Best Proximity Point Theorems for MT-K and MT-C Rational Cyclic Contractions in Metric Spaces

M. R. Yadav, A. K. Sharma, B. S. Thakur

***Abstract*—**The purpose of this paper is to present a best proximity point theorems through rational expression for a combination of contraction condition, Kannan and Chatterjea nonlinear cyclic contraction in what we call MT-K and MT-C rational cyclic contraction. Some best proximity point theorems for a mapping satisfy these conditions have been established in metric spaces. We also give some examples to support our work.

***Keywords*—**Cyclic contraction, rational cyclic contraction, best proximity point and complete metric space.

I. Introduction

A fundamental result in fixed point theory is the Banach Contraction Principle. One kind of a generalization of the Banach Contraction Principle is the notation of cyclical maps [15]. Fixed point theorems deeply research into the existence of a solution $Tx = x$ where T is a self-mapping. However, when T is a non-self-mapping, the equation $Tx = x$ does not necessarily have a solution, in which case best approximation theorems explore the existence of an approximate solution whereas best proximity point theorems analyze the existence of an approximate solution that is optimal. Despite the fact that best approximation theorems produce an approximate solution to the equation $Tx = x$, they may not render an approximate solution that is optimal. On the contrary, best proximity point theorems are intended to furnish an approximate solution x that is optimal in the sense that the error d(x, Tx) is minimum. The notation of best proximity point is introduced in [9]. This definition is more general than the notation of cyclical maps, in sense that if the sets intersect, then every best proximity point is a fixed point. A sufficient condition for the uniqueness of the best proximity points in uniformly convex Banach spaces is given in [9]. Let A and B be non-empty subsets of a metric space (X, d) and $T: A \cup B \to A \cup B$. Then T is called cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$, where $d(A,B) = \inf\{d(x, y): x \in A, y \in B\}$.

A mapping T of X into itself is called a contraction if there exists a positive real number $\alpha < 1$ with the property

$$d(Tx, Ty) \le \alpha\, d(x, y) \qquad (1)$$

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for all $x, y \in X$. The well known Banach's [2] contraction mapping theorem may be stated as follows: Every contraction mapping of complete metric spaces X into itself has a unique fixed point. Many mathematicians worked on this principal.

Banach's contraction principle has played an important role in the development of various results connected with fixed point and approximation theory. Kanan [11] proved that If T is self mapping of a complete metric space X into itself satisfying:

$$d(Tx, Ty) \le \eta\ [d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$; where $\eta \in [0, 1/2]$. Then T has unique fixed point in X. Fisher [10] proved the result with

$$d(Tx, Ty) \le \mu[d(x, Tx) + d(y, Ty)] + \delta\, d(x, y)$$

for all $x, y \in X$; where $\mu,\ \delta \in [0, 1/2]$. Then T has unique fixed point in X. A similar conclusion was also obtained by Chaterjee [3] proved the result with

$$d(Tx, Ty) \le \mu[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$; where $\mu \in [0, 1/2]$. Then T has unique fixed point in X.

The condition (1) entails $A \cap B$ being nonempty. Eldred and Veeramani [9] modified the condition (1) for the case $A \cap B = \phi$ as follows:

$$d(Tx, Ty) \le \alpha d(x, y) + (1 - \alpha)\, d(A, B), \qquad (2)$$

For some $\alpha \in (0,1)$ and for all $x \in A$ and $y \in B$ $d(A, B) = \inf\{d(x, y): x \in A, y \in B\}$.

The mapping T satisfying condition (3) is called a cyclic contraction. Eldred and Veeramani ([9], 3.10) obtained a unique best proximity point for the mapping T in a uniformly convex Banach space setting. Subsequently, a number of extensions and generalizations of their results appeared in [1], [4], [12], [13] and many others. In 1975, Khan [14] extended Banach's theorem through a symmetric rational expression. Khan introduced the following mapping:

$$d(Tx, Ty) \le \alpha \left[\frac{d(x, Tx)(x, Ty) + d(y, Ty) d(y, Tx)}{d(x, Ty) + d(y, Tx)} \right] \qquad (3)$$

for all and $x, y \in X$ $0 \le \alpha < 1$. Then T has a unique fixed point in X. The aim of this paper is to give a best proximity point theorems for satisfying (3) in the context of metric spaces.

