On the Approximate Solution of Continuous Coefficients for Solving Third Order Ordinary Differential Equations

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Abstract**—**This paper derived four newly schemes which are combined in order to form an accurate and efficient block method for parallel or sequential solution of third order ordinary differential equations of the form $y''' = f(x, y, y', y'')$, $y(\alpha) = y_0$, $y'(\alpha) = \beta$, $y''(\alpha) = \eta$ with associated initial or boundary conditions. The implementation strategies of the derived method have shown that the block method is found to be consistent, zero stable and hence convergent. The derived schemes were tested on stiff and non – stiff ordinary differential equations, and the numerical results obtained compared favorably with the exact solution.

Keywords**—**Block Method, Hybrid, Linear Multistep, Self starting, Third Order Ordinary Differential Equations.

I. Introduction

LET us consider the numerical solution of the third order ordinary differential equation of the form

$$y''' = f(x, y, y', y''),\ y(\alpha) = y_0,\quad y'(\alpha) = \beta\ ,\ y''(\alpha) = \mu \qquad (1)$$

with associated initial or boundary conditions. In the field of science and engineering and some other area of applications, we come across physical and natural phenomena which when represented by mathematical models, happen to be differential equations, for example, equation of motion, simple harmonic motion and deflection of a beam are represented by differential equations. In real – life situations, the differential equations that models the problem is too complicated to solve exactly and one of two approaches is taken to approximate the solution by reducing it to systems of first order which lead to computational cost or by direct approximation. This paper developed an alternative approach for direct solution of type (1) based on linear multistep collocation method. Four point directly implicit block hybrid method of order four using constant step size strategies was adopted in this approach.

Solutions to initial value problem of type (1) according to Fatunla [1], [2] are often highly oscillatory in nature and such system often occurs in mechanical systems without dissipation, satellite tracking, and celestial mechanics.

Lambert [3] and several authors such as Onumanyi *et al* [4], Brown [5], and Awoyemi [6], have written on conventional linear multistep method:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j}\ , k \geq 2 \qquad (2)$$

or compactly in the form

$$\rho(E) y_n = h^3 \delta(E) f_n \qquad (3)$$

where E is the shift operator specified by $E^j y_n = y_{n+j}$ while ρ and δ are characteristics polynomials and are given as

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j,\ \delta(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \qquad (4)$$

y_n is the numerical approximation to the theoretical solution $y(x)$ and $f_n = f(x_n, y_n)$.

In the present consideration, our motivations for the study of this approach is a further advancement in efficiency, that is obtaining the most accuracy per unit of computational effort, that can be secured with the method proposed in this paper.

A. Definition: Consistent, Lambert [3]

The linear multistep method (2) is said to be consistent if it has order $p \geq 1$, that is if

$$\sum_{j=0}^{k} \alpha_j = 0 \text{ and } \sum_{j=0}^{k} j\, \alpha_j - \sum_{j=0}^{k} \beta_j = 0 \qquad (5)$$

Introducing the first and second characteristics polynomials (4), we have from (5) LMM type (2) is consistent if

$$\rho(1) = 0 \quad \rho^1(1) = \delta(1)$$

B. Definition: Zero Stability, Lambert [3]

A linear multistep method type (2) is zero stable provided the roots $\xi_j, j = 0(1)k$ of first characteristics polynomial $\rho(\xi)$ specified as $\rho(\xi) = \det \sum_{i=0}^{k} A(i)\xi^{(k-i)} = 0$ satisfies $|\xi_j| \leq 1$ and for those roots with $|\xi_j| = 1$ the multiciplicity must not exceed two. The principal root of $\rho(\xi)$ is denoted by $\xi_1 = \xi_2 = 1$.

C. Definition: Convergence, Lambert [3]

The necessary and sufficient conditions for the linear multistep method type (2) is said to be convergent if it is consistent and zero stable.

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D. Definition: Order and Error Constant, Lambert [3]

The linear multistep method type (2) is said to be of order p if $c_0 = c_1 = c_2 \ldots c_P = 0$ but $c_{p+1} \neq 0$ and c_{p+1} is called the error constant, where

$$c_0 = \sum_{j=0}^{k} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k$$

$$c_1 = \sum_{j=0}^{k} j\,\alpha_j = (\alpha_1 + 2\,\alpha_2 + 3\,\alpha_3 + \ldots + k\,\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$c_2 = \sum_{j=0}^{k} \frac{1}{2!} j^2 \alpha_j - \sum_{j=0}^{k} \beta_j = \left\{ \begin{array}{l} \frac{1}{2!}(\alpha_1 + 2^2\,\alpha_2 + 3^3\,\alpha_3 + \ldots + k^2\,\alpha_k) \\ -(\beta_1 + 2\beta_2 + 3\beta_3 + \ldots + k\beta_k) \end{array} \right\}$$

