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A new splitting H^1-Galerkin mixed method for pseudo-hyperbolic equations

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Abstract—A new numerical scheme based on the H^1-Galerkin mixed finite element method for a class of second-order pseudo-hyperbolic equations is constructed. The proposed procedures can be split into three independent differential sub-schemes and does not need to solve a coupled system of equations. Optimal error estimates are derived for both semidiscrete and fully discrete schemes for problems in one space dimension. And the proposed method dose not requires the LBB consistency condition. Finally, some numerical results are provided to illustrate the efficacy of our method..

Keywords—Pseudo-hyperbolic equations, Splitting system, H^1-Galerkin mixed method, Error estimates.

I. Introduction

In this paper, we consider the following initial-boundary value problem of pseudo-hyperbolic system

$$\begin{cases} u_{tt} - (a(x)u_{tx} + a(x)u_x)_x + u_t = f(x,t), (x,t) \in \Omega \times J, \\ u(0,t) = u(1,t) = 0, t \in \bar{J}, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \quad (1)$$

where $\Omega = (0,1),\ J = (0,T]$ is the time interval with $0 < T < \infty$. $a(x)$ is smooth functions with bounded derivatives, $f(x,t)$, $u_0(x)$ and $u_1(x)$ are given functions, and

$$0 < a_{min} \le a(x) \le a_{max}, x \in \Omega$$

for positive constants a_{min} and a_{max}.

The pseudo-hyperbolic equations are a class of high-order hyperbolic partial differential equations with mixed partial derivative with respect to time and space, which describe heat and mass transfer, reaction-diffusion and nerve conduction, and other physical phenomena [1], [2], [3], [4], [5]. In [6], Guo and Rui used two least-squares Galerkin finite element schemes to solve pseudo-hyperbolic equations. Moreover, the two methods get the approximate solutions with first-order and second-order accuracy in time increment, respectively. Liu et al. [7] proposed two splitting definite mixed finite element schemes for the pseudo-hyperbolic equation and gave semi-discrete and fully discrete error estimates.

In recent years, a lot of researchers have studied mixed finite element methods for elliptic, parabolic and hyperbolic partial differential equations [10]-[17]. Pani [18] (in 1998) proposed the H^1-Galerkin mixed finite element method which is not subject to the LBB consistency condition. Since then, the method has been applied to many problems [19], [20], [21], [22].

In this paper, a new numerical scheme based on the H^1-Galerkin mixed finite element method for pseudo-hyperbolic equations is constructed. The proposed procedures can be split into three independent differential sub-schemes and does not need to solve a coupled system of equations. Optimal error estimates are derived for both semidiscrete and fully discrete schemes for problems in one space dimension. And some numerical results are provided to illustrate the efficacy of our method. Throughout this paper, C will denote a generic positive constant which does not depend on the spatial mesh parameter h and time discretization parameter Δt. At the same time, we give a important integral inequality

$$\int_0^t \int_0^\tau |\psi(s)|^2 ds d\tau \le C \int_0^t |\psi(s)|^2 ds, \quad (2)$$

where ψ is a integrable function in $[0,t], t \in [0,T]$.

II. Semidiscrete scheme and error estimates

If our concern is to approximate $q = a(x)u_x,\ \sigma = u_t - q_x$ accurately, we reformulate the pseudo-hyperbolic equation (1) as the first-order system

$$\begin{cases} (a)\ q = a(x)u_x, \\ (b)\ \sigma = u_t - q_x, \\ (c)\ \sigma_t + \sigma = f(x,t). \end{cases} \quad (3)$$

To derive the splitting H^1-Galerkin mixed method, we consider the following weak formulation of (3): find $\{u, q, \sigma\} : [0,T] \mapsto H_0^1 \times H^1 \times L^2$ satisfying

$$\begin{cases} (a)\ (u_x, v_x) = (\alpha q, v_x), \forall\ v \in H_0^1, \\ (b)\ (\alpha q_t, w) + (q_x, w_x) = -(\sigma, w_x), \forall\ w \in H^1, \\ (c)\ (\sigma_t, z) + (\sigma, z) = (f, z), \forall\ z \in L^2. \end{cases} \quad (4)$$

where $\alpha = 1/a$, for (4b), we have used integration by parts and the Dirichlet boundary conditions $u_t(0,t) = u_t(1,t) = 0$ to get

$$(u_t, w_x) = (u_t, w)\Big|_0^1 - (u_{tx}, w) = -(u_{tx}, w) = -(\alpha q_t, w). \quad (5)$$