II. Preliminaries

In this section, we first define in what follows the MT-function which will be used throughout the paper to get best proximity point theorems.

Definition 1. (See [5]) A function $\varphi:[0,\infty)\to[0,1)$ is said to be an MT-function if $\lim_{s\to t^+}\varphi(s)<1$ for all $t\in[0,\infty)$, (Mizoguchi-Takahashi.s condition [7]).

It is obvious that if $\varphi:[0,\infty)\to[0,1)$ is a non-decreasing function or a nonincreasing function, then φ is an MT-function. So, the set of MT-functions is a rich class, but it is worth to mention that there exist functions which are not MT-function.

Example 1. (See [7]) $\varphi:[0,\infty)\to[0,1)$ be defined by

$$\varphi(t)=\begin{cases}\dfrac{\sin t}{t} & t\in(0,\pi/2]\\ 0 & \text{otherwise}\end{cases}$$

Since $\lim_{s\to t^+}\varphi(s)=1$, φ is not an MT-function. Very recently, Du [7] first proved some characterizations of MT-functions.

Theorem 1. $\varphi:[0,\infty)\to[0,1)$ be a function. Then the following statements are equivalent.

(a) φ is an MT-function,

(b) For each $t\in[0,\infty)$, there exists $r_t^{(1)}$ and $\varepsilon_t^{(1)}>0$ such that $\varphi(s)<r_t^{(1)}$ for all $s\in(t,\ t+\varepsilon_t^{(1)})$,

(c) For each $t\in[0,\infty)$, there exists $r_t^{(2)}$ and $\varepsilon_t^{(2)}>0$ such that $\varphi(s)<r_t^{(2)}$ for all $s\in(t,\ t+\varepsilon_t^{(2)})$,

(d) For each $t\in[0,\infty)$, there exists $r_t^{(3)}$ and $\varepsilon_t^{(3)}>0$ such that $\varphi(s)<r_t^{(3)}$ for all $s\in(t,\ t+\varepsilon_t^{(3)})$,

(e) For each $t\in[0,\infty)$, there exists $r_t^{(4)}$ and $\varepsilon_t^{(4)}>0$ such that $\varphi(s)<r_t^{(4)}$ for all $s\in(t,\ t+\varepsilon_t^{(4)})$,

(f) For any non-increasing sequence $\{x_n\}_{n\in N}$ in $[0,\infty)$, we have $0\le\sup_{n\in N}\varphi(x_n)<L$,

(g) φ is a function of contractive factor [6]; that is, for any strictly decreasing sequence $\{x_n\}_{n\in N}$ in $[0,\infty)$, we have $0\le\sup_{n\in N}\varphi(x_n)<1$.

Definition 2. [8] Let A and B be nonempty subsets of a metric space (X, d). If a map $T:A\cup B\to A\cup B$ satisfies:

(a). $T(A)\subseteq B$ and $T(B)\subseteq A$;

(b). there exists an MT-function $\varphi:[0,\infty)\to[0,1)$ such that

$$d(Tx,Ty)\le\ \varphi(d(x,y))d(x,y)+(1-\varphi(d(x,y)))d(A,B),\qquad(4)$$

for all $x\in A$ and $y\in B$. Then T is called an MT-cyclic contraction with respect to φ on $A\cap B$.

III. MT-K Rational Cyclic Contraction

In this section, first, we introduce the concept of MT-K rational cyclic contractions and establish the following existence theorem for MT-K Rational Cyclic Contraction, which is one of the main results in this paper.

