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The error analysis of an upwind difference approximation for a singularly perturbed problem

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Abstract—An upwind difference approximation is used for a singularly perturbed problem in material science. Based on the discrete Green's function theory, the error estimate in maximum norm is achieved, which is first-order uniformly convergent with respect to the perturbation parameter. The numerical experimental result is verified the valid of the theoretical analysis.

Keywords—singularly perturbed, upwind difference, uniform convergence.

I. Introduction

SINGULARLY perturbed problems have been used to describe pattern formation in developmental material science. We consider the following singularly perturbed problem

$$Lu\,(x) := -\varepsilon\, u''(x) - p(x)u'(x) = f(x) \tag{1}$$

for $x \in (0,1)$ with the boundary condition

$$u(0) = 0, \quad u(1) = 0, \tag{2}$$

where ε is a constant of diffusion satisfying $0 < \varepsilon \leq 1$. We assume that f is sufficiently smooth. It is also assumed that $p \in C^1[0,1]$ and that there are constants β and $\overline{\beta}$ such that

$$0 < \beta \leq p(x) \leq \overline{\beta}, \quad \text{and} \quad |p'(x)| \leq \overline{\beta}, \quad \forall x \in [0,1].$$

From [1], we know that the following result holds true for the solution of the problem (1)-(2):

$$\left|u^{(k)}(x)\right| \leq C\left(1 + \varepsilon^{-k}\, e^{-\beta\, x/\varepsilon}\right), \quad k = 0,1,2. \tag{3}$$

Combining the estimate (3) with the equations (1)-(2), we will get a similar estimate of the third derivative of the solution

$$\left|u^{'''}(x)\right| \leq C\left(1 + \varepsilon^{-3}\, e^{-\beta\, x/\varepsilon}\right). \tag{4}$$

The problem (1)-(2) has a steep layer of order $\mathcal{O}(\varepsilon)$ at the left-hand boundary $x = 0$. It is very hard to approximate efficiently by most numerical methods on an even grid. To approximate the problem (1)-(2) reliably when $\varepsilon \ll 1$, we construct a nonuniform mesh that concentrates nodes in the boundary layer by equidistributing a monitor function ([2], [3], [4], [5], [6], [7]) over the domain of the problem.

In [6] and [7], uniform convergence estimates of order $\gamma \in (0,1)$ of the upwind difference scheme basing on a priori estimates of the solution of problem are derived. In [2] a priori estimates and discrete Green's function are used to improve the accuracy that writes $\max\limits_{0\leq i\leq N} |u(x_i) - u_i^N| \leq CN^{-1}lnN$. In the present work, we analyze the error in the maximum norm of the upwind difference scheme basing on a priori estimates and the discrete Green's function and improve the convergence order to 1.

This paper is organized as follows. In section II, the discretised difference scheme and the mesh are given. In section III, we study the discrete Green's function of the operator. Section IV is for the error analysis. The numerical experiment is given in section V. The last part is the conclusions.

Throughout the paper, C, sometimes subscripted, denotes a generic positive constant that is independent of ε and any mesh used.

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II. An upwind difference method and the mesh

Let $\omega = \{x_0, x_1, \ldots, x_N\}$ be an arbitrary mesh, where $0 = x_0 < x_1 < \ldots < x_N = 1$. The mesh sizes are $h_i = x_i - x_{i-1}, i = 1, 2, \cdots, N$. Then an upwind difference discretization of (1)-(2) is :

$$L^N\, u_i^N \; := -\varepsilon\, D^+\, D^- u_i^N \; - \; p_i\, D^+ u_i^N \; = \; f_i, \tag{5}$$

for $1 \leq i \leq N-1$ with

$$u_0^N = 0, \quad u_N^N = 0, \tag{6}$$

where $p_i = p(x_i)$, $f_i = f(x_i)$, $\{u_i^N\}$ is the solution computed on the mesh $\{x_i\}$ and

$$D^+ u_i^N = \frac{u_{i+1}^N - u_i^N}{h_{i+1}}, \quad D^- u_i^N = \frac{u_i^N - u_{i-1}^N}{h_i},$$
$$D^+ D^- u_i^N = \frac{2}{h_i + h_{i+1}}(D^+ u_i^N - D^- u_i^N).$$

