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Dynamic Analysis of Reduced Order Large Rotating Vibro-Impact Systems

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Abstract—Large rotating systems, especially gear drives and gearboxes, occur as parts of many mechanical devices transmitting the torque with relatively small loss of power. With the increased demand for high speed machinery, mathematical modeling and dynamic analysis of gear drives gained importance. Mathematical description of such mechanical systems is a complex task evolving for several decades. In gear drive dynamic models, which include flexible shafts, bearings and gearing and use the finite elements, nonlinear effects due to gear mesh and bearings are usually ignored, for such models have large number of degrees of freedom (DOF) and it is computationally expensive to analyze nonlinear systems with large number of DOF. Therefore, these models are not suitable for simulation of nonlinear behavior with amplitude jumps in frequency response. The contribution uses a methodology of nonlinear large rotating system modeling which is based on degrees of freedom (DOF) number reduction using modal synthesis method (MSM). The MSM enables significant DOF number reduction while keeping the nonlinear behavior of the system in a specific frequency range. Further, the MSM with DOF number reduction is suitable for including detail models of nonlinear couplings (mainly gear and bearing couplings) into the complete gear drive models. Since each subsystem is modeled separately using different FEM systems, it is advantageous to parameterize models of subsystems and to use the parameterization for optimization of chosen design parameters. Final complex model of gear drive is assembled in MATLAB and MATLAB tools are used for dynamical analysis of the nonlinear system. The contribution is further focused on developing of a methodology for investigation of behavior of the system by Nonlinear Normal Modes with combination of the MSM using numerical continuation method. The proposed methodology will be tested using a two-stage gearbox including its housing.

Keywords—Vibro-impact system, rotating system, gear drive, modal synthesis method, numerical continuation method, periodic solution.

I. Introduction

VIBRO-IMPACT dynamic phenomena are common in many engineering applications [1]. Gear drives and gearboxes belong among such applications. They are used for transmitting rotatory motion and torque. Therefore, it is necessary to ensure their proper functionality and to avoid all dynamic phenomena which can lead to increasing of emitting noise during the operation and opposite to decreasing of lifetime period of the system. Vibro-impact motions, which can arise mainly because of the tooth clearance, belong to undesirable ones and can lead up to the loss of stability of the gear drive motion. To detect the nonlinear phenomena numerically, it is necessary to have precise mathematical models including all key factors influencing the dynamic behavior of the system.

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Recently, the modeling and subsequent dynamic analysis of gear drives takes several tendencies investigating a specific dynamic phenomenon on a certain system leading to creation of mathematical models with small number of degrees of freedom. In the work [2] a system of three shafts supported on rigid bearings connected with two gear coupling is modeled. The tooth clearance is taken into account. Litak and Friswell [3] formulated a torsion model of a shaft system with one degree of freedom representing the relative torsion displacement of the shaft respectively. They analyzed the behavior of the system influenced by the parametric excitation - time-dependent gearing stiffness and by the gear clearance. The model of a shaft system containing 16 DOF with a view to the course of the gearing stiffness is formulated in [4] supposing a crack of the teeth. An analytical study [5] shows the influence of friction forces in gearing with spur and helical gears. The derived model respects pinion and gear as two rigid bodies connected with gear coupling, which includes all the mesh lines in the action plane and the friction forces are evaluated.

On the other hand, complex models of gear drives are created providing elastic shafts, bearings and housing and the gear couplings are simplified into a point contact and bearings are supposed to be isotropic defined by stiffness and clearances in main radial direction [6]. Large rotating systems influenced by gearing clearance and elastic bearings respecting real number of rolling elements are modeled using modal synthesis method [7], [8] to decrease time demands. It has been shown [9] that also very simple models can be used for investigating the existence of periodic solutions and their stability in different operational states by theoretical way. Authors Hongler and Streit [10] gave a survey of the origin of chaos in gearbox models. The approach to global modeling of gear drives used in this study is based on the modal synthesis method with DOF number reduction. This method is one of sub-structuring method [11] and enables to combine the flexibility of finite element method in modeling of large linear subsystems by using different finite element computer programs with more accurate treatment of a geared rotor system. The gearing model, which includes the influence of the possibility of gear mesh interruption, is used. The bearing model has been developed [7] and it is used here.

The contribution presents a methodology of nonlinear large rotation system modelling which is based on degrees of freedom (DOF) number reduction using modal synthesis method [7], [8]. The modal synthesis method (MSM) further enables significant DOF number reduction while keeping the nonlinear behavior of the system in a specific frequency range. The presence of gear coupling in a system is typical representative of piecewise-smooth dynamical system [12]. Further, the MSM with DOF number reduction is suitable for including detail models of nonlinear couplings into the complete gear drive models. Since each subsystem is modeled separately using different FEM systems, it is advantageous to parameterize models of subsystems and to use the parametrization for optimization of chosen design parameters. Models of shafts with gears are based on Timoshenko beam theory extended for rotating systems and together with coupling models are implemented in MATLAB while housing model is created by commercial FEM system. Final complex model of gear drive is assembled in MATLAB and MATLAB tools are used for dynamical analysis of the nonlinear system. Finally, a numerical continuation method of periodic orbits is developed to analyze the periodic motions with respect to a control parameter [13], [14], [15]. The numerical continuation method is rearranged to be suitable for harmonically forced systems included strong nonlinearities such as clearances and their vibro-impact effects. The method is based on predictor-corrector technique. Corrector part uses the general shooting technique to calculate solutions. Continuation is performed with these methods by incrementing the forcing frequency. There exist well developed continuation packages for nonlinear system calculations, AUTO [16], MATCONT [17]. These packages are suited towards calculating periodic solutions of autonomous systems.

