World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:12, 2011

I-Vague Normal Groups

Zelalem Teshome Wale

Abstract—The notions of I-vague normal groups with membership and non-membership functions taking values in an involutary dually residuated lattice ordered semigroup are introduced which generalize the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval [0, 1]. Various operations and properties are established.

Keywords—Involutary dually residuated lattice ordered semigroup, I-vague set, I-vague group and I-vague normal group.

I. Introduction

VAGUE groups are studied by M. Demirci[2]. R. Biswas[1] defined the notion of vague groups analogous to the idea of Rosenfeld [4]. He defined vague normal groups of a group and studied their properties. N. Ramakrishna[3] studied vague normal groups and introduced vague normalizer and vague centralizer.

In his paper, T. Zelalem [9] studied the concept of I-vague groups. In this paper using the definition of I-vague groups, we defined and studied I-vague normal groups where I is an involutary DRL-semigroup. To be self contained we shall recall some basic results in [5], [6], [7], [9] in this paper.

II. Dually Residuated Lattice Ordered Semigroup

Definition 2.1: [5] A system $A = (A, +, \leq, -)$ is called a dually residuated lattice ordered semigroup(in short DRL-semigroup) if and only if

i) $A = (A,\ +)$ is a commutative semigroup with zero"0";

ii) $A = (A,\ \leq)$ is a lattice such that $a + (b \cup c) = (a + b) \cup (a + c)$ and $a + (b \cap c) = (a + b) \cap (a + c)$ for all $a,\ b,\ c \in$ A;

iii) Given $a,\ b \in$ A, there exists a least x in A such that $b + x \geq a$, and we denote this x by a - b (for a given a, b this x is uniquely determined);

iv) (a - b) $\cup$ 0 + b $\leq a \cup b$ for all a, b $\in$ A;

v) a - a $\geq$ 0 for all $a \in$ A.

Theorem 2.2: [5] Any DRL-semigroup is a distributive lattice.

Definition 2.3: [10] A DRL-semigroup A is said to be involutary if there is an element $1 (\neq 0)$(0 is the identity w.r.t. +) such that

(i) $a + (1 - a) = 1 + 1$;

(ii) $1 - (1 - a) = a$ for all $a \in$ A.

Theorem 2.4: [6] In a DRL-semigroup with 1, 1 is unique.

Theorem 2.5: [6] If a DRL-semigroup contains a least element x, then x = 0. Dually, if a DRL-semigroup with 1 contains a largest element α, then α = 1.

Zelalem Teshome: Department of Mathematics, Addis Ababa University, Addis Ababa, Ethiopia.
e-mail: zelalemwale@yahoo.com or zelalem.wale@gmail.com
Manuscript received October xx, 2011; revised January 11, 2007.

Throughout this paper let $I = (I, +, -, \vee, \wedge, 0, 1)$ be a dually residuated lattice ordered semigroup satisfying $1 - (1 - a) = a$ for all $a \in$ I.

Lemma 2.6: [10] Let 1 be the largest element of I. Then for $a,\ b \in$ I

(i) a + (1- a) = 1 .

(ii) 1 - a = 1 - b $\Longleftrightarrow$ a = b.

(iii)1 - (a $\cup$ b) = (1 -a) $\cap$ (1- b).

Lemma 2.7: [10] Let I be complete. If $a_\alpha \in$ I for every $\alpha \in \Delta$, then

(i) $1 - \bigvee_{\alpha\in\Delta} a_\alpha = \bigwedge_{\alpha\in\Delta} (1 - a_\alpha)$.

(ii) $1 - \bigwedge_{\alpha\in\Delta} a_\alpha = \bigvee_{\alpha\in\Delta} (1 - a_\alpha)$.

III. I-Vague Sets

Definition 3.1: [10] An I-vague set A of a non-empty set G is a pair $(t_A,\ f_A)$ where $t_A : G \to I$ and $f_A : G \to I$ with $t_A(x) \leq 1 - f_A(x)$ for all $x \in G$.

Definition 3.2: [10] The interval $[t_A(x),\ 1 - f_A(x)]$ is called the I-vague value of $x \in G$ and is denoted by $V_A(x)$.