The numerical mesh is constructed by equidistributing the standard arc-length function

$$M(u(x), x) = \sqrt{1 \,+\, [u'(x)]^2}$$

over the domain $[0,1]$. This gives rise to a mapping $x = x(\xi)$:

$$\frac{dx}{d\xi} = \frac{L}{\sqrt{1 \,+\, [u'(x)]^2}}, \quad \xi \in (0,1),$$

where L is the total arc-length of $u(x)$ over $[0,1]$. That is to say, mesh points are given by

$$x_i \;\; = \;\; \int_0^{\xi} \frac{L}{\sqrt{1 \,+\, [u'(x)]^2}} d\xi, \quad \xi_i = \frac{i}{N} \tag{7}$$

for $0 \leq i \leq N$.

Thus, the solution of the difference equations (5)-(6) on the mesh (7) produces the numerical approximation to (1)-(2).

III. The discrete Green's function of the difference operator

First, we introduce the following comparison principle.

Lemma 3.1: If $L^N v_i \le L^N w_i, \quad 1 \le i \le N-1$ and $v_0 \le w_0, v_N \le w_N$, then $v_i \le w_i, \quad 0 \le i \le N$.

Proof. The matrix associated with the discretised operator L^N is diagonally dominant and has non-positive off diagonal terms. Hence the matrix is M-matrix, and it has a positive inverse. Thus, $v_i \le w_i, \quad 0 \le i \le N$.

Next, for $j = 1, \cdots, N-1$, we define the discrete Green's function $G(x_i, x_j)$ with respect to the difference operator L^N (with the Dirichlet boundary condition) associated with the point x_j by

$$L^N G(x_i, x_j) = \frac{\delta_{ij}}{h_{i+1}}, \quad i = 1, \cdots, N-1,$$
$$G(0, x_j) = G(1, x_j) = 0,$$

where the Kronecker function δ_{ij} is 1 if $i = j$ and 0 otherwise. Then for each i, we have

$$u_i^N = \sum_{j=1}^{N-1} h_{j+1} G(x_i, x_j) f_j.$$

The comparison principle of Lemma 3.1 with the test function $v = 0$ and

$$w = \frac{2}{\beta} \begin{cases} 1 & \text{if} \quad 0 \le i \le j \le N \\ \prod_{k=j+1}^{i} \left(1 + \frac{\beta h}{2\varepsilon}\right)^{-1} & \text{if} \quad 0 \le j < i \le N \end{cases}$$

yield the following property (see [8], [9] for more details) of the discrete Green's function:

$$0 \le G(x_i, x_j) \le 2/\beta, \quad 1 \le i \le N, \quad 1 \le j \le N-1. \quad (8)$$

IV. The error analysis

During the analysis, we need to introduce the local truncation error. The local truncation error of (5) at the node x_i $(i = 1, 2, \cdots, N-1)$ is defined as:

$$\begin{aligned} \tau_i &= L^N u_i^N - L\, u(x_i) \\ &= -\varepsilon(D^+ D^- u_i - u''(x_i)) - p_i(D^+ u_i - u'(x_i)). \end{aligned}$$

It is easily shown that this reduces to

$$\begin{aligned} \tau_i = & -\frac{\varepsilon}{h_i + h_{i+1}} \Big\{ \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} (s - x_{i+1})^2 u'''(s) ds \\ & -\frac{1}{h_i} \int_{x_{i-1}}^{x_i} (s - x_{i-1})^2 u'''(s) ds \Big\} \\ & + \frac{p_i}{h_{i+1}} \int_{x_i}^{x_{i+1}} (s - x_{i+1}) u''(s) ds, \end{aligned}$$

from which we obtain the bound

$$|\tau_i| \le \varepsilon \int_{x_{i-1}}^{x_{i+1}} |u'''(s)| ds + \overline{\beta} \int_{x_{i-1}}^{x_{i+1}} |u''(s)| ds.$$