II. Complex Modeling of Gear Drives

Gear drives represent a class of large rotating mechanical systems which employ variety of nonlinear dynamic phenomena. As mentioned above, many works focus on modeling of particular part of the gear drive only, e.g. dynamics of gearing and dynamics of shafts (rotors) influenced by nonlinear journal bearing. The approach presented here shows a way to complex modeling and analysis of arbitrary large rotating mechanical system including the nonlinear effects in couplings. The method of modeling has been developed to investigate what rate the nonlinear phenomena arising due to nonlinearities in couplings have with respect to specified operational conditions.

The methodology developed in [7], [8] has been adopted here. The modeling is performed in following steps: sub-structuring, DOF number reduction, expressing linear/nonlinear coupling forces and reassembling the complete model. The sub-structuring uses the fact that the system is divided into subsystems in accordance with a certain number of flexible bodies which are mutually connected by discrete either linear or nonlinear flexible couplings. Then the sub-structures are defined by the naturalness of the system. However, the gear drive contains rotating shafts with rigid toothed wheels and housing, whereas shafts are mutually coupled with discrete flexible gear couplings and shafts and housing are coupled with discrete bearing couplings. After decoupling, conservative models of non-rotating (in case of shafts) sub-structures are created in the corresponding generalized coordinates of sub-structures and modal analysis is performed. The reduction of DOF number is defined by transformation of generalized coordinates of each sub-structure which is defined by modal normalized matrix. The level of DOF number reduction depends on the number of mode shapes taken into the modal transformation matrices. Model of the complex structure respecting all couplings is then transformed into a new configuration space with reduced dimension, where the solution is performed. By using the backward transformation, the solution (displacements, deformations, coupling forces and torques) is described in the original configuration space of generalized coordinates.

The gear drive parts can be generally divided into two sections. The first one includes all rotating parts and the second one includes all stator parts. Mostly, the stator parts constitute the housing only. Let us suppose the system can be divided into S subsystems, then we can order all subsystems in such a way, that the rotating subsystems are labeled by indices $s = 1, \ldots, S-1$ and the stator part by index $s = S$.

Rotating parts are modeled as one dimensional shafts with attached rigid discs which constitute geared wheels. The shafts are discretized by Timoshenko beam finite elements with two nodes at its end. Each node has six DOF - three displacements u_i, v_i, w_i and three rotations φ_i, ϑ_i, ψ_i. Then, nodal displacements of subsystem s rotating with angular velocity ω_s are described in its fixed local generalized coordinate space defined by vector

$$\mathbf{q}_s(t) = [\ldots u_i \, v_i \, w_i \, \varphi_i \, \vartheta_i \, \psi_i \, \ldots]^T \tag{1}$$

with a system of N_s ordinary differential equations in matrix form [7]

$$\begin{aligned} \mathbf{M}_s\ddot{\mathbf{q}}_s(t) + (\mathbf{B}_s + \omega_s\mathbf{G}_s)\dot{\mathbf{q}}_s(t) + (\mathbf{K}_s + \omega_s\mathbf{C}_s)\mathbf{q}_s(t) = \\ = \mathbf{f}_s^E(t) + \mathbf{f}_s^B + \mathbf{f}_s^G, \quad s = 1, 2, \ldots, S-1, \end{aligned} \tag{2}$$

where $\mathbf{M}_s$, $\mathbf{B}_s$ and $\mathbf{K}_s$ are symmetrical mass, damping and stiffness matrices of the uncoupled subsystems of order N_s and $\mathbf{G}_s$ and $\mathbf{C}_s$ are skew symmetrical matrix of gyroscopic effects and circulatory matrix, respectively. These matrices are usually created by means of finite element method combined with lumped parameters representing masses of rigid gear discs. External forced excitation is described by vector $\mathbf{f}_s^E(t)$. Vector $\mathbf{f}_s^B$ expresses the coupling forces transmitted by rolling-element bearings and vector $\mathbf{f}_s^G$ represents the forces in helical gear couplings. All force effects described in vectors above are acting on the subsystem s.

The mathematical model of the housing is expressed in a similar way

$$\mathbf{M}_S\ddot{\mathbf{q}}_S(t) + \mathbf{B}_S\dot{\mathbf{q}}_S(t) + \mathbf{K}_S\mathbf{q}_S(t) = \mathbf{f}_S^E(t) + \mathbf{f}_S^B. \tag{3}$$

Mass, damping and stiffness matrices $\mathbf{M}_S$, $\mathbf{B}_S$, $\mathbf{K}_S$ of order N_S are created after discretization of the housing geometry by finite element method. Vector $\mathbf{f}_S^E(t)$ is a possible external excitation and force effect of bearing couplings acting on the housing is expressed by vector $\mathbf{f}_S^B$. In this case, the damping matrix is assumed to be proportional to stiffness matrix $\mathbf{B}_S = \beta_S\mathbf{K}_S$.