Definition 3.3: [10] Let $B_1 = [a_1,\ b_1]$ and $B_2 = [a_2,\ b_2]$ be two I-vague values. We say $B_1 \geq B_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 3.4: [10] An I-vague set $A = (t_A,\ f_A)$ of G is said to be contained in an I-vague set $B = (t_B,\ f_B)$ of G written as $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$ and $f_A(x) \geq f_B(x)$ for all $x \in$ G. A is said to be equal to B written as A = B if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 3.5: [10] An I-vague set A of G with $V_A(x) = V_A(y)$ for all $x,\ y \in$ G is called a constant I-vague set of G.

Definition 3.6: [10] Let A be an I-vague set of a non empty set G. Let $A_{(\alpha,\ \beta)} = \{x \in G : V_A(x) \geq [\alpha,\ \beta]\}$ where $\alpha,\ \beta \in I$ and $\alpha \leq \beta$. Then $A_{(\alpha,\ \beta)}$ is called the $(\alpha,\ \beta)$ cut of the I-vague set A.

Definition 3.7: Let S $\subseteq$G. The characteristic function of S denoted as $\chi_S = (t_{\chi_S},\ f_{\chi_S})$, which takes values in I is defined as follows:

$$t_{\chi_S}(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{\chi_S}(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{otherwise.} \end{cases}$$

χ_S is called the I-vague characteristic set of S in I. Thus

$$V_{\chi_S}(x) = \begin{cases} [1,\ 1] & \text{if } x \in S\ ; \\ [0,\ 0] & \text{otherwise.} \end{cases}$$

Definition 3.8: [10] Let $A = (t_A,\ f_A)$ and $B = (t_B,\ f_B)$ be I-vague sets of a set G.

(i) Their union $A \cup B$ is defined as $A \cup B = (t_{A\cup B},\ f_{A\cup B})$ where $t_{A\cup B}(x) = t_A(x) \vee t_B(x)$ and

$f_{A\cup B}(x) = f_A(x) \wedge f_B(x)$ for each $x \in$ G.
(ii) Their intersection $A \cap B$ is defined as $A \cap B = (t_{A\cap B},\ f_{A\cap B})$ where $t_{A\cap B}(x) = t_A(x) \wedge t_B(x)$ and $f_{A\cap B}(x) = f_A(x) \vee f_B(x)$ for each $x \in$ G.

Definition 3.9: [10] Let $B_1 = [a_1,\ b_1]$ and $B_2 = [a_2,\ b_2]$ be I-vague values. Then
(i) isup$\{B_1,\ B_2\} = [\sup\{a_1,\ a_2\}, \sup\{b_1,\ b_2\}]$.
(ii) iinf$\{B_1,\ B_2\} = [\inf\{a_1,\ a_2\}, \inf\{b_1,\ b_2\}]$.

Lemma 3.10: [10] Let A and B be I-vague sets of a set G. Then $A \cup B$ and $A \cap B$ are also I-vague sets of G.
Let $x \in$ G. From the definition of $A \cup B$ and $A \cap B$ we have
(i) $V_{A\cup B}(x) =$ isup$\{V_A(x),\ V_B(x)\}$;
(ii) $V_{A\cap B}(x) =$ iinf$\{V_A(x),\ V_B(x)\}$.

Definition 3.11: [10] Let I be complete and $\{A_i$: i $\in \triangle\}$ be a non empty family of I-vague sets of G where
$A_i = (t_{A_i},\ f_{A_i})$. Then
(i) $\bigcap\limits_{i\in\triangle} A_i = (\bigwedge\limits_{i\in\triangle} t_{A_i},\ \bigvee\limits_{i\in\triangle} f_{A_i})$
(ii) $\bigcup\limits_{i\in\triangle} A_i = (\bigvee\limits_{i\in\triangle} t_{A_i},\ \bigwedge\limits_{i\in\triangle} f_{A_i})$

Lemma 3.12: [10] Let I be complete. If $\{A_i$: i $\in \triangle\}$ is a non empty family of I-vague sets of G, then $\bigcap\limits_{i\in\triangle} A_i$ and $\bigcup\limits_{i\in\triangle} A_i$ are I-vague sets of G.

Definition 3.13: [10] Let I be complete and
$\{A_i = (t_{A_i},\ f_{A_i}) :\ i \in \triangle\}$ be a non empty family of I vague sets of G. Then for each $x \in$ G,
(i) isup$\{V_{A_i}(x) :\ i \in \triangle\} = [\bigvee\limits_{i\in\triangle} t_{A_i}(x),\ \bigvee\limits_{i\in\triangle} (1 - f_{A_i})(x)]$.
(ii) iinf$\{V_{A_i}(x) :\ i \in \triangle\} = [\bigwedge\limits_{i\in\triangle} t_{A_i}(x),\ \bigwedge\limits_{i\in\triangle} (1 - f_{A_i})(x)]$.