Definition 3. Let A and B be nonempty subsets of a metric space (X, d). If a map $T:A\cup B\to A\cup B$ satisfies:

(a). $T(A)\subseteq B$ and $T(B)\subseteq A$;

(b). there exists an MT-function $\varphi:[0,\infty)\to[0,1)$ such that

$$d(Tx,Ty)\le\varphi(d(x,y))\left[\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Tx)+d(y,Ty)}\right]+(1-\ \varphi(d(x,y)))d(A,B),\qquad(5)$$

for all $x\in A$ and $y\in B$. Then T is called an MT-K-rational cyclic contraction with respect to φ on $A\cap B$ and $d(x,Tx)+d(y,Ty)\neq0$.

Theorem 2. Let A and B be two nonempty, closed subsets of a metric space (X, d) and the mappings $T:A\cup B\to A\cup B$ satisfy MT-K-rational cyclic condition, then there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty}d(x_n,x_{n+1})=d(A,B).$$

Proof. Letting $x_0\in A\cup B$ be given and define a sequence $\{x_n\}$ by $x_n=Tx_{n-1}$ for all $n\in N$. Since T is satisfy MT-K rational cyclic contraction with MT-function $\varphi:[0,\infty)\to[0,1)$, it follows from inequality (5),

$$\begin{aligned}d(x_1,x_2)&=d(Tx_0,Tx_1)\\&\le\varphi(d(x_0,x_1))\left[\frac{d(x_0,Tx_0)d(x_0,Tx_1)+d(x_1,Tx_1)d(x_1,Tx_0)}{d(x_0,Tx_0)+d(x_1,Tx_1)}\right]+(1-\ \varphi(d(x_0,x_1)))d(A,B),\\&\le\varphi(d(x_0,x_1))\left[\frac{d(x_0,x_1)d(x_0,x_2)+d(x_1,x_2)d(x_1,x_1)}{d(x_0,x_1)+d(x_1,x_2)}\right]+(1-\ \varphi(d(x_0,x_1)))d(A,B),\\&\le\varphi(d(x_0,x_1))\left[\frac{d(x_0,x_1)d(x_0,x_2)+0)}{d(x_0,x_2)}\right]+(1-\varphi(d(x_0,x_1)))d(A,B),\end{aligned}$$

$$\leq \varphi(d(x_0, x_1))\, d(x_0, x_1) + (1 - \varphi(d(x_0, x_1)))d(A,B),$$

Since φ is an MT-function, then from Theorem 1,

$$0 \leq \sup_{n \in N} \varphi(d(x_n, x_{n+1})) < 1.$$

So that we have $0 \leq \sup_{n \in N} \varphi(d(x_0, x_1)) < 1$. Suppose $k = \sup_{n \in N} \varphi(d(x_0, x_1)) < 1$. Therefore $0 \leq k < 1$, since $\varphi(d(x_0, x_1)) \leq k$, we get

$$d(x_1, x_2) \leq k\, d(x_0, x_1) + (1-k)\, d(A,B), \tag{6}$$

Similarly, from inequality (5), we have

$$\begin{aligned}
d(x_2, x_3) &= d(Tx_1, Tx_2) \\
&\leq \varphi(d(x_1, x_2))\left[\frac{d(x_1, Tx_1)d(x_1, Tx_2) + d(x_2, Tx_2)d(x_2, Tx_1)}{d(x_1, Tx_1) + d(x_2, Tx_2)}\right] \\
&\quad + (1 - \varphi(d(x_1, x_2)))d(A,B), \\
&\leq \varphi(d(x_1, x_2))\left[\frac{d(x_1, x_2)d(x_1, x_3) + d(x_2, x_3)d(x_2, x_2)}{d(x_1, x_2) + d(x_2, x_3)}\right] \\
&\quad + (1 - \varphi(d(x_1, x_2)))d(A,B), \\
&\leq \varphi(d(x_1, x_2))\left[\frac{d(x_1, x_2)d(x_1, x_3) + 0)}{d(x_1, x_3)}\right] + (1 - \varphi(d(x_1, x_2)))d(A,B), \\
&\leq \varphi(d(x_1, x_2))\, d(x_1, x_2) + (1 - \varphi(d(x_1, x_2)))d(A,B),
\end{aligned}$$

which implies that,

$$d(x_2, x_3) \leq k\, d(x_1, x_2) + (1-k)\, d(A,B), \tag{7}$$

From (6) and (7), it follows

$$\begin{aligned}
d(x_2, x_3) &\leq k\, d(x_1, x_2) + (1-k)d(A,B) \\
&\leq k[k\, d(x_0, x_1) + (1-k)d(A,B)] + (1-k)d(A,B) \\
&\leq k^2 d(x_0, x_1) + (k - k^2)d(A,B) + (1-k)d(A,B) \\
&\leq k^2 d(x_0, x_1) + (1 - k^2)d(A,B),
\end{aligned} \tag{8}$$

