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Ψ-Eventual Stability of Differential System with Impulses

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Abstract—In this paper, the criteria of Ψ-eventual stability have been established for generalized impulsive differential systems of multiple dependent variables. The sufficient conditions have been obtained using piecewise continuous Lyapunov function. An example is given to support our theoretical result.

Keywords—Impulsive differential equations, Lyapunov function, Eventual stability.

I. INTRODUCTION

Many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. The impulsive system of differential equation are an adequate apparatus for the mathematical simulation of numerous processes and phenomena studied in biology, economics and technology etc. That is why, in recent years, the study of such systems has been very intensive (See [2-11]).

Akinyele [7] introduced the notion of Ψ-stability of degree k with respect to a function $\Psi \in C(R_+, R_+)$, increasing and differentiable on R_+, where $R_+ = [0, \infty)$ and such that $\Psi(t) \geq 1$ for $t \geq 0$ and $\lim_{t\to\infty} \Psi(t) = b, b \in [1, \infty)$. In [6], Morachalo introduced the notions of Ψ-stability, Ψ-uniform stability and Ψ-asymptotic stability of trivial solution of the nonlinear system $x' = f(t, x)$. Then Diamandescu [1] proved some sufficient conditions for Ψ-stability of the zero solution of a nonlinear Volterra integro-differential system.

The main purpose of this work is to investigate the sufficient conditions for the existence of Ψ-eventual stability of trivial solution for generalized impulsive differential system of multiple dependent variables, where Ψ is a matrix function defined in the section below.

The paper is organized as follows. In Section 2, we introduce some preliminary definitions and notations which will be used throughout the paper. In Section 3, we investigate some sufficient conditions for Ψ-uniform eventual stability and Ψ-uniform asymptotic eventual stability of trivial solution of the impulsive differential systems. In Section 4, an example to support our theoretical result has been discussed.

II. PRILIMINARIES

Let R^n denote the Euclidean n-space. Elements of this space are denoted by $x = (x_1, x_2, ..., x_n)^T$ and their norm is given by $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$. For $n \times n$ real matrices, we define the norm $|A| = \sup_{\|x\|\leq 1} \|Ax\|$. Let $\Psi_i : R_+ \to (0, \infty), i = 1, 2, ..., n$ where $R_+ = [0, \infty)$ be continuous functions and let $\Psi = diag[\Psi_1, \Psi_2, ..., \Psi_n]$.

Let R^s_H be the s-dimensional Euclidean space with a suitable norm $\|.\|$ and $R^s_H = \{x \in R^s : \|x\| < H\}$.

Consider the system

$$\begin{aligned}
\dot{x} &= f(t,x) + g(t,y) + h(t,z), t \neq \tau_k,\\
\dot{y} &= u(t,x,y) + v(t,y,z) + w(t,x,z), t \neq \tau_k,\\
\dot{z} &= l(t,x,y,z), t \neq \tau_k, \qquad (1)\\
\Delta x|_{t=\tau_k} &= A_t(x) + B_t(y) + C_t(z),\\
\Delta y|_{t=\tau_k} &= D_t(x,y) + E_t(y,z) + F_t(z,x),\\
\Delta z|_{t=\tau_k} &= G_t(x,y,z), k = 1, 2, ...,
\end{aligned}$$

where $t \in R_+,\ x \in R^n,\ y \in R^m,\ z \in R^p,\ f : R_+ \times R^n_H \to R^n,\ g : R_+ \times R^m_H \to R^n,\ h : R_+ \times R^p_H \to R^n,\ u : R_+ \times R^n_H \times R^m_H \to R^m,\ v : R_+ \times R^m_H \times R^p_H \to R^m,\ w : R^+ \times R^n_H \times R^p_H \to R^m,\ l : R^+ \times R^n_H \times R^m_H \times R^p_H \to R^p,\ A_t : R^n_H \to R^n,\ B_t : R^m_H \to R^n,\ C_t : R^p_H \to R^n,\ D_t : R^n_H \times R^m_H \to R^m,\ E_t : R^m_H \times R^p_H \to R^m,\ F_t : R^n_H \times R^p_H \to R^m,\ G_t : R^n_H \times R^m_H \times R^p_H \to R^p$.

$\Delta x|_{t=\tau_k} = x(\tau_k) - x(\tau_k^-),\ \Delta y|_{t=\tau_k} = y(\tau_k) - y(\tau_k^-),\ \Delta z|_{t=\tau_k} = z(\tau_k) - z(\tau_k^-)$.

Let $t_0 \in R_+, x_0 \in R^n, y_0 \in R^m, z_0 \in R^p$.

Let $x(t, t_0, x_0, y_0, z_0), y(t, t_0, x_0, y_0, z_0), z(t, t_0, x_0, y_0, z_0)$ be the solution of the system (1) satisfying the initial conditions

$$x(t_0^+) = x_0,\ y(t_0^+) = y_0, z(t_0^+) = z_0. \qquad (2)$$

Throughout this article, we assume the following conditions:

(a) The functions $f(t,x), g(t,y), h(t,z), u(t,x,y), v(t,y,z), w(t,x,z)$ and $l(t,x,y,z)$ are continuous in their definition domains, $f(t,0) = g(t,0) = h(t,0) = 0; u(t,0,0) = v(t,0,0) = w(t,0,0) = 0$ and $l(t,0,0,0) = 0$ for $t_0 \in R_+$.

