A Simplified Higher-Order Markov Chain Model
Chao Wang, Ting-Zhu Huang, Chen Jia

Abstract—In this paper, we present a simplified higher-order Markov chain model for multiple categorical data sequences also called as simplified higher-order multivariate Markov chain model. The number of the parameters of the new model is only \(O((n + s)m^2)\) which is less than \(O(nsm^2)\) in the higher-order multivariate Markov chain model. Numerical experiments illustrate the benefits of our new model.

Keywords—Higher-order multivariate Markov chain model, Categorical data sequences, Multivariate Markov chain.

I. INTRODUCTION

MARKOV chains are of interest in a wide range of applications, for example, telecommunication systems, manufacturing systems and inventory systems, see for instance [6] and the references therein. In recent years, the prediction of categorical data sequences [4] has become more and more useful in many real world applications such as sales demand predictions [1]-[3], DNA sequencing [5] and credit data modeling [8]-[11]. Improving the models for exploring these relationships among the given categorical data sequences is an important research area in these years.

Different models have been proposed for multiple categorical data sequences predictions. A multivariate Markov chain model has been presented in [1]. They constructed a new matrix by means of the transition probability matrices among different sequences. For effectiveness, an improved multivariate Markov chain model has been exhibited to speed up the convergent rate of computing the stationary or steady state solutions. In the improved model, they incorporated the positive and negative association parts [8]. The number of the parameters is \(O((sm^2 + s^2))\). A more advanced, namely, higher-order multivariate Markov chain model has been studied in [7]. Moreover, there are some other papers contribute to the multivariate Markov chain model, e.g., [5], [13]-[15], [17].

However, with the development of science technologies and their applications, the data sequences will be longer, the prediction results needed to be more precise. It is inevitable that the computation from a large categorical data sequence group will cause high computational cost. Now, the higher-order multivariate Markov chain model performs the best in the prediction of multivariate discrete-time Markov chains. Thus, it is useful to simplify the higher-order multivariate Markov chain model. For the above purpose, we propose a simplified higher-order multivariate Markov chain model in this paper.

The rest of the paper is organized as follows. In Section II, we review the Markov chain model [3], the multivariate Markov chain model [1], the higher-order multivariate Markov chain model [7] and two lemmas [16], [18]. In Section III, we present a simplified higher-order multivariate Markov chain model for multiple categorical data sequences. Moreover, some properties of the simplified higher-order multivariate Markov chain model are also analyzed. Section IV gives estimation method for the parameters of the simplified higher-order multivariate Markov chain model. Numerical experiments on two examples demonstrate the benefits of our new model in Section V. Finally, concluding remarks are given in Section VI.

II. A REVIEW ON THE MARKOV CHAIN MODELS

In this section, we briefly introduce some definitions of the Markov chain from [1], [12]. Consider the state set of the categorical data sequence be \(M = \{1, 2, \cdots, m\}\). The discrete-time Markov chain with \(m\) states satisfies the following relationships:

\[
\text{Prob}(x_{t+1} = \theta_{ij} | x_t = \theta_i) = \text{Prob}(x_{t+1} = \theta_{ij} | x_t = \theta_i),
\]

where \(\theta_i \in M, \quad t \in \{1, 2, \cdots\}\). The conditional probabilities \(\text{Prob}(x_{t+1} = \theta_{ij} | x_t = \theta_i)\) are called the one-step transition probabilities of the Markov chain. If we rewrite the transition probabilities as

\[
p_{ij} = \text{Prob}(x_{t+1} = \theta_{ij} | x_t = \theta_i), \quad i, j \in M,
\]

the Markov chain model can be represented as follows:

\[
X_{t+1} = PX_t,
\]

where

\[
P = \{p_{ij}\}, \quad 0 \leq p_{ij} \leq 1, \sum_{i=1}^{m} p_{ij} = 1, \forall i, j \in M,
\]

and \(X_0\) is the initial probability distribution, \(X_t = (x_t^1, x_t^2, \cdots, x_t^m)^T\) is the state probability distribution.