Hence inductively, we concluded that from (8), we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq k d(x_n, x_{n-1}) + (1-k)d(A,B) \\
&\leq k^2 d(x_{n-1}, x_{n-2}) + (1-k^2)d(A,B) \\
&\leq \ldots \\
&\leq k^n d(x_0, x_1) + (1-k^n)d(A,B).
\end{aligned}$$

Since $0 \leq k < 1$, we obtain $\lim_{n \to \infty} k^n = 0$, that is

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A,B).$$

This completes the proof.

Theorem 3. Let A and B be two nonempty, closed subsets of a metric space (X, d) and the mappings $T: A \cup B \to A \cup B$ satisfy condition (5), let $x_0 \in A$ and defined $x_n = Tx_{n-1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then the sequence $\{x_n\}$ is bounded.

Proof. It follows from Theorem 2, that $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded. Suppose the sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element x in A. Since $T: A \cup B \to A \cup B$ satisfy MT-K rational cyclic contraction, we get

$$\begin{aligned}
d(x_{2n_k}, Tx_0) &= d(Tx_{2n_k - 1}, Tx_0) \\
&\leq \varphi(d(x_{2n_k - 1}, x_0))\left[\frac{d(x_{2n_k-1}, Tx_{2n_k-1})d(x_{2n_k-1}, Tx_0) + d(x_0, Tx_0)d(x_0, Tx_{2n_k-1})}{d(x_{2n_k-1}, Tx_{2n_k-1}) + d(x_0, Tx_0)}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x_0)))d(A,B), \\
&\leq \varphi(d(x_{2n_k - 1}, x_0))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx_0) + d(x_0, Tx_0)d(x_0, x_{2n_k})}{d(x_{2n_k-1}, x_{2n_k}) + d(x_0, Tx_0)}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x_0)))d(A,B), \\
&\leq \varphi(d(x_{2n_k - 1}, x_0))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx_0) + 0}{d(x_{2n_k-1}, x_{2n_k}) + d(x_0, Tx_0)}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x_0)))d(A,B), \\
&\leq \varphi(d(x_{2n_k - 1}, x_0))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx_0)}{d(x_{2n_k-1}, x_{2n_k}) + d(x_0, Tx_0)}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x_0)))d(A,B), \\
&\leq \varphi(d(x_{2n_k - 1}, x_0))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx_0)}{d(x_{2n_k-1}, Tx_0) + d(x_0, x_{2n_k})}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x_0)))d(A,B), \\
&\leq \varphi(d(x_{2n_k - 1}, x_0))d(x_{2n_k-1}, x_{2n_k}) \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x_0)))d(A,B),
\end{aligned}$$

From Theorem 2, that $\{d(x_{2n-1}, x_{2n})\}$ converges to $d(A,B)$. So that last inequality implies that

$$d(x_{2n_k}, Tx_0) \leq d(A,B).$$

Therefore, the sequence $\{x_{2n}\}$ is bounded. Hence the sequence $\{x_n\}$ is also bounded. This completes the proof.

Theorem 4. Let A and B be two nonempty, closed subsets of a metric space (X, d) and the mappings $T: A \cup B \to A \cup B$ satisfy condition (5), let $x_0 \in A$ and defined $x_n = Tx_{n-1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.