IV. I-Vague Groups

Definition 4.1: [9] Let G be a group. An I-vague set A of a group G is called an I-vague group of G if
(i) $V_A(xy) \geq$ iinf$\{V_A(x), V_A(y)\}$ for all $x,\ y \in G$ and
(ii) $V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$.

Lemma 4.2: [9] If A is an I-vague group of a group G, then $V_A(x) = V_A(x^{-1})$ for all $x \in G$.

Lemma 4.3: [9] If A is an I-vague group of a group G, then $V_A(e) \geq V_A(x)$ for all $x \in G$.

Lemma 4.4: [9] A necessary and sufficient condition for an I-vague set A of a group G to be an I-vague group of G is that $V_A(xy^{-1}) \geq$ iinf$\{V_A(x),\ V_A(y)\}$ for all $x,\ y \in G$.

Lemma 4.5: [9] Let H be a subgroup of G and $[\gamma,\ \delta] \leq [\alpha,\ \beta]$ with $\alpha, \beta, \gamma, \delta \in$ I where $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set A of G defined by

$$V_A(x) = \begin{cases} [\alpha,\ \beta] & \text{if } x \in H \\ [\gamma,\ \delta] & \text{otherwise} \end{cases}$$

is an I-vague group of G.

Lemma 4.6: [9] Let H $\neq \emptyset$ and $H \subseteq G$. The I-vague characterstic set of H, χ_H is an I-vague group of G iff H is a subgroup of G.

Lemma 4.7: [9] If A and B are I-vague groups of a group G, then $A \cap B$ is also an I-vague group of G.

Lemma 4.8: [9] Let I be complete. If $\{A_i$: i $\in \triangle\}$ is a non empty family of I-vague groups of G, then $\bigcap\limits_{i\in\triangle} A_i$ is an I-vague group of G.

Lemma 4.9: [9] Let A be an I-vague group of G and B be a constant I-vague group of G. Then A∪B is an I-vague group of G.

Theorem 4.10: [9] An I-vague set A of a group G is an I-vague group of G if and only if for all $\alpha,\ \beta \in$ I with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha,\ \beta)}$ is a subgroup of G whenever it is non empty.

Theorem 4.11: [9] Let A be an I-vague group of a group G. If $V_A(xy^{-1}) = V_A(e)$ for $x,\ y \in G$, then $V_A(x) = V_A(y)$.

Lemma 4.12: [9] Let A be an I-vague group of a group G. Then $GV_A = \{x \in G :\ V_A(x) = V_A(e)\}$ is a subgroup of G.

V. I-Vague Normal Groups

Definition 5.1: Let G be a group. An I-vague group A of a group G is called an I-vague normal group of G if for all $x,\ y \in G$, $V_A(xy) = V_A(yx)$.
If the group G is abelian, then every I-vague group of G is an I-vague normal group of G.

Lemma 5.2: Let A be an I-vague group of a group G. A is an I-vague normal group of G if and only if $V_A(x) = V_A(yxy^{-1})$ for all $x,\ y \in G$.

Proof: Let A be an I-vague group of a group G.
Suppose that A is an I-vague normal group of G.
Let $x,\ y \in G$. Then
$V_A(x) = V_A(xy^{-1}y) = V_A(yxy^{-1})$. Thus
$V_A(x) = V_A(yxy^{-1})$.
Conversely, suppose that $V_A(x) = V_A(yxy^{-1})$ for all $x,\ y \in G$.
Then $V_A(xy) = V_A(y(xy)y^{-1}) = V_A(yx)$.
We have $V_A(xy) = V_A(yx)$. Hence the lemma follows.

Lemma 5.3: Let H be a normal subgroup of G and $[\gamma,\ \delta] \leq [\alpha,\ \beta]$ for $\alpha,\ \beta,\ \gamma,\ \delta \in$ I with $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set A of G defined by

$$V_A(x) = \begin{cases} [\alpha,\ \beta] & \text{if } x \in H \\ [\gamma,\ \delta] & \text{otherwise} \end{cases}$$

is an I-vague normal group of G.