B. The Multivariate Markov Chain Model

A multivariate Markov chain model has been proposed in [1]...
where the number of categorical data sequences is $s > 1$ and it has the following form:

$$x_{r+1}^{(j)} = \lambda_{r} P^{(j)} x_{r}^{(j)}, \forall j \in \{1, 2, \ldots, s\}, \ r \in \{0, 1, \ldots\} \tag{3}$$

where

$$\lambda_{r} \geq 0, \sum_{k=1}^{s} \lambda_{r} = 1, \forall j, k \in \{1, 2, \ldots, s\} \tag{4}$$

Here, $x_{0}^{(j)}$ is the initial probability distribution of the $j$th sequence. $x_{r}^{(j)}$ is the state probability distribution of the $k$th sequence at time $r$. $x_{r+1}^{(j)}$ is the state probability distribution of the $jth$ sequence at time $r + 1$. Here, $P^{(j)}$ is the one-step transition probability matrix from the states in the $kth$ sequence at time $r$ to the states in the $jth$ sequence at time $t + 1$. In matrix form, we have

$$\begin{pmatrix}
  x_{r+1}^{(1)} \\
  x_{r+1}^{(2)} \\
  \vdots \\
  x_{r+1}^{(s)}
\end{pmatrix} =
\begin{pmatrix}
  \lambda_{1} P^{(1)} & \lambda_{2} P^{(2)} & \cdots & \lambda_{s} P^{(s)} \\
  \lambda_{1} P^{(2)} & \lambda_{2} P^{(2)} & \cdots & \lambda_{s} P^{(s)} \\
  \vdots & \vdots & \ddots & \vdots \\
  \lambda_{1} P^{(s)} & \lambda_{2} P^{(s)} & \cdots & \lambda_{s} P^{(s)}
\end{pmatrix}
\begin{pmatrix}
  x_{r}^{(1)} \\
  x_{r}^{(2)} \\
  \vdots \\
  x_{r}^{(s)}
\end{pmatrix}, \tag{5}$$

where $P^{(j)}$ can be obtained directly from the $s$ categorical data sequences and the parameters $\lambda_{r}$ can be got by the linear programming, details in [1].

**C. The Higher-Order Multivariate Markov Chain Model**

Let the number of categorical data sequences be $s > 1$, $m$ be the number of states in every sequence. The higher-order multivariate Markov chain model [7] can be presented as follows:

$$x_{r+1}^{(j)} = \sum_{h=1}^{m} \lambda^{(h)}_{r,j} P^{(h,j)} x_{r-h+1}^{(j)}, \forall r = n - 1, \cdots$$

where $x_{0}^{(1)}, x_{0}^{(2)}, \cdots, x_{0}^{(s)}$ are the initial probability distributions, and $\lambda^{(h)}_{r,j}$ satisfies

$$\sum_{k=1}^{m} \lambda^{(h)}_{r,j,k} = 1, \lambda^{(h)}_{r,j,1} \geq 0, \forall r \leq j, k \leq s, 1 \leq h \leq n. \tag{7}$$

Here, $x_{r-h+1}^{(j)}$ is the state probability distribution of the $k$th sequence at time $r - h + 1$, $P^{(h,j)}$ is the $h$th-step transition probability matrix from the states in the $k$th sequence at time $t = r - h + 1$ to the states in the $j$th sequence at time $t = j - 1$, $x_{r+1}^{(j)}$ is the state probability distribution of the $j$th sequence at time $r + 1$. Moreover, it has the following form:

$$X_{r+1}^{(j)} = (x_{r+1}^{(j)} \ T, x_{r}^{(j)} \ T, \cdots, x_{r+n+2}^{(j)} \ T) ^{T}, \ r \in \mathbb{R}^{m \times 1}. \tag{8}$$