Proof. Suppose the sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element x in A. In light of the fact that the sequence $\{x_{2n_k}\}$ has a subsequence of $\{x_{2n}\}$ converging to some element x in A. So, it results from Theorem 2 that $d(x_{2n_k}, x_{2n_k-1}) \to d(A,B)$. Since T is a MT-K rational cyclic contraction, it follows that

$$\begin{aligned}
d(x_{2n_k}, Tx) &= d(Tx_{2n_k-1}, Tx) \\
&\le \varphi(d(x_{2n_k-1}, x))\left[\frac{d(x_{2n_k-1}, Tx_{2n_k-1})d(x_{2n_k-1}, Tx) + d(x, Tx)d(x, Tx_{2n_k-1})}{d(x_{2n_k-1}, Tx_{2n_k-1}) + d(x, Tx)}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx) + d(x, Tx)d(x, x_{2n_k})}{d(x_{2n_k-1}, x_{2n_k}) + d(x, Tx)}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx) + 0}{d(x_{2n_k-1}, x_{2n_k}) + d(x, Tx)}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx)}{d(x_{2n_k-1}, x_{2n_k}) + d(x, Tx)}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx)}{d(x_{2n_k-1}, Tx) + d(x, x_{2n_k})}\right] \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x))d(x_{2n_k-1}, x_{2n_k}) \\
&\quad + (1 - \varphi(d(x_{2n_k-1}, x)))d(A,B),
\end{aligned}$$

Taking $n \to\!\infty$ above inequality, then we obtain

$$d(x, Tx) \le d(A,B).$$

So that the last inequality implies that

$$d(x, Tx) = d(A,B).$$

that is x is a best proximity point of T. This completes the proof.

IV. MT-C Rational Cyclic Contraction

In this section, first, we introduce the concept of MT-C rational cyclic contractions and establish the following existence theorem for MT-C Rational Cyclic Contraction, which is one of the main results in this paper.

Definition 4. Let A and B be nonempty subsets of a metric space (X, d). If a map $T: A \cup B \to A \cup B$ satisfies:

(a). $T(A) \subseteq B$ and $T(B) \subseteq A$;

(b). there exists an MT-function $\varphi: [0,\infty) \to [0,1)$ such that

$$d(Tx, Ty) \le \varphi(d(x, y))\left[\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}\right] + (1 - \varphi(d(x, y)))d(A, B), \tag{9}$$

for all $x \in A$ and $y \in B$. Then T is called an MT-C-rational cyclic contraction with respect to φ on $A \cap B$ and $d(x, Ty) + d(y, Tx) \ne 0$.

Theorem 5. Let A and B be two nonempty, closed subsets of a metric space (X, d) and the mappings $T: A \cup B \to A \cup B$ satisfy MT-C-rational cyclic condition, then there exists a sequence $\{x_n\}$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A,B).$$

Proof. Letting $x_0 \in A \cup B$ be given and define a sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in N$. Since T is satisfy MT-C rational cyclic contraction with MT-function $\varphi: [0,\infty) \to [0,1)$, it follows from inequality (9),

$$\begin{aligned}
d(x_1, x_2) &= d(Tx_0, Tx_1) \\
&\le \varphi(d(x_0, x_1))\left[\frac{d(x_0, Tx_0)d(x_0, Tx_1) + d(x_1, Tx_1)d(x_1, Tx_0)}{d(x_0, Tx_1) + d(x_1, Tx_0)}\right] \\
&\quad + (1 - \varphi(d(x_0, x_1)))d(A,B), \\
&\le \varphi(d(x_0, x_1))\left[\frac{d(x_0, x_1)d(x_0, x_2) + d(x_1, x_2)d(x_1, x_1)}{d(x_0, x_2) + d(x_1, x_1)}\right] \\
&\quad + (1 - \varphi(d(x_0, x_1)))d(A,B), \\
&\le \varphi(d(x_0, x_1))\left[\frac{d(x_0, x_1)d(x_0, x_2))}{d(x_0, x_2) + 0}\right] + (1 - \varphi(d(x_0, x_1)))d(A,B), \\
&\le \varphi(d(x_0, x_1))\, d(x_0, x_1) + (1 - \varphi(d(x_0, x_1)))d(A,B),
\end{aligned}$$