If we invoke the derivative bounds given in (4), the above estimate may be simplified to

$$|\tau_i| \le C \int_{x_{i-1}}^{x_{i+1}} |u''(s)| ds. \quad (9)$$

We are now in a position to derive an error estimate as follows.

Lemma 4.1:

$$\sum_{i=0}^{N-1} h_{i+1} |\tau_i| \le C \max_{i=0,1,\cdots,N-1} \int_{x_i}^{x_{i+1}} \sqrt{1 + [u'(x)]^2} dx. \quad (10)$$

Proof. (i) $\max_i h_{i+1} > \varepsilon$: By (3), we have

$$\begin{aligned} \sum_{i=0}^{N-1} h_{i+1} |\tau_i| \le & \sum_{i=0}^{N-1} \Big[h_{i+1} \frac{\varepsilon}{h_i} |D^+ u_i - D^- u_i| \\ & + p_i h_{i+1} \frac{|u(x_{i+1}) - u(x_i)|}{h_{i+1}} \\ & + \varepsilon h_{i+1} |u''(x_i)| + p_i h_{i+1} |u'(x_i)| \Big] \\ \le & \sum_{i=0}^{N-1} \Big[2h_{i+1} \max_{x_{i-1} \le x \le x_{i+1}} |u'(x)| \\ & + p_i \big(|u(x_{i+1})| + |u(x_i)| \big) + h_{i+1} |u'(x_i)| \\ & + p_i h_{i+1} |u'(x_i)| \Big] \\ \le & C \max_{i=0,1,\cdots,N-1} \int_{x_i}^{x_{i+1}} \sqrt{1 + [u'(x)]^2} dx. \end{aligned}$$

(ii) $\max_i h_{i+1} \le \varepsilon$: From (9), we get

$$\begin{aligned} \sum_{i=0}^{N-1} h_{i+1} |\tau_i| \le & C \sum_{i=0}^{N-1} h_{i+1} \int_{x_{i-1}}^{x_{i+1}} |u''(x)| dx \\ \le & C \sum_{i=0}^{N-1} h_{i+1} \varepsilon^{-1} \int_{x_{i-1}}^{x_{i+1}} |u'(x)| dx \\ \le & C \max_{i=0,1,\cdots,N-1} \int_{x_i}^{x_{i+1}} \sqrt{1 + [u'(x)]^2} dx. \end{aligned}$$

Combining all the above inequalities, the desired result is achieved.

Let $e_i = u(x_i) - u_i^N$ denote the error function, where $u(x)$ is the solution of (1)-(2), and u_i^N is the solution of (5)-(6) on the mesh (7). Substituting $u_i^N = e_i + u(x_i)$ into (5)-(6), we see that e_i is the solution of the following problem

$$L^N e_i = -\varepsilon D^+ D^- u_i - p_i D^+ u_i - f_i = \tau_i \quad (11)$$

for $1 \le i \le N-1$ with

$$e_0 = 0, \quad e_N = 0. \quad (12)$$

Note that the matrix in the discrete scheme is an M-matrix. We may derive the following estimate.

Theorem 4.1:

$$|e_i| \le \frac{2}{\beta} \sum_{j=0}^{N-1} h_{j+1} |\tau_j|, \quad 0 \le i \le N. \quad (13)$$

Proof. By using the discrete Green's function for (11)-(12) and using (8), we have

$$|e_i| = \Big| \sum_{j=0}^{N-1} h_{j+1} G(x_i, x_j) \tau_j \Big| \le \frac{2}{\beta} \sum_{j=0}^{N-1} h_{j+1} |\tau_j|$$

for $0 \le i \le N$.