A. Modeling of couplings

The decomposition method used here assumes that all couplings are supposed to be discrete. Following this assumption, bearing and gearing couplings are derived. In the case of bearing, the couplings between shafts and housing are concentrated into contact points between bearing rolling-elements and the outer race fixed with the housing. The discrete finite element mesh at the surface of the outer races is uniformly divided around the circumference according to the number of rolling-elements. Therefore, the rolling motion of bearing elements is neglected. Similarly, the gear mesh is concentrated into the central meshing point on the normal pressure line, details on coupling models are in [7], [8].

B. Reduced mathematical model of the gear drive

It is advantageous to assemble reduced mathematical model of the system with reduced DOF number, because mainly the housing subsystem can have too large number of DOF and this fact can hinder consecutive performing of various dynamical analyzes. The DOF number reduction is based on modal transformation of generalized coordinates of each subsystem

$$\mathbf{q}_s(t) = {}^m\mathbf{V}_s\mathbf{x}_s(t), \qquad s = 1,\, 2,\, \dots,\, S. \tag{4}$$

Matrices ${}^m\mathbf{V}_s \in \mathbb{R}^{n_s,m_s}$ are modal submatrices satisfying the orthonormality conditions ${}^m\mathbf{V}_s^T\mathbf{M}_s{}^m\mathbf{V}_s = \mathbf{E}_s$. They were obtained from modal analysis of mutually uncoupled, undamped and non-rotating subsystems. The number of chosen, further called master, eigenvectors included in modal submatrices is denoted m_s ($m_s \leq N_s$). The matrix $\mathbf{E}_s$ denotes identity matrix of order s. The new configuration space of dimension m is defined by vector

$$\mathbf{x} = [\,\mathbf{x}_1^T,\ \mathbf{x}_2^T,\ \dots\ ,\ \mathbf{x}_S^T\,]^T, \qquad m = \sum_{s=1}^{S} m_s. \tag{5}$$

The models (2) and (3) can be then rewritten in the global condensed form [8]

$$\begin{aligned}\ddot{\mathbf{x}}(t) + \Big(\mathbf{B} + \omega_0\mathbf{G} + \mathbf{V}^T\left(\mathbf{B}_B + \mathbf{B}_G\right)\mathbf{V}\Big)\dot{\mathbf{x}}(t) + \\ + \Big(\boldsymbol{\Lambda} + \omega_0\mathbf{C} + \mathbf{V}^T\left(\mathbf{K}_B + \mathbf{K}_G\right)\mathbf{V}\Big)\mathbf{x}(t) = \\ = \mathbf{V}^T\Big(\sum_i\sum_j(\mathbf{c}_{i,j}f_{i,j}(\mathbf{q}) + \\ \mathbf{c}_{i,j}^{ax}f_{i,j}^{ax}(\mathbf{q})) + \sum_{z=1}^{Z}\mathbf{c}_z f_z(t,\mathbf{q},\dot{\mathbf{q}}) + \mathbf{f}_G(t) + \mathbf{f}_E(t)\Big),\end{aligned} \tag{6}$$

where $\mathbf{f}_E(t) = [\,(\mathbf{f}_1^E(t))^T,\ (\mathbf{f}_2^E(t))^T,\ \dots\ ,\ (\mathbf{f}_S^E(t))^T\,]^T$ is the global force vector of external excitation,

$$\mathbf{B} = \operatorname{diag}\left({}^m\mathbf{V}_s^T\mathbf{B}_s{}^m\mathbf{V}_s\right), \mathbf{G} = \operatorname{diag}\left(\frac{\omega_s}{\omega_0}\,{}^m\mathbf{V}_s^T\mathbf{G}_s{}^m\mathbf{V}_s\right),$$
$$\mathbf{C} = \operatorname{diag}\left(\frac{\omega_s}{\omega_0}\,{}^m\mathbf{V}_s^T\mathbf{C}_s{}^m\mathbf{V}_s\right), \quad \mathbf{V} = \operatorname{diag}\left({}^m\mathbf{V}_s\right) \tag{7}$$

are block diagonal matrices ($\omega_S = 0$ holds for the stator subsystem) and $\boldsymbol{\Lambda} = \operatorname{diag}\left({}^m\boldsymbol{\Lambda}_s\right)$ is diagonal matrix composed of spectral submatrices ${}^m\boldsymbol{\Lambda}_s \in \mathbb{R}^{m_s,m_s}$ of the subsystems satisfying the conditions ${}^m\mathbf{V}_s^T\mathbf{K}_s{}^m\mathbf{V}_s = {}^m\boldsymbol{\Lambda}_s$.