In matrix form, we have

$$\begin{pmatrix}
  X_{r+1}^{(1)} \\
  X_{r+1}^{(2)} \\
  \vdots \\
  X_{r+1}^{(s)}
\end{pmatrix} =
\begin{pmatrix}
  B^{(1,1)} & B^{(1,2)} & \cdots & B^{(1,s)} \\
  B^{(2,1)} & B^{(2,2)} & \cdots & B^{(2,s)} \\
  \vdots & \vdots & \ddots & \vdots \\
  B^{(s,1)} & B^{(s,2)} & \cdots & B^{(s,s)}
\end{pmatrix}
\begin{pmatrix}
  X_{r}^{(1)} \\
  X_{r}^{(2)} \\
  \vdots \\
  X_{r}^{(s)}
\end{pmatrix}, \tag{9}$$

where

$$B^{(j,k)} = \begin{pmatrix}
  \lambda^{(1)}_{j,k} P^{(1)}_{j,k} & \lambda^{(2)}_{j,k} P^{(2)}_{j,k} & \cdots & \lambda^{(m)}_{j,k} P^{(m)}_{j,k} \\
  0 & \cdots & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & \cdots & 0
\end{pmatrix}, \tag{10}$$

else,

$$B^{(j,k)} = \begin{pmatrix}
  \lambda^{(1)}_{j,k} P^{(1)}_{j,k} & \lambda^{(2)}_{j,k} P^{(2)}_{j,k} & \cdots & \lambda^{(m)}_{j,k} P^{(m)}_{j,k} \\
  0 & \cdots & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & \cdots & 0
\end{pmatrix}, \tag{11}$$

Each column sum of the iterative matrix $B$ is not necessary equal to one while each the column sum of transition probability matrices $P^{(h,j)}$ is equal to one.

For analyzing the properties of the simplified higher-order multivariate Markov chain model that will be presented in next section, two Lemmas are given at first.

**Lemma 1.** (Perron-Frobenius Theorem [12]) Let $A \in \mathbb{R}^{m \times m}$ be a non-negative and irreducible matrix. Then;

1. $A$ has a positive real eigenvalue $\lambda$ equal to its spectral radius, i.e., $\lambda = \max_{k} \lambda_{k}(A)$ where $\lambda_{k}(A)$ denotes the $k$th eigenvalue of $A$.
2. To $\lambda$ there is an eigenvector $z$ of its entries being real and positive, such that $Az = \lambda z$.
3. $\lambda$ is a simple eigenvalue of $A$. 

International Scholarly and Scientific Research & Innovation 7(1) 2013 175

ISNI:0000000091950263
**Lemma 2.** ([15]) Let $B$ be a state distribution at time $t$. If $B$ is irreducible and aperiodic, then there is a unique stationary distribution $\pi$ satisfying $\pi = \pi B$. If $\pi_0 \neq \pi$, then the matrix $\pi$ is given as follows:

$$
\pi = \pi_0 B + \pi B + \cdots = \sum_{n=0}^{\infty} \pi_0 B^n
$$

where $\pi_0$ is the initial probability distribution of the first state at time 0, and $B$ is the transition matrix.

**III. SIMPLIFIED HIGHER-ORDER MULTIVARIATE MARKOV CHAIN MODEL**

In this section, for reducing the number of the parameters, we propose a simplified higher-order multivariate Markov chain model for $s$ categorical sequences and briefly discuss some properties of the new model.

In this new model, the state probability distribution of the $j$th sequence at time $t$ depends on the state probability distribution of all the sequences at time $t = r, r-1, \ldots, r-n+1$. For $j = 1, \ldots, s$, $r = 1, \ldots, n-1$, the simplified higher-order multivariate Markov chain model is given as follows:

$$
X_r^{(j)} = \sum_{k=1}^{n} \pi_{r,k} X_{r+1}^{(j)} + \sum_{h=1}^{n} \pi_{h} \sum_{k=1}^{n} \pi_{r,h,k} X_{r+1}^{(j)},
$$

where $X_0^{(1)}, X_0^{(2)}, \ldots, X_0^{(s)}$ are the initial probability distributions, $X_r^{(j)}, P_r^{(j)}$ have been defined in Section II.C and $\pi_{r,h,k}$ satisfies

$$
\sum_{h=1}^{n} \pi_{h} \sum_{k=1}^{n} \pi_{r,h,k} = 1, \quad \sum_{j=1}^{n} \pi_{j} \sum_{k=1}^{n} \pi_{r,j,k} \geq 0,
$$

where $\forall j, k \in \{1, \ldots, s\}, h \in \{1, 2, \ldots, n\}$.