Let V_h, W_h and L_h be finite dimensional subspaces of H_0^1, H^1, and L^2, respectively, with the following approximation properties: for $1 \le p \le \infty$ and $k,\ r,\ l$ positive integers [20]

$$\inf_{v_h \in V_h} \{||v - v_h||_{L^p} + h||v - v_h||_{W^{1,p}}\} \le Ch^{k+1}||v||_{W^{k+1,p}},$$
$v \in\ H_0^1 \cap W^{k+1,p}$,

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$$\inf_{w_h\in W_h}\{||w-w_h||_{L^p}+h||w-w_h||_{W^{1,p}}\}\le Ch^{r+1}||w||_{W^{r+1,p}},$$
$w\in H^1\cap W^{r+1,p}$,

$$\inf_{z_h\in L_h}||z-z_h||_{L^p}\le Ch^{l+1}||w||_{W^{r+1,p}}, z\in L^2\cap W^{l+1,p}.$$

The semidiscrete splitting H^1-Galerkin mixed finite element scheme for (4) consists in determining $\{u_h,q_h,\sigma_h\}:[0,T]\mapsto V_h\times W_h\times L_h$ such that

$$\begin{cases}(a)\ (u_{hx},v_{hx})=(\alpha q_h,v_{hx}),\forall\ v_h\in V_h,\\ (b)\ (\alpha q_{ht},w_h)+(q_{hx},w_{hx})=-(\sigma_h,w_{hx}),\forall\ w_h\in W_h,\\ (c)\ (\sigma_{ht},z_h)+(\sigma_h,z_h)=(f,z_h),\forall\ z_h\in L_h.\end{cases}\tag{6}$$

with given $u_h(0)$, $q_h(0)$ and $\sigma_h(0)$.
For use in the error analysis, we define the Ritz projection $\tilde{u}_h\in V_h$ by

$$(u_x-\tilde{u}_{hx},v_{hx})=0,v_h\ \in V_h.\tag{7}$$

Further, we also define a elliptic projection $\tilde{q}_h\in W_h$ of q as the solution of

$$A(q-\tilde{q}_h,w_h)=0,w_h\ \in W_h.\tag{8}$$

where $A(q,w)=(q_x,w_x)+\lambda(q,w)$. Here λ is chosen appropriately so that A is H^1-coercive, i.e.,

$$A(w,w)\ge\mu_0||w||_1^2,w\ \in H^1,$$

where μ_0 is a positive constant. Moreover, it is not hard to check that $A(\cdot,\cdot)$ is bounded.
We also define the L^2-projection $\tilde{\sigma}_h\in L_h$ by

$$(\sigma-\tilde{\sigma}_h,z_h)=0,z_h\ \in L_h.\tag{9}$$

With $\rho=q-\tilde{q}_h$, $\eta=u-\tilde{u}_h$ and $\delta=\sigma-\tilde{\sigma}_h$, the following estimates are well known [23]: for $j=0,1$

$$||\frac{\partial^i\eta}{\partial t^i}||_j\le Ch^{k+1-j}||\frac{\partial^i u}{\partial t^i}||_{k+1},i=0,1,2,\tag{10}$$

$$||\rho||_j\le Ch^{r+1-j}||q||_{r+1},\ ||\rho_t||_j\le Ch^{r+1-j}||q_t||_{r+1}.\tag{11}$$

$$||\delta||\le Ch^{l+1}||\sigma||_{l+1},\ ||\delta_t||\le Ch^{l+1}||\sigma_t||_{l+1}.\tag{12}$$

Moreover, for $j=0,1$, and $1\le p\le\infty$, we have

$$||\eta||_{W^{j,p}}\le Ch^{k+1-j}||u||_{W^{k+1,p}},\tag{13}$$

$$||\rho||_{W^{j,p}}\le Ch^{r+1-j}||q||_{W^{r+1,p}}.\tag{14}$$

Using the projections $\{\tilde{u}_h,\tilde{q}_h,\tilde{\sigma}_h\}$, we write $u-u_h=u-\tilde{u}_h+\tilde{u}_h-u_h=\eta+\varsigma$, $q-q_h=q-\tilde{q}_h+\tilde{q}_h-q_h=\rho+\xi,\sigma-\sigma_h=\sigma-\tilde{\sigma}_h+\tilde{\sigma}_h-\sigma_h=\delta+\gamma$. From (4)-(9), we then obtain

$$\begin{cases}(a)\ (\varsigma_x,v_{hx})=(\alpha\rho,v_{hx})+(\alpha\xi,v_{hx}),\forall\ v_h\in V_h,\\ (b)\ (\alpha\xi_t,w_h)+A(\xi,w_h)=-(\alpha\rho_t,w_h)\\ \quad-(\delta+\gamma,w_{hx})+\lambda(\rho+\xi,w_h),\forall\ w_h\in W_h,\\ (c)\ (\gamma_t,z_h)+(\gamma,z_h)=-(\delta_t,z_h)-(\delta,z_h),\forall\ z_h\in L_h.\end{cases}\tag{15}$$