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$$c_q = \sum_{j=1}^{k} \left\{ \frac{1}{q!} j^q \alpha_j - \frac{1}{(q-2)!} j^{q-2} \beta_j \right\} = \left\{ \begin{array}{c} \frac{1}{q!}(\alpha_1 + 2^q\,\alpha_2 + 3^q\,\alpha_3 + \ldots + k^q\,\alpha_k) \\ -\frac{1}{(q-1)!}(\beta_1 + 2^{(q-1)}\beta_2 + 3^{(q-1)}\beta_3 + \cdots + k^{(q-1)}\beta_k) \end{array} \right\} \quad (6)$$

E. Theorem: Lambert, [3]

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $\{(x, y) : a \le x \le b, -\infty < y < \infty\}$ where *a and b* finite, and let there exist a constant L such that for every x, y, y^* such that (x, y) *and* (x, y^*) are both in D:

$$|f(x, y) - f(x, y^*)| \le L|y - y^*| \quad (7)$$

Then if η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1), where $y(x)$ is continuous and differentiable for all (x, y) *in D*. The inequality (7) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

II. Derivation of the Proposed Method

We proposed an approximate solution to (1) in the form

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{t+m-1} \beta_j(x) f_{n+j}, \quad (8)$$

$$f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j}, y''_{n+j}) \text{ and } x \in (x_n, x_{n+k}) \quad (9)$$

$$V = y_n, y_{n+1}, y_{n+2}, f_n, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f_{n+3} \quad (10)$$

With m = 4, t = 3 and p = m+t-1 where the $a_j, j = 0,1,(m+t-1)$ are the parameters to be determined, t and m are points of interpolation and collocation respectively; where P, is the degree of the polynomial interpolant of our choice. Consider a power series solution of a single variable of the form

$$P(x) = \sum_{j=0}^{m+t-1} a_j x^j \quad (11)$$

$$P'(x) = \sum_{j=0}^{m+t-1} j a_j x^{j-1} \quad (12)$$

$$P''(x) = \sum_{j=0}^{m+t-1} j(j-1) a_j x^{j-2} \quad (13)$$

$$P'''(x) = \sum_{j=0}^{m+t-1} j(j-1)(j-2) a_j x^{j-3} = f(x, y, y', y'') \quad (14)$$

Specifically, we collocate (14) at $x = x_{n+j}, j = 0(1)k$ and interpolate (8) at $x = x_{n+j}, j = 0(1)k-1$ using the method described above. Also for a given mesh $\{x_n : x_n = a + nh, n = 0, 1, (N)\}$; where $h = x_{n+1} - x_n$, $h = (b-a)/n$ is a step size.

The matrix equation form of the proposed method is expressed as:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{3}{2}} & 60x_{n+\frac{3}{2}}^2 & 120x_{n+\frac{3}{2}}^3 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \\ \beta_{\frac{3}{2}} \\ \beta_3 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+3} \end{bmatrix} \quad (15)$$

where

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{3}{2}} & 60x_{n+\frac{3}{2}}^2 & 120x_{n+\frac{3}{2}}^3 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 \end{bmatrix} \quad (16)$$

Matrix D in (16), which when solved either by inversion techniques or Gaussian elimination method we obtain the columns of D^{-1} which form the matrix C. The elements of C are used to generate the continuous coefficients of the method as:

$$\alpha_0(x) = \frac{1}{2h^2}\{2h^2 - 3h(x - x_n) - (x - x_n)^2\}$$

$$\alpha_1(x) = \frac{1}{h^2}\{2h(x - x_n) - (x - x_n)^2\}$$

$$\alpha_2(x) = \frac{1}{2h^2}\{-h(x - x_n) - (x - x_n)^2\}$$

$$\beta_0(x) = \frac{1}{540h^3}\{-8h^5(x - x_n) - 121h^4(x - x_n)^2 + 90h^3(x - x_n)^3 - 45h^2(x - x_n)^4 - 5h(x - x_n)^5\}$$

$$\beta_1(x) = \frac{1}{360h^3}\{39h^5(x - x_n) - 76h^4(x - x_n)^2 - 9h(x - x_n)^5 + (x - x_n)^6\}$$

$$\beta_{\frac{3}{2}}(x) = \frac{1}{225h^3}\{8h^5(x - x_n) - 16h^4(x - x_n)^2 + 15h^2(x - x_n)^4 - 8h(x - x_n)^5 + (x - x_n)^6\}$$