Proof: Let H be a normal subgroup of G. By lemma 4.5, A is an I-vague group of G.
We show that $V_A(x) = V_A(yxy^{-1})$ for every $x,\ y \in G$.
Let $x,\ y \in G$.
If $x \in H$, then $yxy^{-1} \in H$. Thus $V_A(x) = V_A(yxy^{-1})$.
If $x \notin H$, then $yxy^{-1} \notin H$. Thus $V_A(x) = V_A(yxy^{-1})$.
Hence $V_A(x) = V_A(yxy^{-1})$ for every $x,\ y \in G$.
Therefore A is an I-vague normal group of G.

Lemma 5.4: Let H $\neq \emptyset$. The I-vague characteristic set of H, χ_H is an I-vague normal group of a group G iff H is a normal subgroup of G.

Proof: Suppose that H is a normal subgroup of G. By Lemma 5.3, χ_H is an I-vague normal group of G since

$$V_{\chi_H}(x) = \begin{cases} [1,\ 1] & \text{if } x \in H \\ [0,\ 0] & \text{otherwise.} \end{cases}$$

Conversely, suppose that χ_H is an I-vague normal group of G.
We show that H is a normal subgroup of G.
By lemma 4.6, H is a subgroup of G. Let $y \in$ H and $x \in$ G.

Now we prove that $xyx^{-1} \in$H.
$V_{\chi_H}(xyx^{-1}) = V_{\chi_H}(y) = [1,\ 1]$. This implies $xyx^{-1} \in$ H.
It follows that H is a normal subgroup of G.
Hence the lemma holds true.

Theorem 5.5: If A and B are I-vague normal groups of G, then $A \cap B$ is also an I-vague normal group of G.

Proof: If A and B are I-vague groups of a group G, then $A\cap B$ is also an I-vague group of G by lemma 4.7.
Now it remains to show that $V_{A\cap B}(xy) = V_{A\cap B}(yx)$ for every $x,\ y \in$ G. Let $x, y \in$ G. Then
$V_{A\cap B}(xy)$ = iinf$\{V_A(xy),\ V_B(xy)\}$
= iinf$\{V_A(yx),\ V_B(yx)\}$
$=\ V_{A\cap B}(yx)$.
Hence $V_{A\cap B}(xy) = V_{A\cap B}(yx)$ for each $x,\ y \in$ G.
Therefore $A \cap B$ is an I-vague normal group of G.

Lemma 5.6: Let I be complete. If $\{A_i$: i $\in \triangle\}$ is a non empty family of I-vague normal groups of G, then $\bigcap_{i\in\triangle} A_i$ is an I-vague normal group of G.

Proof: Let A = $\bigcap_{i\in\triangle} A_i$. Then A is an I-vague group of G by lemma 4.8.
Now we prove that $V_A(xyx^{-1}) = V_A(y)$ for every $x,\ y \in$ G.
Let $x,\ y \in$ G. Then
$V_A(xyx^{-1})$= iinf$\{V_{A_i}(xyx^{-1}) :\ i \in \triangle\}$
= iinf$\{V_{A_i}(y) :\ i \in \triangle\}$
= $V_A(y)$
Therefore $\bigcap_{i\in\triangle} A_i$ is an I-vague normal group of G.

Lemma 5.7: Let A be an I-vague normal group of G and B be a constant I-vague group of G. Then A∪B is an I-vague normal group of G.

Proof: Let A be an I-vague normal group of G and B be a constant I-vague group of G. Hence $V_B(x) = V_B(y)$ for all $x,\ y \in$ G. By lemma 4.9, A∪B is an I-vague group of G.
For each $x,\ y \in$ G,
$V_{A\cup B}(yxy^{-1})$= isup$\{V_A(yxy^{-1}),\ V_B(yxy^{-1})\ \}$
= isup$\{V_A(x),\ V_B(x)\}$
= $V_{A\cup B}(x)$
Hence $V_{A\cup B}(yxy^{-1})$ = $V_{A\cup B}(x)$ for every $x,\ y \in$ G.
Therefore A∪B is an I-vague normal group of G.

Remark Even if $V_{A\cup B}(xyx^{-1}) = V_{A\cup B}(y)$ for I-vague normal groups A and B, A∪B is not be an I-vague group of G as we have seen in I-vague groups[9].

Theorem 5.8: An I-vague set A of a group G is an I-vague normal group of G if and only if for all $\alpha,\ \beta \in$ I with $\alpha \le \beta$, the I-vague cut $A_{(\alpha,\ \beta)}$ is a normal subgroup of G whenever it is non-empty.