Theorem 2.1: Assume that $u_h(0)=\tilde{u}_h(0)$, $q_h(0)=\tilde{q}_h(0)$ and $\sigma_h(0)=\tilde{\sigma}_h(0)$ then

$$||u-u_h||_j\le C(u,q,\sigma)h^{min(k+1-j,r+1,l+1)},$$
$$||q-q_h||_j\le C(q,\sigma)h^{min(r+1-j,l+1)},$$
$$||\sigma-\sigma_h||\le C(\sigma)h^{l+1}.$$

and for $1\le p\le\infty$

$$||u-u_h||_{L^p}\le C(u,q,\sigma)h^{min(k+1,r+1,l+1)}$$
$$||q-q_h||_{L^p}\le C(q,\sigma)h^{min(r+1,l+1)}$$

Proof. Since estimates of η, ρ and δ are given by (10)-(12), respectively, it is sufficient to estimate ς, ξ and γ. Choosing $v_h=\varsigma$ in (15(a)), we have

$$(\varsigma_x,\varsigma_x)=(\alpha\rho,\varsigma_x)+(\alpha\xi,\varsigma_x).$$

Using the Cauchy-Schwarz's inequality, we have

$$||\varsigma_x||\le C(||\rho||+||\xi||).\tag{16}$$

We have, from the Poincaré's inequality

$$||\varsigma||\le C||\varsigma_x||,\varsigma\in H_0^1.\tag{17}$$

Taking $z_h=\gamma_t$ in (15c), we have

$$\begin{aligned}||\gamma_t||^2+\frac{1}{2}\frac{d}{dt}||\gamma||^2&=-(\delta_t,\gamma_t)-(\delta,\gamma_t)\\&\le C(||\delta||^2+||\delta_t||^2)+\frac{1}{2}||\gamma_t||^2.\end{aligned}\tag{18}$$

On integrating with respect to t, we obtain

$$\int_0^t||\gamma_t||^2ds+||\gamma||^2\le C\int_0^t(||\delta||^2+||\delta_t||^2)ds.\tag{19}$$

We choose $w_h=\xi_t$ in (15b) to obtain

$$\begin{aligned}&||\alpha^{\frac{1}{2}}\xi_t||^2+\frac{1}{2}\frac{d}{dt}A(\xi,\xi)\\=&-(\alpha\rho_t,\xi_t)+\lambda(\rho+\xi,\xi_t)-(\delta+\gamma,\xi_{xt})\\=&-(\alpha\rho_t,\xi_t)+\lambda(\rho+\xi,\xi_t)+(\delta_t+\gamma_t,\xi_x)-\frac{d}{dt}(\delta+\gamma,\xi_x).\end{aligned}\tag{20}$$

On integrating with respect to t and using the Cauchy-Schwarz's inequality, the Young's inequality, we obtain

$$\begin{aligned}&\int_0^t||\xi_t(s)||^2ds+||\xi||_1^2\\\le&C\int_0^t(||\gamma_t||^2+||\delta_t||^2+||\rho||^2+||\rho_t||^2+||\xi||_1^2)ds\\&+C(||\gamma||^2+||\delta||^2).\end{aligned}\tag{21}$$

Using the Gronwall's lemma, the integral inequality (2) and (19), we get

$$\begin{aligned}||\xi||^2&\le C\int_0^t||\xi_t(s)||^2ds+||\xi||_1^2\\&\le C\int_0^t(||\delta||^2+||\delta_t||^2+||\rho||^2+||\rho_t||^2)ds+C||\delta||^2.\end{aligned}\tag{22}$$

Use (16), (17), (19), (22), (10)-(12) and the triangle inequality to obtain the L^2 and H^1-norm.
For $1\le p\le\infty$, we have, from the Sobolev embedding theorem,

$$||\xi||_{L^p}\le C||\xi||_1,\xi\in H^1,\ ||\varsigma||_{L^p}\le C||\varsigma_x||,\varsigma\in H_0^1.$$

The use of the convergence results (19) and (17) with (10)-(14) and the triangle inequality completes the proof.

III. Crank-Nicolson-Galerkin Scheme and Error Analysis

In this section, we get the error estimates of fully discrete schemes. For the Crank-Nicolson procedure, let $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ be a given partition of the time interval $[0, T]$ with step length $t_n = n\Delta t,\ \Delta t = T/M$, for some positive integer M. For a smooth function ϕ on $[0, T]$, define $\phi^n = \phi(t_n)$ and $\bar{\partial}_t\phi^n = (\phi^n - \phi^{n-1})/\Delta t, \phi^{n-\frac{1}{2}} = (\phi^n + \phi^{n-1})/2$.

