rest.

Keywords—Couple stress fluid, Generalized Stokes' problems, Laplace transform, Numerical inversion.

I. INTRODUCTION

THE study of non-Newtonian fluids has gained the attention of various researchers in fluid dynamics, due to its importance in science and industrial applications. Stokes' Couple stress fluid theory [1] is one among the category of non-Newtonian fluids which allows for polar effects such as sustenance of couple stresses and body couples in the fluid medium. It is a simple generalization of the classical theory of viscous fluids which adequately describes the flow behaviour of fluids containing a substructure such as lubricants with polymer additives, liquid crystals and animal blood. The concept of couple stresses arises due to the way in which the mechanical interactions in the fluid medium are modeled. The stress tensor here is no longer symmetric. An excellent introduction to this theory is available in the monograph "Theories of Fluids with Microstructure - An Introduction" written by Stokes [2] himself. The works of Naduvinamani et al. [3]- [5], Naduvinamani et al. [6], [7] and Jaw-Ren Lin and Chi-Ren Hung [8] reveal that, the couple stress fluid can be taken as a lubricant as it has larger load carrying capacity and lower coefficient of friction in comparison with the Newtonian fluid.

The present paper is devoted to the study of generalized Stokes' first and second problems with respect to an incompressible couple stress fluid. Let us assume that a fluid is filling the space between two infinite horizontal rigid parallel plates and initially let both the plates and fluid be at rest. Now (i) allow the lower plate to start impulsively with a constant velocity U in its own plane keeping the other plate at rest (ii) allow the lower plate to oscillate with velocity U cos(vt) in its own plane keeping the other one at rest. As we get Stokes' first and second problems [9] respectively when we allow the distance between the plates to tend to infinity, the problems dealing with the flows in these two cases are respectively referred to as generalized Stokes' first and second problems.

M. Devakar, Department of Mathematics, Visvesvaraya National Institute of Technology Nagpur - 440 010, INDIA e-mail: (m_devakar@yahoo.co.in).
T.K.V. Iyengar, Department of Mathematics, National Institute of Technology Warangal - 506 004, INDIA e-mail: (iyengar_nitw@yahoo.co.in).

Several researchers worked on generalized Stokes' problems, though not with this label, for Newtonian and diverse non-Newtonian fluids. The works of Gupta and Arora [10], Hassanien and Mansour [11], Hassanien [12], Hayat et al [13], Erdogan [14], Hayat et al [15], Erdogan and Imrak [16], [17] and Fetecau and Fetecau [18] display the recent interest on the generalized Stokes problems.

In the present work, we formulate the generalized Stokes' problem in a general format for an incompressible couple stress fluid and then consider the individual problems as special cases. Analytical solution of equations governing the flow is obtained in Laplace transform domain for each problem. We use a standard numerical inversion technique [19] to obtain the velocity component in each of the problems. The variation of velocity with respect to various parameters is studied and the results are delineated through graphs.

II. BASIC EQUATIONS AND FORMULATION OF THE PROBLEM

The basic equations that characterize the motion of an incompressible couple stress fluid [1], in the absence of pressure gradient, body forces and body couples are,

$$div(\bar{q}) = 0 \tag{1}$$

$$\rho\frac{d\bar{q}}{dt} = -\mu curl(curl\bar{q}) - \eta curl(curl(curl(curl\bar{q}))) \tag{2}$$

where ρ is the density of the fluid, $\bar{q}$ is the velocity vector, μ is the viscosity coefficient and η is the couple stress viscosity coefficient.

To solve the problem dealing with couple stress fluid flows, in addition to the usual assumption of no-slip condition, it is presumed that the couple stresses vanish at the boundary [2].

Consider two infinite horizontal parallel plates at a distance h apart. Let the region between them be filled with an incompressible couple stress fluid. Let initially the plates and the fluid be at rest. Choose a Cartesian system of coordinates with origin on the lower plate and y-axis perpendicular to the plates. The two plates are denoted by $y = 0$ and $y = h$. At time $t = 0^+$, we allow the lower plate to move with velocity $(U(t), 0, 0)$ along the x direction and keep the plate $y = h$ fixed. With the assumption that the flow is slow, the velocity at any point in the flow field is expected to be in the form $\bar{q} = (u(y, t), 0, 0)$.