Since φ is an MT-function, from Theorem 1,

$$0 \le \sup_{n \in N} \varphi(d(x_n, x_{n+1})) < 1.$$

Suppose $k = \sup_{n\in N} \varphi(d(x_n, x_{n+1}))$. Therefore $0 \le k < 1$, since $\varphi(d(x_n, x_{n+1})) \le k$, we get

$$d(x_1, x_2) \le k\, d(x_0, x_1) + (1-k)\, d(A,B), \qquad (10)$$

Similarly, from inequality (9), we have

$$\begin{aligned}
d(x_2, x_3) &= d(Tx_1, Tx_2) \\
&\le \varphi(d(x_1, x_2))\left[\frac{d(x_1, Tx_2)d(x_1, Tx_2) + d(x_2, Tx_2)d(x_2, Tx_1)}{d(x_1, Tx_2) + d(x_2, Tx_1)}\right] \\
&\quad + (1-\varphi(d(x_1, x_2)))d(A,B), \\
&\le \varphi(d(x_1, x_2))\left[\frac{d(x_1, x_2)d(x_1, x_3) + d(x_2, x_3)d(x_2, x_2)}{d(x_1, x_3) + d(x_2, x_2)}\right] \\
&\quad + (1-\varphi(d(x_1, x_2)))d(A,B), \\
&\le \varphi(d(x_1, x_2))\left[\frac{d(x_1, x_2)d(x_1, x_3))}{d(x_1, x_3) + 0}\right] + (1-\varphi(d(x_1, x_2)))d(A,B), \\
&\le \varphi(d(x_1, x_2))\, d(x_1, x_2) + (1-\varphi(d(x_1, x_2)))d(A,B),
\end{aligned}$$

which implies that,

$$d(x_2, x_3) \le k\, d(x_1, x_2) + (1-k)\, d(A,B), \qquad (11)$$

From (10) and (11), it follows

$$\begin{aligned}
d(x_2, x_3) &\le k\, d(x_1, x_2) + (1-k)d(A,B) \\
&\le k[k\, d(x_0, x_1) + (1-k)d(A,B)] + (1-k)d(A,B) \\
&\le k^2 d(x_0, x_1) + (k-k^2)d(A,B) + (1-k)d(A,B) \\
&\le k^2 d(x_0, x_1) + (1-k^2)d(A,B), \qquad (12)
\end{aligned}$$

Hence inductively, we concluded that from (12), we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\le k d(x_n, x_{n-1}) + (1-k)d(A,B) \\
&\le k^2 d(x_{n-1}, x_{n-2}) + (1-k^2)d(A,B) \\
&\le \ldots \\
&\le k^n d(x_0, x_1) + (1-k^n)d(A,B).
\end{aligned}$$

Since $0 \le k < 1$, we obtain $\lim_{n\to\infty} k^n = 0$, that is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B).$$

This completes the proof.

Theorem 6. Let A and B be two nonempty, closed subsets of a metric space (X, d) and the mappings $T: A\cup B \to A\cup B$ satisfy condition (9), let $x_0 \in A$ and defined $x_n = Tx_{n-1}$. Assume that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then the sequence $\{x_n\}$ is bounded.

Proof. It follows from Theorem 5, that $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded. Suppose the sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element x in A. Since $T: A\cup B \to A\cup B$ satisfy MT-C rational cyclic contraction, we get

$$\begin{aligned}
&d(x_{2n_k}, Tx_0) \\
&= d(Tx_{2n_k-1}, Tx_0) \\
&\le \varphi(d(x_{2n_k-1}, x_0))\left[\frac{d(x_{2n_k-1}, Tx_{2n_k-1})d(x_{2n_k-1}, Tx_0) + d(x_0, Tx_0)d(x_0, Tx_{2n_k-1})}{d(x_{2n_k-1}, Tx_0) + d(x_0, Tx_{2n_k-1})}\right] \\
&\quad + (1-\varphi(d(x_{2n_k-1}, x_0)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x_0))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx_0) + d(x_0, Tx_0)d(x_0, x_{2n_k})}{d(x_{2n_k-1}, Tx_0) + d(x_0, x_{2n_k})}\right] \\
&\quad + (1-\varphi(d(x_{2n_k-1}, x_0)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x_0))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx_0)}{d(x_{2n_k-1}, Tx_0)}\right] \\
&\quad + (1-\varphi(d(x_{2n_k-1}, x_0)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x_0))d(x_{2n_k-1}, x_{2n_k}) \\
&\quad + (1-\varphi(d(x_{2n_k-1}, x_0)))d(A,B),
\end{aligned}$$