The system (4) has the following equivalent formulation

$$\begin{cases} (\frac{u_x^n + u_x^{n-1}}{2}, v_x) = (\frac{\alpha(q^n + q^{n-1})}{2}, v_x) \\ \quad + (R_1^n, v_x), \forall\ v \in H_0^1, \\ (\alpha\bar{\partial}_t q^n, w) + (\frac{q_x^n + q_x^{n-1}}{2}, w_x) = -(\frac{\sigma^n + \sigma^{n-1}}{2}, w_x) \\ \quad + (R_2^n, w) + (R_3^n, w_x), \forall\ w \in H^1, \\ (\bar{\partial}_t\sigma^n, z) + (\frac{\sigma^n + \sigma^{n-1}}{2}, z) \\ = (f^{n-\frac{1}{2}}, z) + (R_4^n, z), \forall\ z \in L^2, \end{cases} \tag{23}$$

where

$$\begin{aligned} R_1^n &= \frac{u_x^n + u_x^{n-1}}{2} - u_x^{n-\frac{1}{2}} + \alpha q^{n-\frac{1}{2}} - \frac{\alpha(q^n + q^{n-1})}{2}, \\ R_2^n &= (\alpha\bar{\partial}_t q^n - \alpha q_t^{n-\frac{1}{2}}) + \frac{\sigma^n + \sigma^{n-1}}{2} - \sigma^{n-\frac{1}{2}}, \\ R_3^n &= \frac{q_x^n + q_x^{n-1}}{2} - q_x^{n-\frac{1}{2}}, \\ R_4^n &= (\bar{\partial}_t\sigma^n - \sigma_t^{n-\frac{1}{2}}) + \frac{\sigma^n + \sigma^{n-1}}{2} - \sigma^{n-\frac{1}{2}}. \end{aligned}$$

Let U^n, Z^n and Q^n, respectively, be the approximations of u, q and σ at $t = t_n$ which we shall define through the following scheme. Given $\{U^{n-1}, Z^{n-1}, Q^{n-1}\}$ in $V_h \times W_h \times L_h$, we now determine a triple $\{U^n, Z^n, Q^n\}$ in $V_h \times W_h \times L_h$ satisfying

$$\begin{cases} (\frac{U_x^n + U_x^{n-1}}{2}, v_{hx}) = (\frac{\alpha(Q^n + Q^{n-1})}{2}, v_{hx}), \forall\ v_h \in V_h, \\ (\alpha\bar{\partial}_t Q^n, w_h) + (\frac{Q_x^n + Q_x^{n-1}}{2}, w_{hx}) \\ = -(\frac{Z^n + Z^{n-1}}{2}, w_{hx}), \forall\ w_h \in W_h, \\ (\bar{\partial}_t Z^n, z_h) + (\frac{Z^n + Z^{n-1}}{2}, z_h) = (f^{n-\frac{1}{2}}, z_h), \forall\ z_h \in L_h, \end{cases} \tag{24}$$

For fully discrete error estimates, we now split the errors $u(t_n) - U^n = (u(t_n) - \tilde{u}_h(t_n)) + (\tilde{u}_h(t_n) - U^n) = \eta^n + \varsigma^n,\ q(t_n) - Q^n = (q(t_n) - \tilde{q}_h(t_n)) + (\tilde{q}_h(t_n) - Q^n) = \rho^n + \xi^n,\ \sigma(t_n) - Z^n = (\sigma(t_n) - \tilde{\sigma}_h(t_n)) + (\tilde{\sigma}_h(t_n) - Z^n) = \delta^n + \gamma^n$. Using (7)-(9) and (23)-(24), we then obtain

$$\begin{cases} (a)\ (\varsigma_x^{n-\frac{1}{2}}, v_{hx}) = (R_1^n, v_{hx}) + (\alpha\rho^{n-\frac{1}{2}}, v_{hx}) \\ \quad + (\alpha\xi^{n-\frac{1}{2}}, v_{hx}), \forall\ v_h \in V_h, \\ (b)\ (\alpha\bar{\partial}_t\xi^n, w_h) + A(\xi^{n-\frac{1}{2}}, w_h) = -(\alpha\bar{\partial}_t\rho^n, w_h) \\ \quad + (R_2^n, w_h) + (R_3^n, w_{hx}) + (\gamma^{n-\frac{1}{2}} + \delta^{n-\frac{1}{2}}, w_{hx}) \\ \quad + \lambda(\rho^{n-\frac{1}{2}} + \xi^{n-\frac{1}{2}}, w_h), \forall\ w_h \in W_h, \\ (c)\ (\bar{\partial}_t\gamma^n, z_h) + (\gamma^{n-\frac{1}{2}}, z_h) = -(\bar{\partial}_t\delta^n, z_h) \\ \quad + (R_4^n, z_h), \forall\ z_h \in L_h. \end{cases} \tag{25}$$