The mathematical model of gear drives is strongly nonlinear due to the possibility of gear mesh interruption and in consequence of nonlinear bearing couplings that respect loss of contact in all contact points in dependance on position of journal center. To perform the dynamical analysis, the condensed mathematical model (6) represented by a system of nonlinear differential equations of second order, has to be transformed into the state space $\mathbf{u}(t) = [\dot{\mathbf{x}}(t)^T\,\mathbf{x}(t)^T]^T \in \mathbb{R}^{2m}$, i.e. it has to be equivalent replaced with a system of $2m$ nonlinear differential equations of first order

$$\dot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) = \mathbf{f}(t,\mathbf{u}). \tag{8}$$

The model (8) can be solved using a suitable numerical integration method. The time integration is started from the initial state

$$\mathbf{u}(0) = \begin{bmatrix} \dot{\mathbf{x}}(0) \\ \mathbf{x}(0) \end{bmatrix} \tag{9}$$

to minimize the startup transient motions. The initial dynamic displacements $\mathbf{x}(0)$ in the new coordinate space (5) correspond to static equilibrium caused by the vector $\mathbf{f}_E(0)$ which can describe an arbitrary, not necessary static, external excitation at the start of numerical integration. The vector $\dot{\mathbf{q}}(0)$ defines initial velocity condition. If we suppose that the interior parts (shafts with gears) rotate with an initial angular velocity ω_s, then $\dot{\mathbf{q}}(0)$ contains nonzero elements at positions referring to angular velocities around the axial axes of each node placed on the shafts. The angular velocity of driving shaft ω_1 is supposed to be the reference one and the angular velocities of all other shafts are expressed with respect to speed ratios p_s between the reference one and the current shaft s is equal to $\omega_s = p_s\omega_1$.

The reduced mathematical model has to fulfill desired demands on the accuracy. The choice of master eigenvectors is based on the frequency criterion. It is sufficient to include such eigenvectors of each subsystem that correspond to eigenfrequencies which are at least twice higher than the highest excitation frequency (Nyquist-Shannon sampling theorem). Although the reduced model is nonlinear, we compare the modal properties of the linearized reduced model with the modal properties of the non-reduced linearized model using e.g. MAC criterion. The influence of nonlinear part of the model on the accuracy of the model can be investigated by numerical simulation which is performed for different reduction levels through the whole operational area. In this case, we can compare time courses of chosen dynamic quantities. Increasing the reduction level of each subsystem the model becomes more inaccurate and from particular reduction level the difference between chosen dynamic quantities gained from non-reduced and reduced model starts to increase significantly. The lower boundary reduction level agrees for linear and nonlinear reduced model from the accuracy point of view.

III. Analysis of vibro-impact gearbox vibration

The presented methodology was applied to modeling and dynamic analysis of nonlinear vibrations of two-stage gearbox

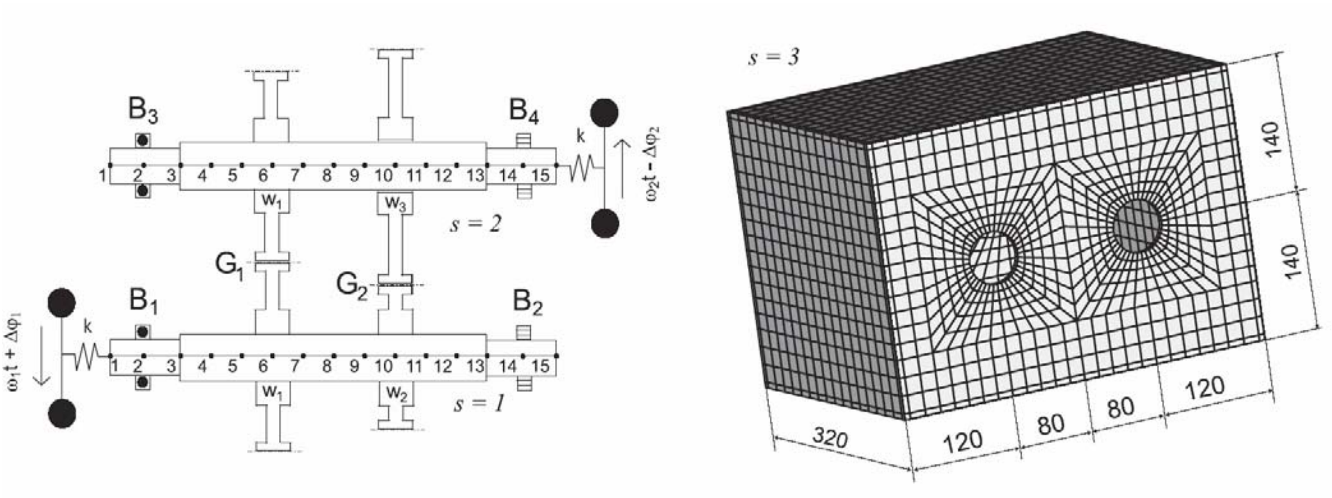

Fig. 1. Scheme of two-stage gearbox

(Fig. 1) whose geometry is based on the real gearbox. We suppose the gearbox as a part of a complex drive. The driving and driven rotating parts of the drive are modeled by discs rotating about corresponding shaft axes, which are characterized by angular momentum and connected with the gearbox by external couplings that have torsional stiffness only. Based on the methodology described in previous section we can formulate a procedure for creation of a particular computational model of the gear drive depicted in Fig. 1. The procedure is described in following steps.