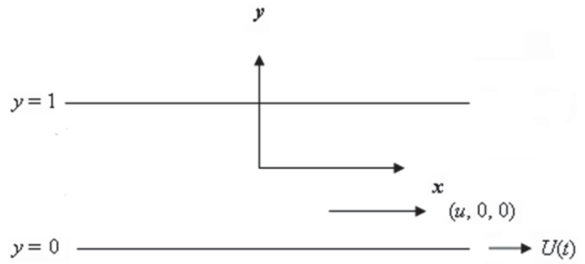

Fig. 1 Flow Configuration

The above choice of this velocity satisfies the equation of continuity (1) and the equation (2) takes the form

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} - \eta\frac{\partial^4 u}{\partial y^4} \tag{3}$$

Introducing non-dimensional variables

$$u^* = \frac{u}{U};\ y^* = \frac{y}{h};\ t^* = \frac{U}{h}t \tag{4}$$

the equation (3) reduces to

$$R\frac{\partial u^*}{\partial t^*} = \frac{\partial^2 u^*}{\partial {y^*}^2} - a^2\frac{\partial^4 u^*}{\partial {y^*}^4} \tag{5}$$

where,

$$R = \frac{\rho U h}{\mu};\ a^2 = \frac{l^2}{h^2};\ l^2 = \frac{\eta}{\mu} \tag{6}$$

Dropping *'s in equation (5), we get the equation for $u(y,t)$ in non dimensional form as

$$R\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - a^2\frac{\partial^4 u}{\partial y^4} \tag{7}$$

We solve this equation subject to respective boundary conditions for each one of the generalizations of Stokes' problems.

III. Details of Problems and their Solutions

A. Generalized Stokes' First Problem

Let the plate $y = 0$ start moving with velocity $(U, 0, 0)$ at time $t = 0^+$ where U is a constant. This implies that $U(t) = U$ for $t > 0$ and in non dimensional form this becomes $U(t) = 1$ for $t > 0$. Thus we notice that the non dimensional velocity $u(y,t)$ has to satisfy the equation (7) subject to the following conditions:

Initial condition:

$$u(y,t) = 0 \text{ for all } y \tag{8}$$

Boundary conditions: For $t > 0$
(Usual no-slip conditions on the boundary)

$$u(0,t) = 1;\ u(1,t) = 0 \tag{9}$$

(vanishing of couple stresses on the boundary)

$$\frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = 0 \text{ and } y = 1 \tag{10}$$

Taking Laplace transform of (7), (9) and (10) and using (8), we obtain

$$(D^4 - \frac{1}{a^2}D^2 + \frac{Rs}{a^2})\bar{u} = 0 \tag{11}$$

where

$$D = \frac{d}{dy};\ \bar{u}(y,s) = \int_0^\infty e^{-st}u\,dt \tag{12}$$

subject to the conditions:
(Usual no-slip conditions on the boundary)

$$\bar{u}(0,s) = \frac{1}{s};\ \bar{u}(1,s) = 0 \tag{13}$$

(vanishing of couple stresses on the boundary)

$$\frac{\partial^2 \bar{u}}{\partial y^2} = 0 \text{ at } y = 0 \text{ and } y = 1 \tag{14}$$

The equation (11) can be written as

$$(D^2 - \alpha^2)(D^2 - \beta^2)\bar{u} = 0 \tag{15}$$

where

$$\alpha^2 + \beta^2 = \frac{1}{a^2};\ \alpha^2\beta^2 = \frac{Rs}{a^2} \tag{16}$$

The solution of equation (15) using conditions (13) and (14) is given by

$$\bar{u}(y,s) = \frac{1}{s(\alpha^2 - \beta^2)}\left[\alpha^2\frac{\sinh(y-1)\beta}{\sinh\beta} - \beta^2\frac{\sinh(y-1)\alpha}{\sinh\alpha}\right] \tag{17}$$

We numerically invert this function $\bar{u}(y,s)$ using the numerical procedure of Honig and Hirdes [19] for various values of time t, distance y and material parameters.