Theorem 3.1: Assume that $Q^0 = \tilde{q}_h(0), Z^0 = \tilde{\sigma}_h(0)$, then there exists some positive constants C independent of h and Δt such that for $0 < \Delta t \le \Delta t_0$ and $J = 0, 1, \cdots, M$

$$\begin{aligned} & (a).||u^{n-\frac{1}{2}} - U^{n-\frac{1}{2}}||_j \\ \le\ & C(u, q, \sigma)(h^{min(k+1-j, r+1, l+1)} + (\Delta t)^2), \\ & (b).||q^J - Q^J|| + h\Big(\Delta t\sum_{n=1}^{J}||q^n - Q^n||_1^2\Big)^{\frac{1}{2}} \\ \le\ & C(q, \sigma)(h^{min(r+1, l+1)} + (\Delta t)^2), \\ & (c).||\sigma^J - Z^J|| \le C(\sigma)(h^{l+1} + (\Delta t)^2). \end{aligned}$$

Proof. Choose $z_h = \gamma^{n-\frac{1}{2}}$ in (25c) and use the Cauchy-Schwarz's inequality and the Young's inequality to get

$$\begin{aligned} & \frac{1}{2\Delta t}[||\gamma^n||^2 - ||\gamma^{n-1}||^2] + ||\gamma^{n-\frac{1}{2}}||^2 \\ =& -(\bar{\partial}_t\delta^n, \gamma^{n-\frac{1}{2}}) + (R_4^n, \gamma^{n-\frac{1}{2}}) \\ \le& C(||\bar{\partial}_t\delta^n||^2 + ||R_4^n||^2) + \frac{1}{2}||\gamma^{n-\frac{1}{2}}||^2. \end{aligned} \tag{26}$$

Multiply (26) by $2\Delta t$ and sum from $n = 1$ to J to obtain

$$\begin{aligned} & ||\gamma^J||^2 + \Delta t\sum_{n=1}^{J}||\gamma^{n-\frac{1}{2}}||^2 \\ \le& ||\gamma^0||^2 + C\Delta t\sum_{n=1}^{J}(||\bar{\partial}_t\delta^n||^2 + ||R_4^n||^2). \end{aligned} \tag{27}$$

Note that

$$||\bar{\partial}_t\delta^n||^2 \le \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}||\delta_t(s)||^2 ds,$$

and

$$||R_4^n||^2 \le C(\Delta t)^4(||\sigma_{ttt}^{n-\frac{1}{2}}||^2 + ||\sigma_{tt}^{n-\frac{1}{2}}||^2).$$

On substitution and noting that $\gamma^0 = 0$, the resulting inequality becomes

$$\begin{aligned} & ||\gamma^J||^2 + \Delta t\sum_{n=1}^{J}||\gamma^{n-\frac{1}{2}}||^2 \\ \le& C\Delta t\sum_{n=1}^{J}\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}||\delta_t(s)||^2 ds + C\Delta t\sum_{n=1}^{J}||R_4^n||^2 \\ \le& \int_0^J||\delta_t(s)||^2 ds + CJ\cdot(\Delta t)^5(||\sigma_{ttt}||^2_{L^\infty(L^2)} + ||\sigma_{tt}||^2_{L^\infty(L^2)}) \\ \le& C(h^{l+1}||\sigma_t||^2_{L^2(H^{l+1})} + (\Delta t)^4(||\sigma_{ttt}||^2_{L^\infty(L^2)} + ||\sigma_{tt}||^2_{L^\infty(L^2)})). \end{aligned} \tag{28}$$

Choose $w_h = \xi^{n-\frac{1}{2}}$ in (25b) and use the Cauchy-Schwarz's inequality and the Young's inequality to get

$$\begin{aligned}&(\alpha\bar{\partial}_t\xi^n, \xi^{n-\frac{1}{2}}) + A(\xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}})\\ =&-(\alpha\bar{\partial}_t\rho^n, \xi^{n-\frac{1}{2}}) + (R_2^n, \xi^{n-\frac{1}{2}}) + (R_3^n, \xi_x^{n-\frac{1}{2}})\\ &+(\gamma^{n-\frac{1}{2}} + \delta^{n-\frac{1}{2}}, \xi_x^{n-\frac{1}{2}}) + \lambda(\rho^{n-\frac{1}{2}} + \xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}})\\ \le& C(||\bar{\partial}_t\rho^n||^2 + ||R_2^n||^2 + ||R_3^n||^2 + ||\rho^{n-\frac{1}{2}}||^2 + ||\xi^{n-\frac{1}{2}}||^2\\ &+||\gamma^{n-\frac{1}{2}}||^2 + ||\delta^{n-\frac{1}{2}}||^2) + \frac{\mu_0}{2}||\xi_x^{n-\frac{1}{2}}||^2.\end{aligned} \quad (29)$$