Proof: By theorem 4.10, an I-vague set A of a group G is an I-vague group of G if and only if for all $\alpha, \beta \in$ I with $\alpha \le \beta$, the I-vague cut $A_{(\alpha,\beta)}$ is a subgroup of G whenever it is non-empty.
Suppose that A is an I-vague normal group of G.
Consider $A_{(\alpha,\ \beta)}$. Let $y \in A_{(\alpha,\ \beta)}$ and $x \in$ G. We prove that $xyx^{-1} \in A_{(\alpha,\ \beta)}$.
$y \in A_{(\alpha,\beta)}$ implies $V_A(y) \ge [\alpha,\ \beta]$. Since $V_A(y) =$
$V_A(xyx^{-1})$, $V_A(xyx^{-1}) \ge [\alpha,\ \beta]$. Hence $xyx^{-1} \in A_{(\alpha,\ \beta)}$, so $A_{(\alpha,\ \beta)}$ is a normal subgroup of G.
Conversely, suppose that for all $\alpha,\ \beta \in$ I with $\alpha \le \beta$, the non empty set $A_{(\alpha,\ \beta)}$ is a normal subgroup of G.
Now it remains to prove that $V_A(y) = V_A(xyx^{-1})$ for all $x,\ y \in G$. Suppose that $V_A(y) = [\alpha,\ \beta]$. Then $y \in A_{(\alpha,\ \beta)}$. Since $A_{(\alpha,\ \beta)}$ is a normal subgroup of G, $xyx^{-1} \in A_{(\alpha,\ \beta)}$. It follows that $V_A(xyx^{-1}) \ge [\alpha,\ \beta] = V_A(y)$ for all $x \in$ G. Hence $V_A(xyx^{-1}) \ge V_A(y)$ for all $x \in$ G. This implies $V_A(x^{-1}yx) \ge V_A(y)$ for all $x,\ y \in G$. Put xyx^{-1} instead of y. Hence $V_A(x^{-1}(xyx^{-1})x) \ge V_A(xyx^{-1})$, so $V_A(y) \ge V_A(xyx^{-1})$. Consequently, $V_A(xyx^{-1}) = V_A(y)$ for all $x,\ y \in G$. Thus A is an I-vague normal group of G. Hence the theorem follows.

Theorem 5.9: If A is an I-vague normal group of G, then GV_A is a normal subgroup of G.

Proof: We prove that GV_A is a normal subgroup of G.
By lemma 4.12, $GV_A = \{x \in G : V_A(x) = V_A(e)\}$ is a subgroup of G. Now we show that $xyx^{-1} \in GV_A$ for $x \in$ G and $y \in GV_A$. Since A is an I-vague normal group of G, $V_A(xyx^{-1}) = V_A(y)$. $y \in GV_A$ implies $V_A(y) = V_A(e)$.
Hence $V_A(xyx^{-1}) = V_A(e)$, so $xyx^{-1} \in GV_A$.
Thus GV_A is a normal subgroup of G.

Theorem 5.10: If A is an I-vague group of a group G and B is an I-vague normal group of G, then $A \cap B$ is an I-vague normal group of GV_A.

Proof: GV_A is a subgroup of G because A is an I-vague group of G. Since A and B are I-vague groups of G, it follows that $A \cap B$ is an I-vague group of G by lemma 4.7. So $A \cap B$ is an I-vague group of GV_A. Now we prove that $V_{A\cap B}(xy) = V_{A\cap B}(yx)$ for all $x,\ y \in GV_A$.
Let $x,\ y \in GV_A$. Then $xy,\ yx \in GV_A$. Hence
$V_A(xy) = V_A(yx) = V_A(e)$. $V_B(xy) = V_B(yx)$ because B is an I-vague normal group of G.
$V_{A\cap B}(xy)$ = iinf$\{V_A(xy),\ V_B(xy)\}$ = iinf$\{V_A(yx),\ V_B(yx)\}$
$=\ V_{A\cap B}(yx)$. It follows that $V_{A\cap B}(xy) = V_{A\cap B}(yx)$ for every $x,\ y \in GV_A$. Therefore $A \cap B$ is an I-vague normal group of GV_A.

Theorem 5.11: Let A be an I-vague group of G. Then A is an I-vague normal group of G iff $V_A([x,\ y]) \ge V_A(x)$ for all $x,\ y \in$ G.