Using the modal synthesis method, the gear drive is decomposed into three subsystems according to Fig. 1: driving shaft, driven shaft and a housing which is fixed in several nodal points at the lower base with the fixed frame. Each shaft is connected with rigid disc by external torsional coupling having torsional stiffness k (bending and longitudinal stiffness are neglected). Models of all three uncoupled subsystems are created using FEM. Rotating shafts are modeled using in-house software developed in MATLAB code. The housing is modeled as three dimensional continuum using the software package ANSYS. It is discretized by finite element mesh whose structure can be clearly seen in Fig. 1. The initial DOF number of the uncoupled subsystems after discretization was $N_1 = 91$ (driving shaft), $N_2 = 92$ (driven shaft with torsional freely mounted gear in nodal point 10) and $N_3 \sim 15000$ (housing). The complete housing model is further used for modal analysis only. More details on the gear drive parameters can be found in [7].

Operational state is given mainly by two control parameters – rotational speed of the driving shaft n [rpm] and external static torsional load of the gearbox which is determined by twist angles $\Delta\varphi_1$ and $\Delta\varphi_2$. The twist angles define relative torsional displacement of the external discs with respect to the nodal point which they are linked to.

The gearbox vibration was investigated for the first transmission degree when gear mesh G_1 transmits the power. The second gear assembly consists of a fixed mounted geared wheel on the driving shaft and of a geared wheel freely rotating about the deformed shaft axis. The vibration is caused by the internal kinematic excitation in gearing G_1. Numerical experiments showed that the reduced (condensed) model (6) of the complete system of order $m = 160$ ($m_1 = 30$, $m_2 = 30$, $m_3 = 100$) is acceptable in the excitation frequency range up to 5000 Hz. The time response of condensed mathematical model (8) to kinematic excitation was calculated by using the numerical simulation that employs Prince-Dormand integration scheme which is a member of Runge-Kutta 5th(4th) order method.

In dependence on particular application where the gearbox is employed, different operational regimes can arise. In cases when the operation is given by a constant velocity or the gearbox velocity changes but between a few constant values, the steady-state response is dominant and it is worth to be investigated to ensure proper dynamic operation of the gearbox. On the other hand, there are many applications where the transient states are dominant, e.g. car gearbox when the car accelerates and slows down. Both of mentioned cases can be covered by the methodology introduced above. In the next, we will give some examples of the test gearbox behavior in different operational regimes.

A. Steady-state nonlinear response - sequential parameter continuation

Here, we are concerned with the qualitative analysis of the gearbox nonlinear vibrations (vibro-impact motion). The aim is to investigate how does the system behave in chosen operational area under assumption that the gearbox undergoes steady-state regime at every operational state from the operational area. The range of revolutions of the driving shaft is defined by particular usage of the gear drive as well as the external static load.

The nonlinear model includes such nonlinear phenomena like impacts in gearing and nonlinear contact forces transmitted by rolling-elements of bearings. Vibro-impact systems are characterized by period-doubling scenario when the period number of the time response increases unexpectedly

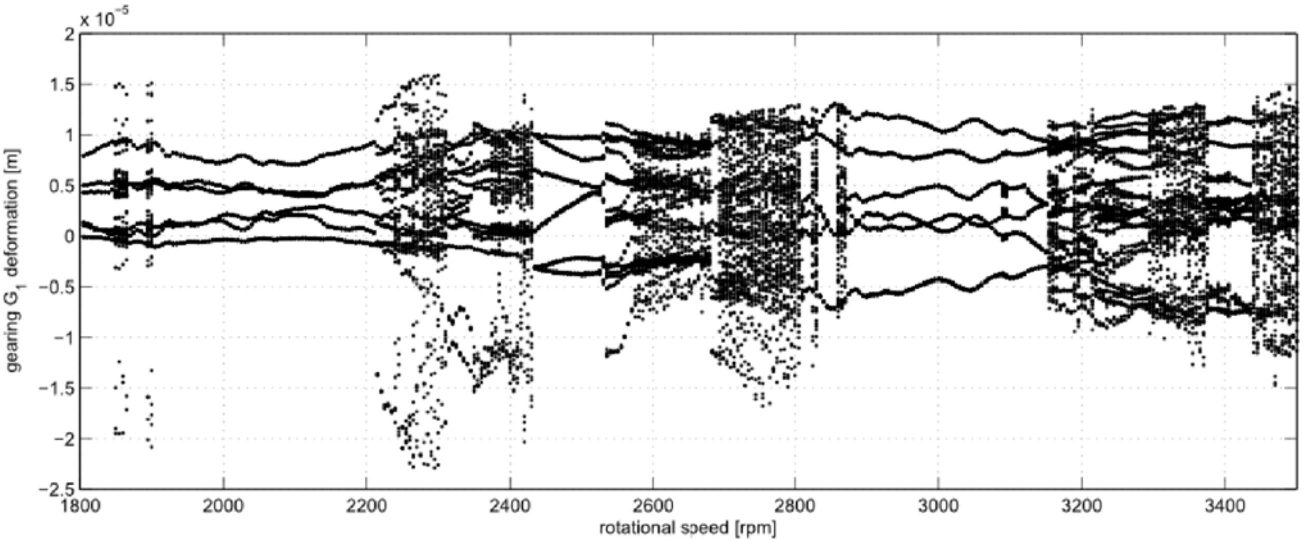

Fig. 2. Bifurcation diagram of gearing deformation in dependence on rotational speed of driving shaft