B. Generalized Stokes' Second Problem

Let the plate $y = 0$ start moving with velocity $(U\cos(vt), 0, 0)$ at time $t = 0^+$ where U is a constant. This implies that $U(t) = U\cos(vt)$ for $t > 0$ and in non dimensional form this becomes $U(t) = \cos(vt)$ for $t > 0$. Thus we notice that the non dimensional velocity $u(y,t)$ has to satisfy the equation (7) subject to the following conditions:

Initial condition:

$$u(y,t) = 0 \text{ for all } y \tag{18}$$

Boundary conditions: For $t > 0$
(Usual no-slip conditions on the boundary)

$$u(0,t) = \cos(vt);\ u(1,t) = 0 \tag{19}$$

(vanishing of couple stresses on the boundary)

$$\frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = 0 \text{ and } y = 1 \tag{20}$$

Taking Laplace transform of (7), (19) and (20) and using (18), it is noted that $\bar{u}(y,s)$ satisfies

$$(D^2 - \alpha^2)(D^2 - \beta^2)\bar{u} = 0 \tag{21}$$

where, α and β are given by (16)
with the conditions:
(Usual no-slip conditions on the boundary)

$$\bar{u}(0,s) = \frac{s}{s^2 + v^2};\ \bar{u}(1,s) = 0 \tag{22}$$

(vanishing of couple stresses on the boundary)

$$\frac{\partial^2 \bar{u}}{\partial y^2} = 0 \text{ at } y = 0 \text{ and } y = 1 \tag{23}$$

The solution $\bar{u}(y, s)$ for this case is given by

$$\bar{u}(y, s) = \frac{s}{(s^2 + v^2)(\alpha^2 - \beta^2)} \left[\alpha^2 \frac{\sinh(y-1)\beta}{\sinh \beta} - \beta^2 \frac{\sinh(y-1)\alpha}{\sinh \alpha} \right] \tag{24}$$

We numerically invert this function $\bar{u}(y, s)$ using the numerical procedure of [19] for various values of time t, distance y and material parameters.

IV. Numerical Inversion of Laplace Transform

In order to invert $\bar{u}(y, s)$, we adopt a numerical inversion technique due to Honig and Hirdes [19]. Using this method, the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t) = \frac{e^{bt}}{t_1} \left[\frac{1}{2} \bar{f}(b) + Re \left(\sum_{k=1}^{N} \bar{f}(b + \frac{ik\pi}{t_1}) exp(\frac{ik\pi t}{t_1}) \right) \right] \tag{25}$$

for $0 < t_1 \leq 2t$, where N is sufficiently large integer chosen such that,

$$e^{bt} Re \left(\bar{f}(b + \frac{iN\pi}{t_1}) exp(\frac{iN\pi t}{t_1}) \right) < \epsilon \tag{26}$$

where ϵ is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter b is a positive free parameter that must be greater than the real parts of all the singularities of $\bar{f}(s)$. The optimal choice of b was obtained by the criteria described in [19].

V. Discussion of Results

The function $u(y, t)$ is evaluated numerically for different values of y and t varying the couple stress parameter a and Reynolds number R in each problem.

A. First Problem

The variation of the velocity u is presented through Fig. 2 to Fig. 4. When a and R are fixed at any y, as time increases the velocity increases as seen in Fig. 2. As y starts form 0 to 1, as expected the velocity starts from 1 and decreases to 0. The same trend is seen at a fixed time t for a fixed R for any value of a.

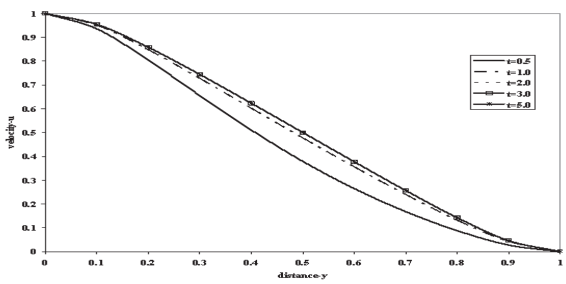

Fig. 2 Variation with time when $a = 0.1$ and $R = 2$

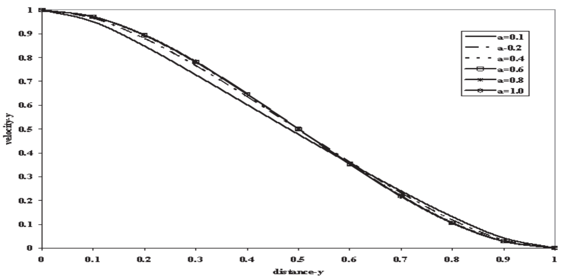

Fig. 3 Variation with a when $t = 1$ and $R = 2$

As the couple stress parameter a increases for a fixed time t, the velocity shows an increasing trend initially near the moving plate and a decreasing trend subsequently near the stationary plate (see Fig. 3). The Reynolds number R seems to have not much influence on the velocity (see Fig. 4).