$$\beta_3(x) = -\frac{1}{1080h^3}\{-5h^5(x - x_n) + 7h^4(x - x_n)^2 - 8h^2(x - x_n)^4 + 5h(x - x_n)^5 - (x - x_n)^6\}$$

$$(17)$$

The values of continuous coefficients (17) are substituted into (8) yields, after some algebraic manipulation and by setting $\psi = (x - x_n)$, the new continuous scheme of Block Hybrid method

$$\bar{y}(x) = \frac{1}{2h^2}\{2h^2 - 3h\psi - \psi^2\}.y_n + \frac{1}{h^2}\{2h\psi - \psi^2\}.y_{n+1} + \frac{1}{2h^2}\{-h\psi - \psi^2\}.y_{n+2} + \frac{1}{540h^3}\{-8h^5\psi - 121h^4\psi^2 + 90h^3\psi^3 - 45h^2\psi^4 - 5h\psi^5\}.f_n + \frac{1}{360h^3}\{39h^5\psi - 76h^4\psi^2 - 9h\psi^5 + \psi^6\}.f_{n+1} + \frac{1}{225h^3}\{8h^5\psi - 16h^4\psi^2 + 15h^2\psi^4 - 8h\psi^5 + \psi^6\}.f_{n+\frac{3}{2}} + -\frac{1}{1080h^3}\{-5h^5\psi + 7h^4\psi^2 - 8h^2\psi^4 + 5h\psi^5 - \psi^6\}.f_{n+3} \quad (18)$$

Evaluating (18) $x = x_{n+3}$ and $x = x_{n+\frac{3}{2}}$ and its first and second derivatives both at $x = x_n$ yield the following four discrete hybrid schemes, which are used as a block integrator;

$$\begin{aligned} y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n &= \frac{h^3}{18}\left\{f_n + 16f_{n+\frac{3}{2}} + f_{n+3}\right\} \\ y_{n+\frac{3}{2}} - \frac{3}{8}y_{n+2} - \frac{3}{4}y_{n+1} + \frac{1}{8}y_n &= \frac{h^3}{4608}\left\{\begin{matrix} -10f_n - 189f_{n+1} \\ -88f_{n+\frac{3}{2}} - f_{n+3} \end{matrix}\right\} \\ hz_o + \frac{1}{2}y_{n+2} - 2y_{n+1} + \frac{3}{2}y_n &= \frac{h^3}{1080}\left\{\begin{matrix} 68f_n + 351f_{n+1} \\ -64f_{n+\frac{3}{2}} + 5f_{n+3} \end{matrix}\right\} \\ hz_0' - y_{n+2} + 2y_{n+1} - y_n &= \frac{h^3}{1080}\left\{\begin{matrix} -356f_n - 963f_{n+1} \\ +256f_{n+\frac{3}{2}} - 17f_{n+3} \end{matrix}\right\} \end{aligned} \quad (19)$$

Equations (19) constitute the member of a zero stable block integrators of order $(4,4,4,4)^T$ with

$$c_{p+3} = \left(-\frac{1}{60}, \frac{37}{122880}, -\frac{33}{7}, -\frac{33}{2}\right).$$

The application of the block integrators with n = 0 gives the accurate values as shown in Tables I and II of fourth section of this paper. To start the IVP integration on the sub interval [X_0, X_3], we use equation (19) when n = 0 to produce simultaneously values for y_1, y_2, y_3 and $y_{\frac{3}{2}}$ without recourse to any predictor – corrector method to provide y_1 and y_2 in the main method. Hence, this is an improvement over other cited works.

III. Stability Analysis

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus, several definitions, which call for the method to posses some "adequate" region of absolute stability, can be found in several literatures. See Lambert [3], Fatunla [1], [2] etc.

Following Fatula [1], [2]; the four discrete schemes proposed in this report in equation (19) are put in matrix equation form and for easy analysis the result was normalized to obtain;

$$\text{i.e. } A^{(0)}Y_m = \sum_{i=1}^{k} A^i\, Y_{m-i} + h\sum_{i=0}^{k} B^{(i)}\, F_{m-i} \quad (20)$$

where h is a fixed mesh size within a block, $A^i, B^i, i = 0(1)k$ are r by r matrix coefficients, $A^{(0)}$ is r by r identity matrix, Y_m, Y_{m-i} and F_{m-i} are vectors of numerical estimates described by