Proof: Let A be an I-vague group of G.
Suppose that A is an I-vague normal group of G.
We prove that $V_A([x,\ y]) \ge V_A(x)$ for $x,\ y \in$ G.
Let $x,\ y \in$ G. Then
$V_A([x,\ y]) = V_A(x^{-1}(y^{-1}xy))$
$\ge$ iinf$\{V_A(x^{-1}),\ V_A(y^{-1}xy)\}$
= iinf$\{V_A(x),\ V_A(x)\}$ since A is an I-vague normal
$= V_A(x)$ group of G.
Hence $V_A([x,\ y]) \ge V_A(x)$.
Conversely, suppose that $V_A([x,\ y]) \ge V_A(x)$ for all $x,\ y \in$ G. We prove that A is an I-vague normal group of G.
Let $x,\ z \in$ G. Then
$V_A(x^{-1}zx) = V_A(ex^{-1}zx)$
$= V_A(zz^{-1}x^{-1}zx)$
$= V_A(z[z,\ x])$
$\ge$iinf$\{V_A(z),\ V_A([z,\ x])\}$
$= V_A(z)$ by our supposition.
Hence $V_A(x^{-1}zx) \ge V_A(z)$ for $x,\ z \in$ G. It implies $V_A(xzx^{-1}) \ge V_A(z)$ for $x,\ z \in$ G. Instead of z put $x^{-1}zx$. Then we get $V_A(z) \ge V_A(x^{-1}zx)$.

Thus $V_A(z) = V_A(x^{-1}zx)$ for every $x,\ z \in$ G.
Therefore A is an I-vague normal group of G.
Hence the theorem follows.

Definition 5.12: : Let A be an I-vague group of a group G. Then the set
$N(A) = \{a \in G : V_A(axa^{-1}) = V_A(x)$ for all $x \in G\}$ is called an I-vague normalizer of A.

Theorem 5.13: Let A be an I-vague group of G. Then
(i) A is an I-vague normal group of N(A).
(ii) I-vague normalizer N(A) is a subgroup of G .
(iii) A is an I-vague normal group of G iff N(A)= G.

Proof: Let A be an I-vague group of G.
(i) We prove that A is an I-vague normal group of N(A). Let $x,\ a \in$ N(A).
By definition, $V_A(axa^{-1}) = V_A(x)$ for all $x,\ a \in$ N(A).
Thus A is an I-vague normal group of N(A).
(ii) Let $a,\ b \in$ N(A). We show that $a^{-1} \in$ N(A) and $ab \in$ N(A).
Let $a \in$ N(A). Then $V_A(axa^{-1}) = V_A(x)$ for all $x \in$ G.
$V_A(x) = V_A(a(a^{-1}xa)a^{-1}) = V_A(a^{-1}xa)$.
Hence $V_A(a^{-1}xa) = V_A(x)$, so $a^{-1} \in$ N(A).
Let $a,\ b \in$ N(A). Then
$V_A(axa^{-1}) = V_A(x)$ and $V_A(bxb^{-1}) = V_A(x)$ for all $x \in$ G.
Then $V_A(abx(ab)^{-1}) = V_A(a(bxb^{-1})a^{-1}) = V_A(bxb^{-1}) = V_A(x)$.
Thus $ab \in$ N(A). Therefore N(A) is a subgroup of G.
(iii) Suppose that A is an I-vague normal group of G.
We prove that N(A) = G.
Let $a \in$ G. Since A is an I-vague normal group of G, $V_A(axa^{-1}) = V_A(x)$ for all $x \in$ G. It follows that $a \in$ N(A).
Hence $G \subseteq$N(A).
Since N(A)$\subseteq$ G, G = N(A).
Conversely, assume that N(A) = G. For all $a,\ x \in$ G,
$V_A(axa^{-1}) = V_A(x)$.
By definition, A is an I-vague normal group of G.

Theorem 5.14: Let A be an I-vague group of a group G. Then GV_A is a normal subgroup of $N(A)$.