Assuming that

$$
X_r^{(j)} = ([X_r^{(j)}]), [X_r^{(j)}] \in \mathbb{R}^{s \times n},
$$

where $\forall j = 1, 2, \ldots, s$. In matrix form, the simplified higher-order multivariate Markov chain model can be presented as

$$
X_r^{(1)} X_r^{(2)} \cdots X_r^{(s)} = \left( \begin{array}{ccc}
B^{(1,1)} & B^{(1,2)} & \cdots & B^{(1,s)} \\
B^{(2,1)} & B^{(2,2)} & \cdots & B^{(2,s)} \\
\vdots & \vdots & \ddots & \vdots \\
B^{(s,1)} & B^{(s,2)} & \cdots & B^{(s,s)}
\end{array} \right) \left( \begin{array}{c}
X_r^{(1)} \\
X_r^{(2)} \\
\vdots \\
X_r^{(s)}
\end{array} \right),
$$

where

$$
\Lambda = \begin{pmatrix}
\Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1s} \\
\Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda_{s1} & \Lambda_{s2} & \cdots & \Lambda_{ss}
\end{pmatrix},
$$

and if $i \neq j$,
IV. ESTIMATING THE PARAMETERS OF SIMPLIFIED HIGHER-ORDER MULTIVARIATE MARKOV CHAIN MODEL

In this section, we will estimate the parameters of the new multivariate Markov chain model. Let’s first estimate the transition matrices $P_{(j,k)}^h$. If the data sequences are given and the state set is $M = \{1, 2, \ldots, m\}$, $F_{(j,k)}^{(i,h)}$ is frequency from the $i_k$ state in the $k$ th sequence at time $t = r - h + 1$ to the $i_j$ state in the $f$ th sequence at time $t = r + 1$ for $\forall 1 \leq i, j \leq s$, $1 \leq h \leq n$ and $1_m = [1, \ldots, 1]^T \in \mathbb{R}^{1 \times m}$, then the transition frequency matrices $F_{(j,k)}^{(i,h)}$ of the data sequences can be constructed as:

$$ F_{(j,k)}^{(i,h)} = \begin{pmatrix} f_{1,1}^{(i,h)} & f_{1,2}^{(i,h)} & \cdots & f_{1,m}^{(i,h)} \\ f_{2,1}^{(i,h)} & f_{2,2}^{(i,h)} & \cdots & f_{2,m}^{(i,h)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}^{(i,h)} & f_{m,2}^{(i,h)} & \cdots & f_{m,m}^{(i,h)} \end{pmatrix}_{m \times m}.$$  

Transition probability matrices $P_{(j,k)}^h$ can be obtained by normalizing the frequency transition matrices as follows:

$$ P_{(j,k)}^h = \begin{pmatrix} p_{1,1}^{(j,k)} & p_{1,2}^{(j,k)} & \cdots & p_{1,m}^{(j,k)} \\ p_{2,1}^{(j,k)} & p_{2,2}^{(j,k)} & \cdots & p_{2,m}^{(j,k)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m,1}^{(j,k)} & p_{m,2}^{(j,k)} & \cdots & p_{m,m}^{(j,k)} \end{pmatrix}_{m \times m},$$

where

$$ P_{(j,k)}^h = \begin{cases} \frac{f_{j,k}^{(i,h)}}{\sum_{i=1}^{m} f_{j,k}^{(i,h)}}, & \text{if } \sum_{i=1}^{m} f_{j,k}^{(i,h)} \neq 0 \\ 1/m, & \text{otherwise.} \end{cases}$$