twice for a certain values of operational parameters. Or the period number can increase by one or more, but not twice, and then we talk about the period-adding scenario. These scenarios could repeat till the motion becomes chaotic. Moreover, the time response of nonlinear model is influenced by impacts in gear meshing with one freely rotating gear. All results introduced here are gained for the condensation level of the gearbox with 160 DOF number.

We can detect the nonlinear phenomena observing time response of many different system parameters. But the most significant is the gear mesh deformation, which realizes the qualitative character of the motion of the whole system. Therefore, we use this quantity to detect nonlinear phenomena in the gear drive vibration. In the next, the nonlinear behavior is displayed using bifurcation diagram for chosen level of external static load $\Delta\varphi = 0.025$ rad.

Fig. 2 shows the bifurcation diagram of gearing deformation of gear mesh G_1 transmitting the power in dependence on driving shaft revolutions. The dots display minima and maxima of gearing deformation. In the diagram, a line corresponding to zero gearing deformation is plotted. If a dot corresponds to negative gearing deformation, it means the gearing reaches the gear mesh interruption and the toothed wheels move within the backlash.

Let us point out some nonlinear phenomena visible in the diagrams. We can observe sudden transitions between chaotic and 1-period motion at 1850 rpm, 1890 rpm, 2210 rpm, 2430 rpm etc. Further, there is an area of chaotic motion which is ranged with slow varying extreme values from 2260 rpm to 2325 rpm. The area e.g. from 2000 rpm to 2210 rpm corresponds to motion with fundamental excitation frequency only. The period-doubling can be clearly seen around 2520 rpm.

B. Pseudo arc-length parameter continuation of periodic orbits

Other and more sophisticated way how to analyze periodic solutions of such a system is to use numerical continuation for harmonically forced nonlinear systems [15], [14].

A shooting method for finding periodic solutions of the system is used. It is based on direct numerical time integration and Newton-Raphson algorithm. According to this method [13], the initial value problem is converted into a two-point boundary value problem. The nonautonomous system can be formulated in the state space

$$\dot{\mathbf{u}}(t) = \mathbf{f}(t, \mathbf{u}). \tag{10}$$

The shooting method allows to find periodic solutions of the system by solving a two-point boundary value problem which is defined by periodicity condition. The periodicity condition is ensured by zero value of so called shooting function

$$\mathbf{F}(\mathbf{u}_0, T) = \mathbf{u}(T, \mathbf{u}_0) - \mathbf{u}_0 \tag{11}$$

where T is the period of the periodic solution and $\mathbf{u}_0$ the initial condition vector.

Once the periodic solution is obtained, the stability analysis can be performed by studying Floquet multipliers, that correspond to eigenvalues of the monodromy matrix evaluated at the period T.

Knowing the initial conditions of the periodic orbit for a given excitation frequency, a time integration over one period is calculated and the amplitude over this time span is extracted to plot a frequency-amplitude diagram.

To develop and test the continuation technique for vibro-impact systems a simplified model of two rotating wheels has been adopted. This model can serve as a first approximation of the reduced complex model of a gear drive system.

To validate the proposed scheme we re-investigated the specific impact pair problem analyzed by Padmanabhan and

Singh [14]. Let us mention that the basic torsional motion of the complex gear drive model can be as a first approximation described by this model to analyze the basic properties of the effect of gear mesh clearance. The model of the impact pair

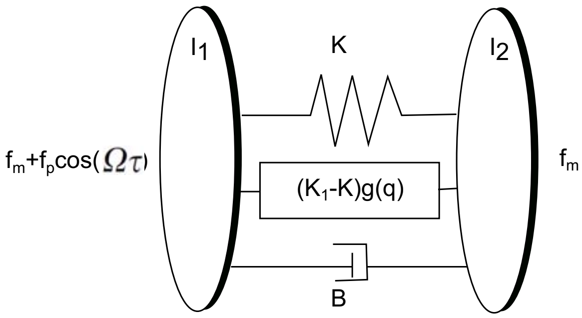

Fig. 3. The model of an impact pair [14]

shown in Fig. 3 rewritten with respect to relative displacement q along with defining parameters $\zeta = B/2I\omega_n, \omega_n = \sqrt{K_1/I}$, $I = I_1 I_2/(I_1 + I_2)$, $\kappa = K/K_1$, $\tau = \omega_n t$, $\Omega = \omega/\omega_n$, $f_m = F_m/I\omega_n^2$ and $f_p = F_p/I_1\omega_n^2$, we have

$$\dot{\mathbf{u}} = \mathbf{K}\mathbf{u} + \mathbf{h}(\mathbf{u}) + \mathbf{f}(\tau) \qquad (12)$$

where $\mathbf{u} = [q\ \dot{q}]^T$ and

$$\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -\kappa & -2\zeta \end{bmatrix}, \mathbf{h}(\mathbf{u}) = \begin{bmatrix} 0 \\ -g(q) \end{bmatrix}, \qquad (13)$$

$$\mathbf{f}(\tau) = \begin{bmatrix} 0 \\ f_m + f_p \cos\Omega\tau \end{bmatrix}. \qquad (14)$$