Proof: Let A be an I-vague group of G. We prove that GV_A is a normal subgroup of $N(A)$.
First we prove that $GV_A \subseteq N(A)$.
Let $x \in GV_A$. Then $x \in GV_A,\ V_A(x) = V_A(e)$.
For $y \in$ G, $V_A(xyx^{-1}) \geq$ iinf$\{V_A(x),\ V_A(yx^{-1})\}$
$\geq$ iinf$\{V_A(x),\ V_A(y)\}$
= iinf$\{V_A(e),\ V_A(y)\}$
$= V_A(y)$.
Hence $V_A(xyx^{-1}) \geq V_A(y)$ for $y \in$ G and $x \in GV_A$.
$x \in GV_A$ implies $x^{-1} \in GV_A$. Thus $V_A(x^{-1}yx) \geq V_A(y)$ where $x \in GV_A$ and $y \in$G. Put xyx^{-1} instead of y.
We have $V_A(x^{-1}(xyx^{-1})x) \geq V_A(xyx^{-1})$ and hence
$V_A(y) \geq V_A(xyx^{-1})$.
Therefore $V_A(y) = V_A(xyx^{-1})$ for each $y \in$ G.
Thus $x \in N(A)$. Therefore $GV_A \subseteq N(A)$.
Since GV_A is a subgroup of G and $GV_A \subseteq N(A)$, GV_A is a subgroup of N(A).
Now we show that $yay^{-1} \in GV_A$ for all $a \in GV_A$ and for all $y \in$ N(A).
Since $y \in$ N(A), $V_A(yay^{-1}) = V_A(a)$. Since $a \in GV_A$, $V_A(a) = V_A(e)$. Hence $V_A(yay^{-1}) = V_A(e)$, so $yay^{-1} \in GV_A$. Therefore GV_A is a normal subgroup of $N(A)$.

Definition 5.15: Let A be an I-vague group of a group G. Then the set
$C(A) = \{a \in G : V_A([a,\ x]) = V_A(e)$ for all $x \in G\}$ is called an I-vague centralizer of A.

Theorem 5.16: Let A be an I-vague group of a group G. Then $C(A)$ is a normal subgroup of G.

Proof: Let A be an I-vague group of G. We prove that $C(A) = \{a \in G : V_A([a,\ x]) = V_A(e)$ for all $x \in G\}$ is a normal subgroup of G.
Step(1) We show that $a \in$ C(A) implies $V_A(xa) = V_A(ax)$ for all $x \in$ G.
Let $a \in$ C(A). Then $V_A([a,\ x]) = V_A(e)$ for all $x \in$ G.
$V_A([a,\ x]) = V_A(e) \Rightarrow V_A(a^{-1}x^{-1}ax) = V_A(e)$
$\Rightarrow V_A((xa)^{-1}ax) = V_A(e)$
$\Rightarrow V_A((xa)^{-1}((ax)^{-1})^{-1}) = V_A(e)$
$\Rightarrow V_A((xa)^{-1}) = V_A((ax)^{-1})$ by thm 4.11
$\Rightarrow V_A(xa) = V_A(ax)$.
Therefore $V_A(xa) = V_A(ax)$ for all $x \in$ G.
Step(2) We show that $a \in$C(A) implies $V_A([x,\ a]) = V_A(e)$ for all $x \in G$.
$V_A([x,\ a]) = V_A(x^{-1}a^{-1}xa) = V_A((x^{-1}a^{-1}xa)^{-1}) = V_A(a^{-1}x^{-1}ax) = V_A([a,\ x]) = V_A(e)$
Hence $V_A([x,\ a]) = V_A(e)$ for each $a \in$ C(A) and for all $x \in$ G.
Step(3) We prove that $C(A)$ is a subgroup of G.
We show that (i) $a \in$ C(A) implies $a^{-1} \in C(A)$.
(ii) $a,\ b \in$ C(A) implies $ab \in$ C(A).
Now proof of (i)
For all $x \in$ G, $V_A([a^{-1},\ x]) = V_A(ax^{-1}a^{-1}x)$
$= V_A(x^{-1}a^{-1}xa)$ by step (1)
$= V_A((x^{-1}a^{-1}xa)^{-1})$
$= V_A(a^{-1}x^{-1}ax)$
$= V_A([a,\ x])$
$= V_A(e)$.
Thus $V_A([a^{-1},\ x]) = V_A(e)$ for all $x \in$ G.
Hence we have that $a^{-1} \in C(A)$.
Proof of (ii) Let $a,\ b \in$ C(A). Then $V_A([a,\ x]) = V_A([b,\ x]) = V_A(e)$ for all $x \in$ G.
$V_A([ab,\ x]) = V_A((ab)^{-1}x^{-1}(ab)x)$
$= V_A(b^{-1}(a^{-1}x^{-1}abx))$
$= V_A((a^{-1}x^{-1}abx)b^{-1})$ by step(1)
$= V_A((a^{-1}x^{-1}ax)(x^{-1}bxb^{-1})$
$= V_A([a,\ x][x,\ b^{-1}])$.
$\geq$ iinf$\{V_A([a,\ x]),\ V_A([x, b^{-1}])\}$
= iinf$\{V_A(e),\ V_A(e)\}$ since $b^{-1} \in$ C(A)
$= V_A(e)$.
This implies $V_A([ab,\ x]) \geq V_A(e)$ for all $x \in$ G.
Since $V_A(e) \geq V_A([ab,\ x])$, $V_A([ab,\ x]) = V_A(e)$ for all $x \in$ G. Hence $ab \in$ C(A).
From (i) and (ii) C(A) is a subgroup of G.
Step(4) Now we show that $g^{-1}ag \in$C(A) for all $a \in$ C(A) and for all $g \in$ G.
That is $V_A([g^{-1}ag,\ x]) = V_A(e)$ for all $g,\ x \in$ G and for all $a \in$ C(A).
$V_A([g^{-1}ag,\ x]) = V_A((g^{-1}ag)^{-1}x^{-1}g^{-1}agx)$
$=V_A(g^{-1}a^{-1}gx^{-1}g^{-1}agx)$