Subsequently, the way of estimating the parameters $\lambda_{ij}^h$ will be introduced. Consider $X$ being a joint stationary probability distribution and $X$ can be presented as

$$ X = \left(\left(X^{(1)}\right)^T, \left(X^{(2)}\right)^T, \ldots, \left(X^{(s)}\right)^T\right)^T_{m \times 1},$$

where

$$ X^{(i)} = \left(\left(x^{(i)}\right)^T, \left(x^{(j)}\right)^T, \ldots, \left(x^{(s)}\right)^T\right)^T_{m \times 1},$$

and $X = \left(\left(X^{(1)}\right)^T, \left(X^{(2)}\right)^T, \ldots, \left(X^{(s)}\right)^T\right)^T_{m \times 1}$.

One would expect that

$$ \begin{pmatrix} B^{(11)} & B^{(12)} & \cdots & 0 \\ B^{(21)} & B^{(22)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(ss)} \end{pmatrix} X \approx \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(s)} \end{pmatrix}. $$

(20)
Certainly, non-equality (20) can be rewritten as:

$$|BX - X| \leq \omega,$$

(21)

where $\omega > 0$ and is as small as possible.

One way of estimating $\lambda_{j,k}^{(h)}$ is to transform (21) into a minimization problem as the following form:

$$
\begin{align*}
& \min_{\lambda_{j,k}^{(h)}} |BX - X|,
& \text{subject to } \sum_{h=1}^{n} \sum_{j,k} \lambda_{j,k}^{(h)} = 1, \\
& \lambda_{j,k}^{(h)} \geq 0, \forall h,k,j.
\end{align*}
$$

(22)

Here, we choose the norm as infinite norm, and thus the above minimization problem (22) can be presented as:

$$
\begin{align*}
& \min \max_{\lambda_{j,k}^{(h)}} \left\{ \sum_{h=1}^{n} \sum_{j,k} \lambda_{j,k}^{(h)} P_n^{(h)}X^{(h)} + \right. \\
& \left. \sum_{h=1}^{n} \lambda_{j,k}^{(h)} x^{(h)} \right\},
& \text{subject to } \sum_{h=1}^{n} \sum_{j,k} \lambda_{j,k}^{(h)} = 1, \\
& \lambda_{j,k}^{(h)} \geq 0, \forall h,k,j.
\end{align*}
$$

(23)

where $[\cdot]$ is the $i$th entry of the vector. Using the idea of [1], the minimization problem (23) becomes a linear programming problem as follows:

$$
\begin{align*}
& \min_{\lambda_{j,k}^{(h)}} \omega_j,
& \text{subject to } \\
& \omega_j \geq x^{(j)} - C, \\
& \omega_j \geq -x^{(j)} + C, \\
& \omega_j \geq 0,
& \sum_{h=1}^{n} \lambda_{j,k}^{(h)} + \sum_{h=1}^{n} \lambda_{j,k}^{(h)} = 1, \lambda_{j,k}^{(h)} \geq 0, \forall k,j,h.
\end{align*}
$$

V. NUMERICAL EXPERIMENTS

In this section, we report on the numerical results with different higher-order multivariate Markov chain models for two examples obtained with a Matlab 7.0.1 implementation of a Windows XP with 2.93GHz 64-bit processor and 2GB memory.

A Simple Example

There are two categorical data sequences

$$
S_1 = \{2,1,3,4,4,3,3,1,3,2,3\},
$$

$$
S_2 = \{2,4,4,4,4,2,3,3,1,4,3\}.
$$

In this example, we aim at constructing a simplified 2th-order multivariate Markov chain model. Counting the first lag transition frequencies, we have

$$
F_{1}^{(1,1)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{pmatrix},
$$

$$
F_{1}^{(2,2)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}.
$$

With the same process, the second lags transition frequencies are given as follows:

$$
F_{2}^{(1,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix},
$$

$$
F_{2}^{(2,2)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 0 & 0 & 1 \end{pmatrix}.
$$