Noting that $c(a+b)(a-b) = ca^2 - cb^2 = (c^{\frac{1}{2}}a)^2 - (c^{\frac{1}{2}}b)^2, c > 0$, we have

$$(\alpha\xi^{n-\frac{1}{2}}, \partial_t\xi^n) = \frac{1}{2\Delta t}[||\alpha^{\frac{1}{2}}\xi^n||^2 - ||\alpha^{\frac{1}{2}}\xi^{n-1}||^2]. \quad (30)$$

On substitution and summing from $n = 1$ to J, we obtain

$$\begin{aligned}&\alpha_{min}||\xi^J||^2 + \mu_0\Delta t\sum_{n=1}^{J}||\xi^n||_1^2\\ \le&\Delta t\sum_{n=1}^{J}(||\bar{\partial}_t\rho^n||^2 + ||R_2^n||^2 + ||R_3^n||^2 + ||\rho^{n-\frac{1}{2}}||^2\\ &+||\xi^{n-\frac{1}{2}}||^2 + ||\gamma^{n-\frac{1}{2}}||^2 + ||\delta^{n-\frac{1}{2}}||^2)\\ \le&\Delta t\sum_{n=1}^{J}(||\bar{\partial}_t\rho^n||^2 + ||R_2^n||^2 + ||R_3^n||^2 + ||\rho^{n-\frac{1}{2}}||^2\\ &+||\gamma^{n-\frac{1}{2}}||^2 + ||\delta^{n-\frac{1}{2}}||^2) + \Delta t\sum_{n=1}^{J}||\xi^n||^2.\end{aligned} \quad (31)$$

Choose Δt_0 in such a way that for $0 < \Delta t \le \Delta t_0, (\alpha_{min} - C\Delta t) > 0$. Then an application of Gronwall's lemma to obtain

$$\begin{aligned}&||\xi^J||^2 + \Delta t\sum_{n=1}^{J}||\xi^n||_1^2\\ \le&\Delta t\sum_{n=1}^{J}(||\bar{\partial}_t\rho^n||^2 + ||R_2^n||^2 + ||R_3^n||^2\\ &+||\rho^{n-\frac{1}{2}}||^2 + ||\gamma^{n-\frac{1}{2}}||^2 + ||\delta^{n-\frac{1}{2}}||^2).\end{aligned} \quad (32)$$

Note that

$$||\bar{\partial}_t\rho^n||^2 \le \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}||\rho_t(s)||^2 ds,$$

$$||R_2^n||^2 \le C(\Delta t)^4(||q_{ttt}^{n-\frac{1}{2}}||^2 + ||\sigma_{tt}^{n-\frac{1}{2}}||^2),$$

and

$$||R_3^n||^2 \le C(\Delta t)^4||q_{tt}^{n-\frac{1}{2}}||_1^2.$$

On substitution and using (28), we can get

$$\begin{aligned}&||\xi^J||^2 + \Delta t\sum_{n=1}^{J}||\xi^n||_1^2\\ \le&Ch^{l+1}(||\sigma||^2_{L^2(H^{l+1})} + ||\sigma_t||^2_{L^2(H^{l+1})})\\ &+CT\cdot(\Delta t)^4(||\sigma_{ttt}||^2_{L^\infty(L^2)} + ||\sigma_{tt}||^2_{L^\infty(L^2)})\\ &+Ch^{r+1}(||q||^2_{L^2(H^{r+1})} + ||q_t||^2_{L^2(H^{r+1})})\\ &+CT\cdot(\Delta t)^4(||q_{ttt}||^2_{L^\infty(L^2)} + ||q_{tt}||^2_{L^\infty(L^2)}).\end{aligned} \quad (33)$$

TABLE I
$L^\infty(L^2)$-ERRORS AND ORDER OF CONVERGENCE

$h = 2\Delta t$	$\lVert u - u_h\rVert_{L^\infty(L^2)}$	Order	$\lVert q - q_h\rVert_{L^\infty(L^2)}$	Order
1/5	2.7986E-02		5.4168E-02	
1/10	7.1493E-03	1.968830	1.2409E-02	2.126094
1/20	1.7990E-03	1.990618	2.9587E-03	2.068322
1/40	4.5036E-04	1.998030	7.2123E-04	2.036427
1/80	1.1264E-04	1.999360	1.7794E-04	2.019069
1/160	2.8159E-05	2.000051	4.4184E-05	2.009795
1/320	7.0392E-06	2.000113	1.1008E-05	2.004972
1/640	1.7597E-06	2.000082	2.7472E-06	2.002519
1/1280	4.3991E-07	2.000049	6.8617E-07	2.001324