The non-linear function $g(q)$ is defined as

$$g(q) = \begin{cases} (1-\kappa)(q-b_r)0, & q > b_r, \\ 0, & -b_r \leq q \leq b_r, \\ (1-\kappa)(q+b_r)0, & q < -b_r. \end{cases} \qquad (15)$$

Parameter b_r introduces the clearance between two rotating wheels. The system (12) is solved by using the continuation procedure. The results are shown in Figs. 4 – 8. The continuation parameter was the nondimensional period T. The computations started at a high frequency value, i.e. small period value, as the nonlinear effect are small.

Fig. 4 presents an illustration of the continuation technique capability. One can trace the period-one solutions (P1) which means that the response has the same period length as the excitation. The solution overcomes the turning points and change its stability. At approx 0.5 Hz the period-doubling bifurcation can be detected, i.e. P2 solution.

Figs. 5 and 6 display results gained from one simulation. One can see frequency-amplitude characteristic of the system (Fig. 5) and the corresponding minimum values of the periodic solution. It can be clearly seen that after definition of the clearance the response changes its stability and goes through the turning point. After reaching the maximum amplitude, the solution becomes stable with decreasing excitation frequency. Presented plots are gained for the damping value of $\zeta = 0.03$.

Similarly, the Figs. 7 – 8 display the same quantities gained for higher damping coefficient $\zeta = 0.05$. Numerical simulations show, that there exist a critical damping value

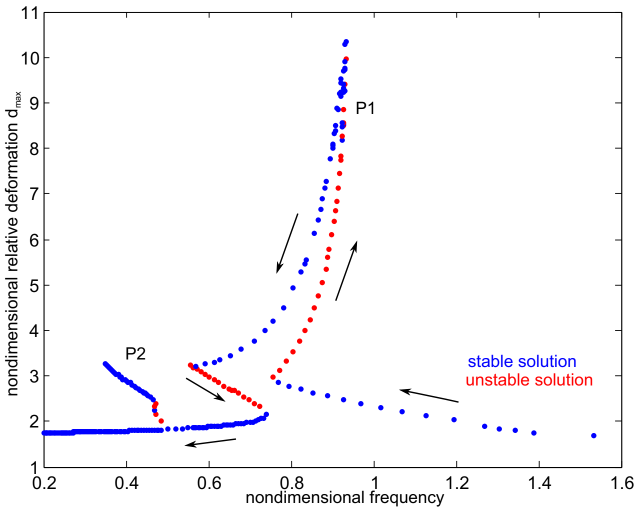

Fig. 4. Amplitude characteristic of nondimensional relative displacement between two impacting wheels, $\zeta = 0.015$

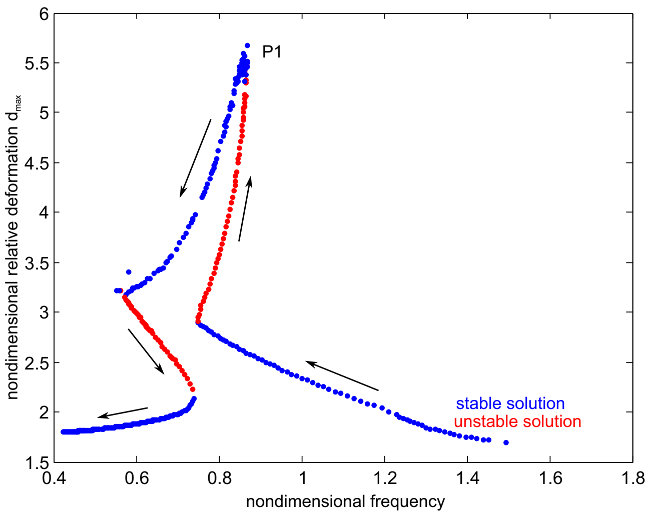

Fig. 5. Amplitude characteristic of nondimensional relative displacement between two impacting wheels, $\zeta = 0.03$, $b_r = 1$

which causes the turning points to be diminished and the response become stable in the whole operational area.

The Figs. 4 – 8 help to reveal what happens during the sequential continuation used in Fig. 2. The sequential continuation can not trace the unstable solutions and therefore, as the parameter (rotational speed) changes, one can see the jumps in the response and period doubling bifurcation which results from the initial conditions used for simulation. To trace all the stable branches is also impossible and one can not detect all the undesired operational states by the sequential continuation technique. The next step is to apply the pseudo arc-length continuation technique to many degrees of freedom model which respects more mode shapes of the system and the response can be more rich for nonlinear phenomena.