Finally, we obtain the main result.

Theorem 4.2: Let $u(x)$ be the solution of (1)-(2). Let u_i^N be the upwind difference approximation obtained by solving (5)-(6) on the mesh (7). Then

$$\max_{0\le i\le N} |u(x_i) - u_i^N| \le C\,N^{-1}.$$

Proof. From (3), we find that $\int_0^1 \sqrt{1+[u'(x)]^2}dx \le C$. Note that (7) indicates that for $j = 1, 2, \cdots, N-1$,

$$\int_{x\ -1}^{x} \sqrt{1+[u'(x)]^2}dx = \int_{x}^{x\ +1} \sqrt{1+[u'(x)]^2}dx.$$

By Theorem 4.1, Lemma 4.1, we know that for $0 \le i \le N$,

$$\begin{aligned}|u(x_i) - u_i^N| &\le \frac{2}{\beta}\sum_{j=0}^{N-1} h_{j+1}|\tau_j| \\ &\le C \max_{j=0,1,\cdots,N-1}\int_{x}^{x\ +1}\sqrt{1+[u'(x)]^2}dx \\ &\le C\,N^{-1}.\end{aligned}$$

The desired result is obtained.

Remark: Theorem 4.2 indicates that the upwind difference approximation is first-order uniformly convergent with respect to the perturbation parameter ε.

V. A Numerical Experiment

Consider the problem (1)-(2) with $p(x) = \frac{1}{1+x}$ and $f(x) = \frac{1}{1+x}$. Its exact solution is $u(x) = \frac{(1+x)^{1-\frac{1}{\ }}-1}{2^{1-\frac{1}{\ }}-1} - x$. It is solved using the upwind difference scheme (5)-(6) on the mesh (7). The numerical results are listed in Table 1, where $\|e\|_\infty = \max_i |u(x_i) - u^N(x_i)|$ is the error in the maximum norm for a fixed N. The convergence rates $r = \log_2(\frac{\eta}{\eta_2})$ for $\varepsilon = 10^{-5}, 10^{-8}, 10^{-11}$ are also presented, where η_N stands for the error with a fixed ε and a fixed N. These convergence rates approximate to 1 as N is increasing, which coincides with Theorem 4.2.

TABLE I
THE ERRORS IN THE MAXIMUM NORM AND THE CONVERGENCE RATES OF THE NUMERICAL APPROXIMATION

N	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-11}$
32	$\|e\|_\infty = 6.12e{-2}$	$\|e\|_\infty = 6.12e{-2}$	$\|e\|_\infty = 6.12e{-2}$
64	$\|e\|_\infty = 3.12e{-2}$ $r = 0.97$	$\|e\|_\infty = 3.12e{-2}$ $r = 0.97$	$\|e\|_\infty = 3.12e{-2}$ $r = 0.97$
128	$\|e\|_\infty = 1.57e{-2}$ $r = 0.99$	$\|e\|_\infty = 1.57e{-2}$ $r = 0.99$	$\|e\|_\infty = 1.57e{-2}$ $r = 0.99$
256	$\|e\|_\infty = 7.86e{-3}$ $r = 0.99$	$\|e\|_\infty = 7.86e{-3}$ $r = 0.99$	$\|e\|_\infty = 7.87e{-2}$ $r = 0.99$
512	$\|e\|_\infty = 3.94e{-3}$ $r = 1.00$	$\|e\|_\infty = 3.94e{-3}$ $r = 1.00$	$\|e\|_\infty = 3.94e{-3}$ $r = 1.00$

VI. Conclusions

In this paper, we first presents an upwind difference scheme with the equidistributing principle for a singularly perturbed problem arise from pattern formation in developmental material science, then obtain the uniformly convergent error estimate basing on the discrete Green's function. The numerical experimental results coincide with the theoretical analysis.

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