Choosing $v_h = \varsigma^{n-\frac{1}{2}}$ in (25a), we have

$$||\varsigma_x^{n-\frac{1}{2}}||^2 = (R_1^n, \varsigma_x^{n-\frac{1}{2}}) + (\alpha\rho^{n-\frac{1}{2}}, \varsigma_x^{n-\frac{1}{2}}) + (\alpha\xi^{n-\frac{1}{2}}, \varsigma_x^{n-\frac{1}{2}}). \quad (34)$$

Use the Cauchy-Schwarz, the Young's inequality as well as the Poincaré's inequality and (33) to obtain

$$\begin{aligned}&||\varsigma^{n-\frac{1}{2}}||^2 \le C||\varsigma_x^{n-\frac{1}{2}}||^2\\ \le&C(||\rho^{n-\frac{1}{2}}||^2 + ||\xi^{n-\frac{1}{2}}||^2 + ||R_1^n||^2)\\ \le&C(||\rho^n||^2 + ||\xi^n||^2) + CT\cdot(\Delta t)^4(||u_{tt}^{n-\frac{1}{2}}||_1^2 + ||q_{tt}^{n-\frac{1}{2}}||^2)\\ \le&Ch^{l+1}(||\sigma||^2_{L^2(H^{l+1})} + ||\sigma_t||^2_{L^2(H^{l+1})}) + Ch^{r+1}(||q||^2_{L^2(H^{r+1})}\\ &+||q_t||^2_{L^2(H^{r+1})}) + CT\cdot(\Delta t)^4(||\sigma_{ttt}||^2_{L^\infty(L^2)} + ||\sigma_{tt}||^2_{L^\infty(L^2)}\\ &+||q_{ttt}||^2_{L^\infty(L^2)} + ||q_{tt}||^2_{L^\infty(L^2)} + ||u_{tt}||^2_{L^\infty(H^1)}).\end{aligned} \quad (35)$$

Using (28), (33), (35) and the triangle inequality completes the L^2-error estimate and the H^1-estimate.

IV. Numerical example

In order to illustrate the efficiency of the splitting mixed element method presented in this article, we consider the following initial-boundary value problem of pseudo-hyperbolic system

$$\begin{cases}u_{tt}(x,t) - u_{xxt}(x,t) - u_{xx}(x,t) + u_t(x,t) = f(x,t),\\ (x,t) \in (0,1)\times(0,1],\\ u(0,t) = u(1,t) = 0, t \in [0,1],\\ u(x,0) = \sin(\pi x), u_t(x,0) = -2\sin(\pi x), x \in [0,1],\end{cases} \quad (36)$$

where $f(x,t) = (2-\pi^2)e^{-2t}\sin(\pi x)$.

It is not difficult to verify that the exact solution is $u(x,t) = e^{-2t}\sin(\pi x)$. The corresponding basis functions are piecewise linear functions. The errors in $L^\infty(L^2)$-norm and the accuracy of the approximate solutions u_h, q_h and σ_h are provided in Table I and Table II. Furthermore, the obtained surfaces of the numerical solutions u_h, q_h and σ_h are shown in Figs. 1-3, respectively. And the comparisons of the exact solutions (u, q, σ) and the numerical solutions (u_h, q_h, σ_h) at $t = 0.25, 0.5, 0.75, 1.0$ are shown in Figs. 4-6.

We can see from the above data and figures that the convergence order obtained in numerical simulations are agree with the results obtained in theoretical analysis when the time step and spatial step ratio is 1/2 (that is $h = 2\Delta t$). The numerical results show that the splitting H^1-Galerkin mixed

TABLE II
$L^{\infty}(L^2)$-ERRORS AND ORDER OF CONVERGENCE

$h = 2\Delta t$	$\|\|\sigma - \sigma_h\|\|_{L^{\infty}(L^2)}$	Order
1/5	5.3670E-02	
1/10	1.3694E-02	1.970541
1/20	3.4661E-03	1.982183
1/40	8.7231E-04	1.990414
1/80	2.1882E-04	1.995096
1/160	5.4802E-05	1.997444
1/320	1.3713E-05	1.998684
1/640	3.4297E-06	1.999390
1/1280	8.5763E-07	1.999655

finite element method introduced in this article is efficient for second-order pseudo-hyperbolic problem.