IV. Conclusion

The paper describes the new methodology of the gear drive modeling and the analysis of their nonlinear vibrations focused

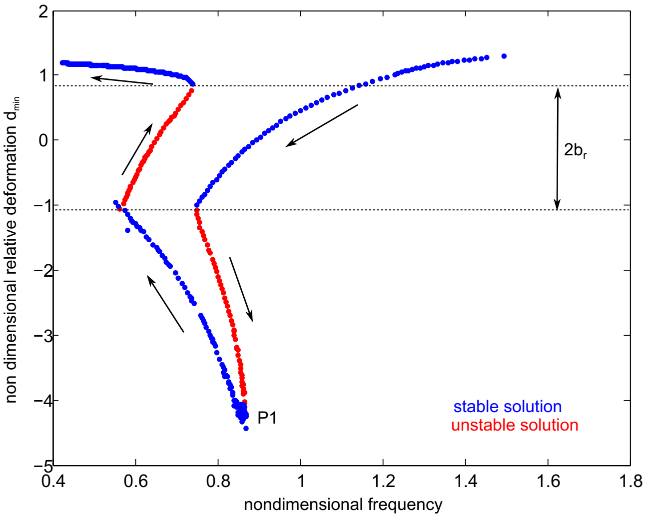

Fig. 6. Minima of nondimensional relative displacement between two impacting wheels, $\zeta = 0.03$, $b_r = 1$

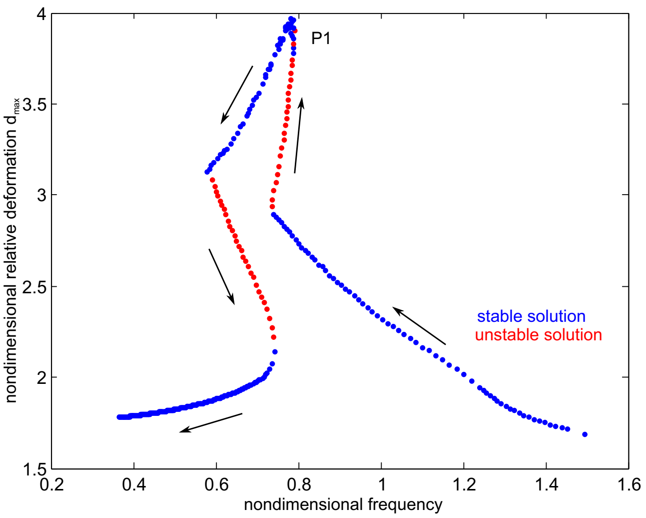

Fig. 7. Amplitude characteristic of nondimensional relative displacement between two impacting wheels, $\zeta = 0.05$, $b_r = 1$

on coupling impact motion analysis. The models of these systems suppose a flexible stator (housing) and nonlinear gear couplings between rotating shaft with gears supported by nonlinear rolling-element bearings. The mathematical model of the whole system created by means of the modal synthesis method, which allows significant DOF number reduction. The methodology of modeling is applied to test gear drive and to the two-stage gearbox. Nonlinear vibrations excited by kinematic transmission errors accompanied by nonlinear phenomena like bifurcation, periodic and quasi-periodic solutions and chaos are investigated using maps of impact number during one period of motion, phase trajectories and and bifurcation diagrams of gearing deformations. The method enables to detect operational areas determining non-stable and chaotic motion of the studied system. The developed software in MATLAB code based on presented methodology is effective tool for modeling and simulation of gear drive nonlinear

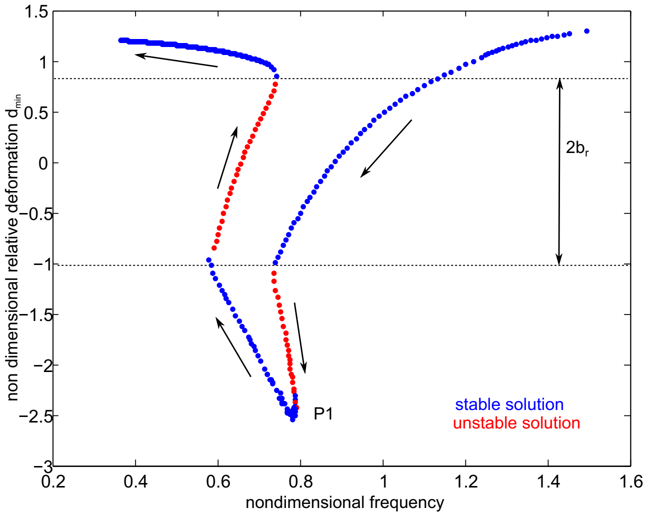

Fig. 8. Minima of nondimensional relative displacement between two impacting wheels, $\zeta = 0.05$, $b_r = 1$

vibro-impact motions and can be further used for either linear or nonlinear complex dynamic analysis of real multi-stage gearboxes.

Further, the numerical continuation method of period solution has been employed to reveal the basic principles of the nonlinear phenomena detected using sequential continuation without the possibility to trace all the solution branches either stable or unstable. The outlook of the current state is to modify the proposed continuation technique for periodic solution to reduced order rotating systems and to analyze more complex dynamical system using this method.

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