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ . \\ . \\ . \\ y_{n+r} \end{bmatrix}, Y_{m-i} = \begin{bmatrix} y_{n-r} \\ . \\ . \\ . \\ y_{n+1} \\ y_n \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ . \\ . \\ . \\ f_{n+r} \end{bmatrix}, F_{m-i} = \begin{bmatrix} f_{n-r} \\ . \\ . \\ . \\ f_{n+1} \\ f_n \end{bmatrix}$$

For n = mr and for some integer m ≥ 0. This give rise to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_n \end{bmatrix} + h^3\left\{\begin{bmatrix} \frac{121}{1000} & -\frac{59}{1000} & 0 & \frac{167}{5000} \\ \frac{19}{40} & -\frac{197}{1000} & 0 & \frac{11}{1000} \\ \frac{1133}{1000} & -\frac{89}{250} & 0 & \frac{11}{500} \\ \frac{1519}{500} & 0 & 0 & \frac{11}{100} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{51}{500} \\ 0 & 0 & 0 & \frac{631}{250} \\ 0 & 0 & 0 & \frac{533}{1000} \\ 0 & 0 & 0 & \frac{27}{20} \end{bmatrix}\begin{bmatrix} f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_n \end{bmatrix}\right\} \quad (21)$$

with $y_0 = \binom{y_0}{hz_0}$ usually giving along the initial value problem. The first characteristics polynomial of the proposed block hybrid method is given by

$$\rho(\lambda) = \det\,[\lambda I - A_1^{(1)}] \quad (22)$$

$$\rho(\lambda) = \det\begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 1 & \lambda - 1 \end{bmatrix} \quad (23)$$

Solving the determinant of (23), yields $\rho(\lambda) = \lambda^3(\lambda - 1)$, which implies, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ or $\lambda_4 = 1$

By definition of zero stable and (23), the hybrid method is zero stable and is also consistent as its order P = $(4,4,4,4)^T > 1$, thus, it is convergent following Henrici [7] and Fatunla [2].

IV. Implementation of the Method

This section deals with numerical experiments by considering the derived discrete schemes in block form for solution of differential equations of third order initial value problems. The idea is to enable us see how the proposed methods performs when compared with exact solutions. The results are summarized in Tables I & II.

A. Numerical Experiment

I. Consider the I.V.P $y''' - y'' + y' - y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1, h = 0.01,$ whose exact solution is $y(x) = \cos x$

TABLE I
RESULTS FOR THE PROPOSED METHOD OF PROBLEM I

x	Exact Solution	Approximate Solution	Absolute Error of Proposed Method
0.01	0.9999500004	0.9999502003	1.9990E-07
0.02	0.9998000067	0.9998002023	1.9560E-07
0.03	0.9995500337	0.9995501702	1.3651E-07
0.04	0.9992001067	0.9992003588	2.5210E-07
0.05	0.9987502604	0.9987515643	1.3039E-06
0.06	0.9982005399	0.9982035679	3.0280E-06
0.07	0.9975510003	0.9975543456	3.3453E-06
0.08	0.9968017063	0.9968004658	1.2405E-06
0.09	0.9959527330	0.9959540620	1.3290E-06
0.10	0.9950041653	0.9950213456	1.7180E-05

B. Numerical Experiment

II. Consider the I.V.P $y''' + 5y'' + 7y' + 3y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$, $h = 0.1$, whose exact solution is $y(x) = e^{-x} + xe^{-x}$

TABLE II
RESULTS FOR THE PROPOSED METHOD OF PROBLEM II

x	Exact Solution	Approximate Soln.	Absolute Error of Proposed Method
0.1	0.9953211598	0.9953212241	6.4300E-08
0.2	0.9824769037	0.9824768765	2.7200E-08
0.3	0.9630636869	0.9630636564	3.0500E-08
0.4	0.9384480644	0.9384481542	8.9800E-08
0.5	0.9097959895	0.9097955469	4.4260E-07
0.6	0.8780986178	0.8780978452	7.7260E-07
0.7	0.8441950165	0.8441930642	1.9523E-06
0.8	0.8087921354	0.8087911080	1.0274E-06
0.9	0.7724823534	0.7724810025	1.3509E-06
1.0	0.7357588824	0.7357454120	1.3470E-05

V. CONCLUSION

In this paper, a new block method with uniform order was developed. The resultant numerical integrator posses the following desirable properties:

- Being self – starting as such it eliminate the use of predictor – corrector method
- Convergent schemes
- Facility to generate solutions at 4 points simultaneously
- Produce solution over sub intervals that do not overlaps.
- Zero stability

In addition, the results of new schemes are more accurate when compared with the theoretical solution, see Tables I and II respectively. Hence, this work is an improvement in terms of efficiency and stability analysis.

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