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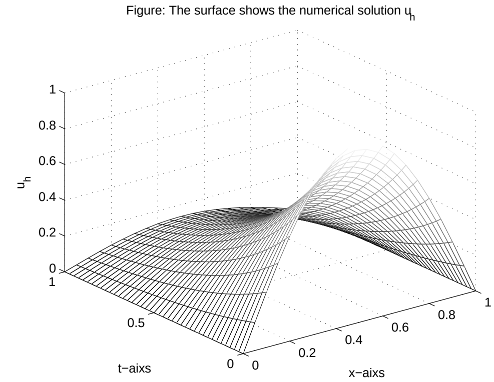

Fig. 1. the surface of u_h with $t \in [0,1], x \in [0,1], h = 2\Delta t = 0.05$

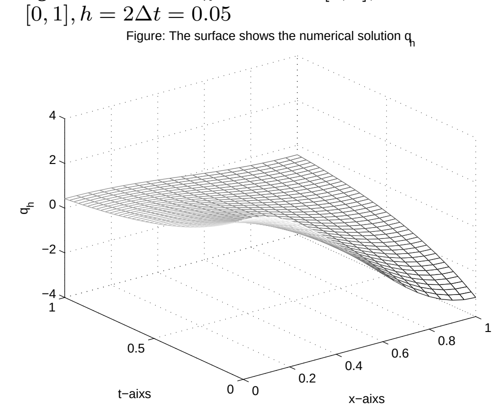

Fig. 2. the surface of q_h with $t \in [0,1], x \in [0,1], h = 2\Delta t = 0.05$

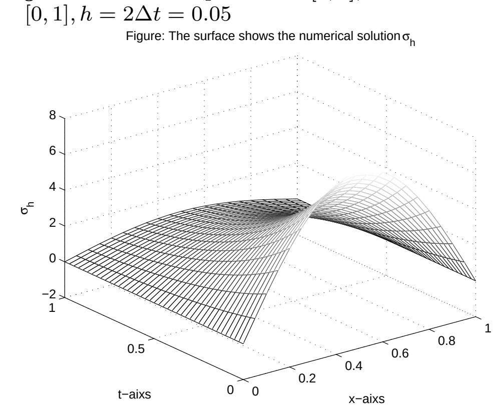

Fig. 3. the surface of σ_h with $t \in [0,1], x \in [0,1], h = 2\Delta t = 0.05$

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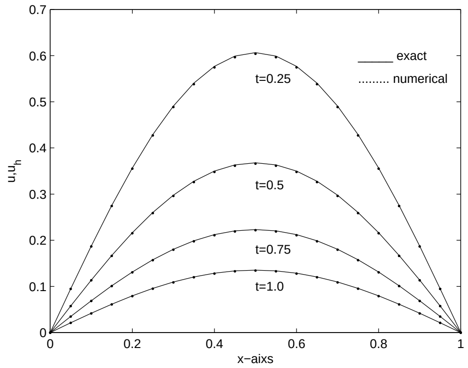

Fig. 4. Comparison between the numerical solutions u_h and the exact solutions u with $t \in [0,1], x \in [0,1], h = 2\Delta t = 0.05$

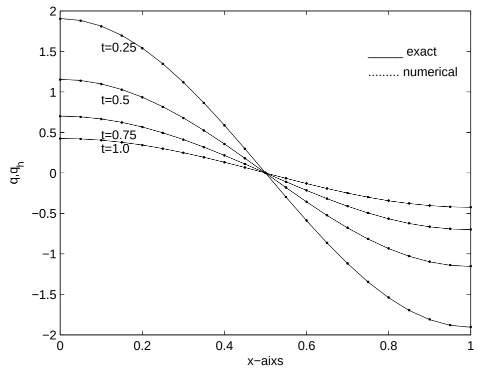

Fig. 5. Comparison between the numerical solutions q_h and the exact solutions q with $t \in [0,1], x \in [0,1], h = 2\Delta t = 0.05$

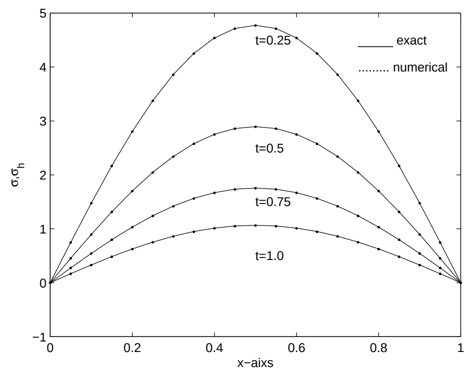

Fig. 6. Comparison between the numerical solutions σ_h and the exact solutions σ with $t \in [0,1], x \in [0,1], h = 2\Delta t = 0.05$