Counting the inner first lags transition frequencies, it has

$$
F_{2}^{(2,3)} = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix},
$$

$$
F_{2}^{(1,2)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 0 & 0 & 1 \end{pmatrix}.
$$

Normalizing the transition frequency matrices, we obtain the transition probability matrices:

$$
\tilde{p}_{1}^{(1,1)} = \begin{pmatrix} 0 & 1/2 & 1/5 & 0 \\ 0 & 0 & 1/5 & 1 \\ 1/2 & 2/5 & 1/2 \\ 0 & 1/5 & 1/2 \end{pmatrix},
$$

$$
\tilde{p}_{1}^{(2,2)} = \begin{pmatrix} 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \\ 1/2 & 2/3 & 0 \\ 1/2 & 0 & 3/4 \end{pmatrix}.
$$
The initial state probability distributions can be obtained by computing the proportion of the occurrence of each state in each of the sequence

\[ x_0^{(1)} = \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6} \right), \quad x_0^{(2)} = \left( \frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{5}{12} \right). \]

and \( \lambda_k^{(b)} \) can be calculated by linear programming problem which corresponds to the simplified second-order multivariate Markov chain model of the two categorical data sequences. With the results of \( \lambda_k^{(b)} \) the simplified 2-order multivariate Markov chain model is presented as follows:

\[
\begin{align*}
\lambda^{(1,1)} &= x_0^{(1)} + 0.6647 P_{1}^{(1,1)} x_r^{(1)} + 0.3353 P_{2}^{(1,1)} x_r^{(1)}, \\
\lambda^{(2,2)} &= x_0^{(2)} + 0.3747 P_{1}^{(2,2)} x_r^{(2)} + 0.3199 P_{2}^{(2,2)} x_r^{(2)} + 0.3054 P_{2}^{(2,2)} x_r^{(2)}. 
\end{align*}
\]

**B. An Application to Sales Demand Prediction**

In this part, the sales demand sequences are presented to show the effectiveness of the simplified higher-order multivariate Markov chain model for multivariate Markov chains. Since the requirement of the market fluctuates heavily, the production planning and the inventory control directly affect the estate cost. Thus, studying the interplay between the storage space requirement and the overall growing sales demand is a pressing issue for the company. Here, our goal is to predict the sales demand of the market for minimizing the estate cost. Consider the products can be classified into six possible states (1, 2, 3, 4, 5, 6), e.g., 1 = no sales volume, 2 = vary low sales volume, 3 = low sales volume, 4 = standard sales volume, 5 = fast sales volume, 6 = vary fast sales volume. The customer’s sales demand of five important products of the company for a year has been given in [1].

We choose the simplified 8-order multivariate Markov chain model to model five categorical data sequences. By computing the proportion of the occurrence of each state in each of the sequence, the initial probability distributions of the five categorical data sequences are

\[ x_0^1 = (0.0818, 0.4052, 0.0483, 0.0335, 0.0037, 0.4275)^T, \]
\[ x_0^2 = (0.3680, 0.1970, 0.0335, 0.0000, 0.0037, 0.3978)^T, \]
\[ x_0^3 = (0.1450, 0.2045, 0.0186, 0.0000, 0.0037, 0.6283)^T, \]
\[ x_0^4 = (0.0000, 0.3569, 0.1338, 0.1896, 0.0632, 0.2565)^T, \]
\[ x_0^5 = (0.0000, 0.3569, 0.1227, 0.2268, 0.0520, 0.2416)^T. \]