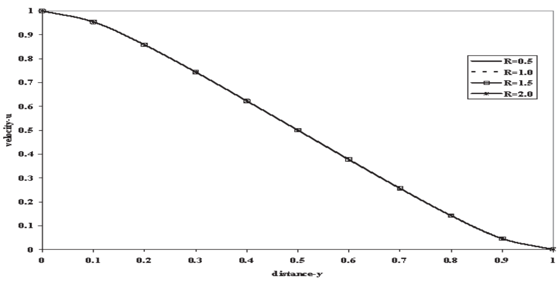

Fig. 4 Variation with R when $a = 0.1$ and $t = 1$

B. Second Problem

The oscillatory behavior of $u(y, t)$ is seen in Fig. 5 for a fixed a and fixed R. Fig. 6 indicates that, for a fixed time and fixed Reynolds number R, as the couple stress parameter a increases, there is a decrease in velocity at each y. Unlike the case of Generalized Stokes' first problem, here the velocity is influenced by variation in R and it can be observed from Fig. 7 that an increase in R at any y shows an increasing trend in velocity.

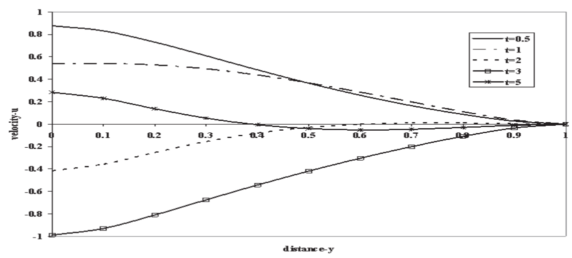

Fig. 5 Variation with time when $a = 0.1$ and $R = 2$

VI. Conclusion

Generalized Stokes' first and second problems for an incompressible couple stress fluid are studied in this paper.

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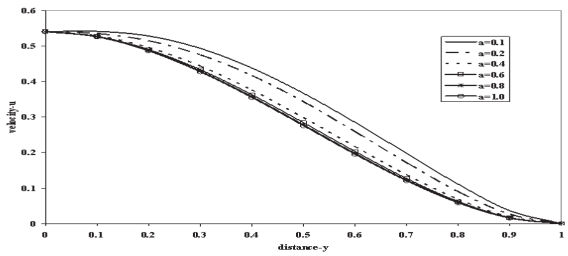

Fig. 6: Variation with a when $t = 1$ and $R = 2$

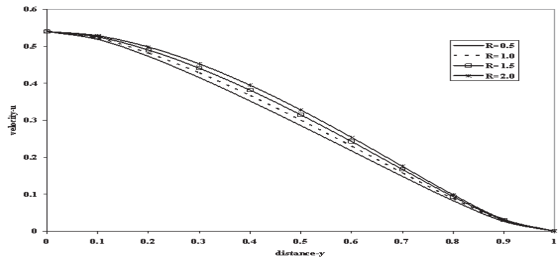

Fig. 7: Variation with R when $a = 0.1$ and $t = 1$

The solution of each of the problems is obtained using the condition that the couple stresses vanish on the boundary.

In first problem, nearer to the moving plate as the couple stress parameter a increases, there is an increase in the velocity for a fixed time t but as we move nearer to the stationary plate, after a certain critical value of y, the velocity decreases. It is in tune with the observation of V.K.Stokes in connection with the variation of velocity in the flow between two parallel plates when the top one is moving with constant velocity and the bottom one is at rest (see page 42, Fig. (3.5.1) in [2]).

In generalized Stokes' second problem, the velocity is having behavior similar to that obtained in Stokes' second problem considered in Devakar and Iyengar [20].

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