$=V_A(g^{-1}a^{-1}gaa^{-1}x^{-1}g^{-1}agx)$
$=V_A([g,\ a]a^{-1}(gx)^{-1}agx)$
$=V_A([g,\ a][a,\ gx])$
$\geq$iinf$\{V_A([g,\ a]),\ V_A([a,\ gx])\}$
$=$iinf$\{V_A(e),\ V_A(e)\}$
$= V_A(e).$

Hence $V_A([g^{-1}ag,\ x]) \geq V_A(e)$.
Since $V_A(e) \geq V_A([g^{-1}ag,\ x]),\ V_A([g^{-1}ag,\ x]) = V_A(e)$.
This implies $g^{-1}ag \in C(A)$.
From step(3) and step(4), we have C(A) is a normal subgroup of G.

Theorem 5.17: Let A be an I-vague normal group of a group G. Then GV_A is a subgroup of C(A).

Proof: Let A be an I-vague normal group of a group G. We prove that GV_A is a subgroup of C(A).
Let $x \in GV_A$. Then $V_A(x) = V_A(e)$. Consider $V_A([x,\ y])$ for each $y \in$ G.

$V_A([x,\ y]) = V_A(x^{-1}(y^{-1}xy)) \geq$iinf$\{V_A(x^{-1}),\ V_A(y^{-1}xy)\}$
$=$ iinf$\{V_A(x),\ V_A(x)\}$
$= V_A(x)$
$= V_A(e).$

Hence $V_A([x,\ y]) \geq V_A(e)$.
Since $V_A(e) \geq V_A([x,y]),\ V_A([x,y]) = V_A(e)$.
By the definition of C(A), $x \in$C(A).
Thus $GV_A \subseteq C(A)$. Since GV_A is a subgroup of G, GV_A is a subgroup of C(A).

Acknowledgment

The author would like to thank Prof. K. L. N. Swamy and Prof. P. Ranga Rao for their valuable suggestions and discussions on this work.

References

[1] R. Biswas, Vague groups, International Journal of Computational Cognition, vol. 4(2), 2006, pp. 20-23. Systems, vol. 79, 1996, pp. 403-405.
[2] M. Demirci, Vague groups, Jou. Math. Anal. Appl., vol. 230, 1999, pp. 142-156.
[3] N. Ramakrishna, A Study of Vague Groups, Vague Universal Algebras and Vague Graphs, Doctoral Thesis, Andhra University, Visakhapatnam, India, March 2009.
[4] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. vol.35, 1971, pp. 512-517.
[5] K. L. N. Swamy, Dually residuated lattice ordered semigroups, Math. Annalen, vol. 159, 1965, pp. 105 -114.
[6] K. L. N. Swamy, Dually residuated lattice ordered semigroups II, Math. Annalen, vol. 160, 1965, pp. 64 -71.
[7] K. L. N. Swamy, Dually residuated lattice ordered semigroups III, Math. Annalen, vol. 167, 1966 , pp.71-74.
[8] L. A. Zadeh, Fuzzy sets, Information and Control, vol. 8, 1965, pp. 338-353.
[9] T. Zelalem, I-Vague Groups, submitted for publication to the International Journal of Computational and Mathematical Sciences.
[10] T. Zelalem, A Theory of I-Vague Sets, Doctoral Thesis, Andhra University, Vishakapatnam, India, July, 2010.