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Fixed Points of Contractive-Like Operators by a Faster Iterative Process

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***Abstract*—**In this paper, we prove a strong convergence result using a recently introduced iterative process with contractive-like operators. This improves andgeneralizes corresponding results in the literature in two ways: iterativeprocess is faster, operators are more general. At the end, we indicatethat the results can also be proved with the iterative process witherror terms.

***Keywords*—**Contractive-like operator, iterative process, fixed point, strong convergence.

I. Introduction

LET C be a nonempty convex subset of a normed space E and $T : C \to C$ a mapping. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, I the identity mapping on C and $F(T)$ the set of all fixed points of T.

The Picard or successive iterative process [12] is defined by the sequence $\{u_n\}$:

$$\begin{cases} u_1 = u \in C, \\ u_{n+1} = Tu_n, \ n \in \mathbb{N}. \end{cases} \tag{1}$$

The Mann iterative process [11] is defined by the sequence $\{v_n\}$:

$$\begin{cases} v_1 = v \in C, \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n T v_n, \ n \in \mathbb{N} \end{cases} \tag{2}$$

where $\{\alpha_n\}$ is in $(0,1)$.

The sequence $\{z_n\}$ defined by

$$\begin{cases} z_1 = z \in C, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \ n \in \mathbb{N} \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$, is known as the Ishikawa iterative process [5].

Ishikawa process can be seen as a "Double Mann iterative process" or "a hybrid of Mann process with itself". Recently, Khan [8] introduced a new process for one mapping by the sequence $\{x_n\}$:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = T y_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \in \mathbb{N} \end{cases} \tag{4}$$

where $\{\alpha_n\}$ is in $(0,1)$. This process is independent of all Picard, Mann and Ishikawa iterative processes since $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. Khan [8] has shown both analytically and numerically that this process converges faster than all Picard, Mann and Ishikawa iterative processes for contractions.

On the other hand, Berinde [1] introduced a new class of quasi-contractive type operators on a normed space E satisfying

$$\|Tx - Ty\| \le \delta \|x - y\| + L\|Tx - x\| \tag{5}$$

for any $x, y \in E$, $0 < \delta < 1$ and $L \ge 0$.

To appreciate this class of operators, we have to go through some definitions in a metric space (X, d) .

A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx, Ty) \le a d(x, y) \text{ for all } x, y \in X,$$

where $0 < a < 1$.

The map T is called Kannan mapping [7] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.$$

A similar definition is due to Chatterjea [2]: there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X.$$

Combining the above three definitions, Zamfirescu [14] proved the following important result.

Theorem 1. Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers a, b and c satisfying $0 < a < 1$, $b \in (0, \frac{1}{2})$, $c \in (0, \frac{1}{2})$,

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such that for each pair $x, y \in X$, at least one of the following conditions holds:

$$(z_1)\ d(Tx,Ty) \le a d(x,y) \text{ for all } x, y \in X$$
$$(z_2)\ d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)] \text{ for all } x, y \in X$$
$$(z_3)\ d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)] \text{ for all } x, y \in X.$$

Then T has a unique fixed point p and the Picard iterative sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n, n \in \mathbb{N}$ converges to p for any arbitrary but fixed $x_1 \in X$.

An operator T satisfying the contractive conditions (z_1), (z_2) and (z_3) in the above theorem is called Zamfirescu operator. The class of Zamfirescu operators is one of the most studied classes of quasi-contractive type operators. In this class, Mann and Ishikawa iterative processes are known to converge to a unique fixed point of T.

This class of mappings is larger than not only contractions but also Kannan mappings and Zamfirescu operators. Berinde [1] used the Ishikawa iterative process (3) to approximate fixed points of this class of operators in a normed space. Actually, the following was his main theorem:

Theorem 2. [1] Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (5). Let $\{z_n\}$ be defined by the iterative process (3). If $F(T) \ne \phi$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$ then $\{x_n\}$ converges strongly to a fixed point of T.

This kind of operators was further studied by Khan [9], [10], for example.

Imoru and Olatinwo [4] gave a more general definition: An operator T is called a contractive-like operator if there exists a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\phi : [0,\infty) \to [0,\infty)$, with $\phi(0) = 0$, such that for each $x, y \in E$,

$$\|Tx - Ty\| \le \delta \|x - y\| + \phi(\|Tx - x\|) \tag{6}$$

Our purpose in this paper is to prove a convergence result for approximating fixed points of the class of contractive-like operators defined in (6) using the iterative process (4) in the setting of normed spaces. In this way, our results improve and generalize corresponding results of [1] in two ways: iterative process used is simpler and faster, and the class of mappings is more general.

II. Approximating Fixed Points in Normed Spaces

Here is our main theorem which deals with the iterative process (4) for the mappings defined in (6) in normed spaces.

Theorem 3. Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (6) and $F(T) \ne \phi$. Let $\{x_n\}$ be defined by the iterative process (4). Let $\{\alpha_n\}$ be such that $0 < \alpha_n < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a point of $F(T)$.

Proof. Let $w \in F(T)$. Then

$$\begin{aligned}\|y_n - w\| &\le (1-\alpha_n)\|x_n - w\| + \alpha_n \|Tx_n - w\| \\ &\le (1-\alpha_n)\|x_n - w\| + \alpha_n \delta \|x_n - w\| \\ &\le (1 - \alpha_n(1-\delta))\|x_n - w\|.\end{aligned}$$

Thus

$$\begin{aligned}\|x_{n+1} - w\| &= \|Ty_n - w\| \\ &\le \delta \|y_n - w\| \\ &\le \delta(1 - \alpha_n(1-\delta))\|x_n - w\| \\ &\le \delta^2 (1 - \alpha_n(1-\delta)) \\ &\quad (1 - \alpha_{n-1}(1-\delta))\|x_{n-1} - w\| \\ &\le \delta^3 (1 - \alpha_n(1-\delta))(1 - \alpha_{n-1}(1-\delta)) \\ &\quad (1 - \alpha_{n-2}(1-\delta))\|x_{n-2} - w\| \\ &\quad \vdots \\ &\le \delta^n \prod_{k=1}^{n} [1 - (1-\delta)\alpha_k] \|x_1 - w\| \\ &= \|x_1 - w\| \delta^n \exp\left(\sum_{k=1}^{n} -(1-\delta)\alpha_k\right) \\ &= \|x_1 - w\| \delta^n \exp\left(-(1-\delta)\sum_{k=1}^{n} \alpha_k\right)\end{aligned}$$

for all $n \in \mathbb{N}$.

Since $0 < \delta < 1, \alpha_n \in (0,1)$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we get

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - w\| &\le \limsup_{n\to\infty} \|x_1 - w\| \delta^n \\ &\quad \exp\left(-(1-\delta)\sum_{k=1}^{n} \alpha_k\right) \\ &\le 0.\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - w\| = 0$. Consequently $x_n \to w \in F(T)$. This completes the proof.

Liu [6] introduced iterative processes with error terms. Later on, Xu [13] improved these processes by giving more satisfactory error terms. Both processes constitute generalizations of Mann and Ishikawa iterative processes. Our iterative process with error terms in the sense of Xu looks like:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Ty_n, \\ y_n = a_n x_n + b_n T x_n + c_n u_n, \ n \in \mathrm{N} \end{cases}$$

where $\{u_n\}$ is a bounded sequence in C. Numerous papers have been produced on Ishikawa and Mann iterative processes with errors and follow a similar computational techniques as those without errors, see, for example, [3]. To avoid repetition, we omit the proof here.

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