The transition probability matrices \( P_{h}^{(j,k)} \) can be obtained after normalizing the transition frequency matrices. By solving the corresponding linear programming problem, \( \lambda_k^{(b)} \) is obtained and the simplified 2nd-order multivariate Markov chain model can be presented as follows:

\[
\begin{align*}
x_{r+1}^{(k)} &= x_0^{(k)} + P_{1}^{(k)} x_r^{(k)} + P_{2}^{(k)} x_r^{(k)}, \\
&= x_0^{(k)} + \begin{pmatrix}
0.6647 P_{1}^{(1,1)} x_r^{(1)} + 0.3353 P_{2}^{(1,1)} x_r^{(1)} & 0.3747 P_{1}^{(2,2)} x_r^{(2)} + 0.3199 P_{2}^{(2,2)} x_r^{(2)} + 0.3054 P_{2}^{(2,2)} x_r^{(2)} \\
0.3636 P_{1}^{(2,2)} x_r^{(2)} + 0.3636 P_{2}^{(2,2)} x_r^{(2)} + 0.6198 P_{1}^{(2,4)} x_r^{(4)} + 0.0993 P_{2}^{(2,4)} x_r^{(4)} & 0.2809 P_{1}^{(4,4)} x_r^{(4)} + 0.1993 P_{2}^{(4,4)} x_r^{(4)} + 0.6198 P_{1}^{(5,5)} x_r^{(5)} + 0.0635 P_{2}^{(5,5)} x_r^{(5)}
\end{pmatrix} x_r^{(k)}.
\end{align*}
\]

Next we use the simplified 8th-order multivariate Markov chain model to predict the next state of the 6th sequence \( x_r^{(k)} \) at time \( t \), which can be taken as the state with the maximum probability, i.e.,

\[
x_r^{(i)} = j, \text{ if } x_r^{(j)} = \max_{1 \leq i \leq m} x_r^{(i)}, \forall 1 \leq i \leq m.
\]

For evaluating the performance and the effectiveness of different multivariate Markov chain models, the prediction accuracy \( r \) is proposed in [1] and defined as:

\[
r = \frac{1}{T-n} \sum_{t=n+1}^{T} \delta_t \times 100\%,
\]

where \( T \) is the length of the data sequence and

\[
\delta_t = \begin{cases} 1, & \text{if } x_r^{(r)} = \theta^{(r)} \\ 0, & \text{otherwise} \end{cases}.
\]

For comparisons, the numerical results of the 8th-order multivariate Markov chain are also presented in Table I, where we denote “time” is the computational time, “\( np \)” is the number of the parameters in the models, “\( H \)” is a higher-order multivariate Markov chain model, “\( SIF \)” is a simplified higher-order multivariate Markov chain model and the prediction accuracies of Product A, Product B, Product C, Product D, Product E are denoted as “A”, “B”, “C”, “D”, “E” respectively. Observing the data in Table I, the prediction
accuracies of our new model are nearly the same as the higher-order multivariate Markov chain model's. In addition, it is obviously that the computational time and the number of the parameters of our new model are much less than those of the higher-order multivariate Markov chain model.

Subsequently, another prediction criterion for Markov chain models is introduced. Note that "\(nA\)" is the number of the categorical data in the sequences, \(\mathcal{C}\) is a predict probability at time \(t\), \(X_t\) is a fact value at time \(t\) and \(X_t = [X^1_t, \ldots, X^n_t]^T\). If \(m_i\) is the fact state at \(t\) in \(i\)th categorical data sequence, \(X^i_t = e_{(m_i)} = \{0, \ldots, 0, 1, \ldots, 0\}^T \in \mathbb{R}^{1 \times n}\). We denote the prediction error in the models as "\(pe\)" which can be estimated by the equation:

\[
pe = \sum_{t=0}^{n-1} \|X_t - \hat{X}_t\|_2.
\]

In Table II, we denote that the prediction error of the simplified higher-order multivariate Markov chain model is "\(pe_1\)", the prediction error of the higher-order multivariate Markov chain model is "\(pe_2\)", the number of the parameters of the simplified higher-order multivariate Markov chain model is "\(np_1\)", the number of the parameters of the higher-order multivariate Markov chain model is "\(np_2\)", the computational time of the simplified higher-order multivariate Markov chain model is "\(time_1\)". the computational time of the higher-order multivariate Markov chain model is "\(time_2\)", respectively. Stop criterion of the linear programming problem can be found in Matlab \texttt{linprog}.