Since the sequence $\{x_{2n_k}\}$ converges to x in A and From Theorem 5, that $\{d(x_{2n-1}, x_{2n})\}$ converges to $d(A,B)$. So that last inequality implies that

$$d(x_{2n_k}, Tx_0) \le d(A,B).$$

Therefore, the sequence $\{x_{2n}\}$ is bounded. Hence the sequence $\{x_n\}$ is also bounded. This completes the proof.

Theorem 7. Let A and B be two nonempty, closed subsets of a metric space (X, d) and the mappings $T: A\cup B \to A\cup B$ satisfy condition (9), let $x_0 \in A$ and defined $x_n = Tx_{n-1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.

Proof. Suppose the sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element x in A. In light of the fact that the sequence $\{x_{2n_k}\}$ has a subsequence of $\{x_{2n}\}$ converging to some element x in A. So, it results from Theorem 5 that is $d(x_{2n_k}, x_{2n_k-1}) \to d(A,B)$. Since T is a

MT-C rational cyclic contraction, it follows that

$$
\begin{aligned}
d(x_{2n_k}, Tx) &= d(Tx_{2n_k-1}, Tx) \\
&\le \varphi(d(x_{2n_k-1}, x))\left[\frac{d(x_{2n_k-1}, Tx_{2n_k-1})d(x_{2n_k-1}, Tx) + d(x, Tx)d(x, Tx_{2n_k-1})}{d(x_{2n_k-1}, Tx) + d(x, Tx_{2n_k-1})}\right] \\
&\quad + (1-\varphi(d(x_{2n_k-1}, x)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx) + d(x, Tx)d(x, x_{2n_k})}{d(x_{2n_k-1}, Tx) + d(x, x_{2n_k})}\right] \\
&\quad + (1-\varphi(d(x_{2n_k-1}, x)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x))\left[\frac{d(x_{2n_k-1}, x_{2n_k})d(x_{2n_k-1}, Tx)}{d(x_{2n_k-1}, Tx)}\right] \\
&\quad + (1-\varphi(d(x_{2n_k-1}, x)))d(A,B), \\
&\le \varphi(d(x_{2n_k-1}, x))d(x_{2n_k-1}, x_{2n_k}) \\
&\quad + (1-\varphi(d(x_{2n_k-1}, x)))d(A,B),
\end{aligned}
$$

Taking $n \to \infty$ above inequality, then we obtain

$$d(x, Tx) \le d(A,B).$$

So that the last inequality implies that

$$d(x, Tx) = d(A,B).$$

that is x is a best proximity point of T. This completes the proof.

Example 2. Consider the complete metric space X = R with the usual metrics. Suppose that A = [0, 3], B = [−3, 0] and define $T: A \cup B \to A \cup B$ by $Tx = \frac{-(x+1)}{2}$ if $x \in (1,3]$, $Tx = 0$ if $x \in [-1,1]$ and $Tx = \frac{(-x+1)}{2}$ for all $x \in A \cup B$. If

$$\varphi: [0,\infty) \to [0,1)$$

is defined by $\varphi(t) = \frac{t}{2(1+t)}$ for all $t \ge 0$. Now, notice that $d(A,B) = 0$ and, $T(A) \subseteq B$ and $T(B) \subseteq A$. Putting $x = 0$ and $y = -1$, then we obtain $d(x, Tx) = d(A,B) = 0$. It is easy to say that T is a generalized MT-K and MT-C rational cyclic contraction maps. Hence $x = 0$ is a best proximity point of T.

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