Table II provides that the performance of the prediction errors in simplified higher-order multivariate Markov chain model which is comparable with the higher-order multivariate Markov chain model. Table II illustrates the benefits of the simplified higher-order multivariate Markov chain model in terms of the computational time and the number of parameters controlling.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(pe)</th>
<th>(np)</th>
<th>(time)</th>
<th>(pe)</th>
<th>(np)</th>
<th>(time)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>372.1810</td>
<td>25</td>
<td>0.0781</td>
<td>372.1810</td>
<td>25</td>
<td>0.0781</td>
</tr>
<tr>
<td>2</td>
<td>355.3360</td>
<td>50</td>
<td>0.0925</td>
<td>365.2160</td>
<td>30</td>
<td>0.0781</td>
</tr>
<tr>
<td>3</td>
<td>352.5338</td>
<td>75</td>
<td>0.1250</td>
<td>352.5689</td>
<td>35</td>
<td>0.0781</td>
</tr>
<tr>
<td>4</td>
<td>353.3630</td>
<td>100</td>
<td>0.1406</td>
<td>353.4019</td>
<td>40</td>
<td>0.0925</td>
</tr>
<tr>
<td>5</td>
<td>357.6212</td>
<td>125</td>
<td>0.1763</td>
<td>353.8105</td>
<td>45</td>
<td>0.0925</td>
</tr>
<tr>
<td>6</td>
<td>356.8549</td>
<td>150</td>
<td>0.2031</td>
<td>353.8025</td>
<td>50</td>
<td>0.1094</td>
</tr>
<tr>
<td>7</td>
<td>357.4291</td>
<td>175</td>
<td>0.2500</td>
<td>353.6237</td>
<td>55</td>
<td>0.1094</td>
</tr>
<tr>
<td>8</td>
<td>354.7548</td>
<td>200</td>
<td>0.2969</td>
<td>355.7788</td>
<td>60</td>
<td>0.1094</td>
</tr>
</tbody>
</table>

VI. CONCLUSION

In real application, it is inevitable that the computation from a large categorical data sequence group which will cause high computational cost. For saving computational cost, we propose a simplified higher-order multivariate Markov chain model in this paper. The number of the parameters of the new model is only \(O((n + s)sm^2)\) which is less than \(O((ns^2m^2))\) of the higher-order multivariate Markov chain model. The results of the prediction accuracy and the prediction error in these two models are comparable or nearly the same. Moreover, numerical experiments illustrate the benefits of our new model in time consuming, the number of the parameters controlling and the storage requirements. Certainly, the simplified higher-order multivariate Markov chain model can also be applied in credit risk and other research areas.

REFERENCES


V. Chvatal, Linear programming, Freeman, 1983.


R. Horn and C. Johnson, Matrix analysis, Cambridge University Press,


Chao Wang was born in 1983 in Jinlin, China. He received his B.S. (2007) in Applied Mathematics from Bei Hua University and his M.S. (2010) in School of Mathematical Sciences from University of Electronic Science and Technology of China. He is working on his Ph.D. at the University of Electronic Science and Technology in Chengdu, China. His current interests include multivariate Markov Chain model, saddle point problems and linear complementary problem.

Ting-Zhu Huang received the B.S., M.S., and Ph.D. degrees in Computational Mathematics from the Department of Mathematics, Xi’an Jiaotong University, Xi’an, China, in 1986, 1992 and 2000, respectively. During 2005, he was a visiting scholar in Dept. of Computer Science, Loughborough University of UK. He is currently a Full Professor in the School of Mathematical Sciences, UESTC. He is currently an editorial of Advances in Numerical Analysis, Journal of Pure and Applied Mathematics: Advances and Applications, Journal of Electronic Science and Technology of China. He is the author or coauthor of more than 100 research papers. His current research interests include numerical linear algebra with applications, iterative methods of linear systems and saddle point problems, preconditioning technologies, and matrix analysis with applications.

Chen Jia received the B.S. degrees from the School of Accounting, Jilin University of Finance and Economics, Chuanzhou, China, in 2008. She is currently a postgraduate in the School of Accounting, Jilin University of Finance and Economics.