(b) The functions $A_t, B_t, C_t, D_t, E_t, F_t$ and G_t are continuous in their definition domains and $A_t(0) = B_t(0) = C_t(0) = D_t(0,0) = E_t(0,0) = F_t(0,0) = G_t(0,0,0) = 0$.

(c) If $x \in R^n_H, y \in R^m_H$ and $z \in R^p_H$, then $\|x + A_t(x) + B_t(y) + C_t(z)\| \leq \|x\|$, $\|y + D_t(x,y) + E_t(x,y) + F_t(x,y)\| \leq \|y\|$ and $\|z + G_t(x,y,z)\| \leq \|z\|$.

(d) $0 \leq \tau_0 < \tau_1 < \tau_2 < ...$ and $\lim_{k\to\infty} \tau_k = \infty$.

(e) For each point $(t_0, x_0, y_0, z_0) \in R_+ \times R^n_H \times R^m_H \times R^p_H$, the solution $x(t, t_0, x_0, y_0, z_0), y(t, t_0, x_0, y_0, z_0), z(t, t_0, x_0, y_0, z_0)$ of the system (1) is defined in (t_0, ∞) and is unique.

Note that $\Psi(t_0) = \Psi_0$.

Now, we have following definitions:

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Definition 2.1: The zero solution of (1) is said to be Ψ-uniformly eventually stable if for $\epsilon > 0$, $\exists\ \delta = \delta(\epsilon) > 0$ and $\tau = \tau(\epsilon) > 0$ such that $\|\Psi(t)x(t)+\Psi(t)y(t)+\Psi(t)z(t)\| < \epsilon$ for $\|\Psi_0 x_0 + \Psi_0 y_0 + \Psi_0 z_0\| < \delta$ and $t \geq t_0 \geq \tau(\epsilon)$.

Definition 2.2: The zero solution of (1) is said to be Ψ-uniformly asymptotically eventually stable if it is uniformly eventually stable and $\exists\ \delta > 0$ such that for $\epsilon > 0$, there exist $\tau = \tau(\epsilon) > 0$ and $T = T(\epsilon) > 0$ such that for $(x_0, y_0, z_0) \in R_H^n \times R_H^m \times R_H^p$ and $\|\Psi_0 x_0 + \Psi_0 y_0 + \Psi_0 z_0\| < \delta$ implies $\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\| < \epsilon$ for $t \geq t_0 + T, t_0 \geq \tau(\epsilon)$.

Definition 2.3: A function $V : R_+ \times R_H^n \times R_H^m \times R_H^p \to R_+$ is said to belong to class $\mathcal{V}_0$ if
(i) V is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times R_H^n \times R_H^m \times R_H^p$;
(ii) $V(t, x, y, z)$ is locally Lipschitizian in all x, y, z on each of the sets $[\tau_{k-1}, \tau_k) \times R_H^n \times R_H^m \times R_H^p$ and $V(t, 0, 0, 0) = 0$ for $t \in R_+$;
(iii) For each $(x, y, z) \in R_H^n \times R_H^m \times R_H^p$, we have, $\lim_{(t,x,y,z)\to(\tau_k^+,x_0,y_0,z_0)} V(t, x, y, z) = V(\tau_k^+, x_0, y_0, z_0)$ exists.

Definition 2.4: Let $V \in \mathcal{V}_0$, for any $(t, x, y, z) \in [\tau_{k-1}, \tau_k) \times R_H^n \times R_H^m \times R_H^p$, the right hand derivative $V'(t, x(t), y(t), z(t))$ along the solution of the problem (1) is defined as

$$\begin{aligned} V'(t, x(t), y(t), z(t)) &= \lim_{s\to 0^+} \frac{1}{s}[V(t+s, x+s\{f(t,x) \\ &+ g(t,y) + h(t,z)\}, y + s\{u(t,x,y) \\ &+ v(t,y,z) + w(t,x,z)\}, z + \\ &sl(t,x,y,z)) - V(t,x,y,z)]. \end{aligned}$$

We define,
$K = \{w \in C(R_+, R_+) : w \text{ is strictly increasing and } w(0) = 0\}$,
$K_1 = \{\phi \in C(R_+, R_+) : \phi \text{ is increasing and } \phi(s) < s \text{ for } s > 0\}$.

III. Main results

In this section we shall present sufficient conditions for the Ψ-uniform eventual stability and Ψ-uniform asymptotic eventual stability of trivial solution of the impulsive differential system (1).

Theorem 3.1 Assume that there exist functions $V \in \mathcal{V}_0, a, b \in K, \phi \in K_1$ such that
(i) $b(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|) \leq V(t, x, y, z) \leq a(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|)$, $(t, x, y, z) \in R_+ \times R_H^n \times R_H^m \times R_H^p$;
(ii) $V'(t, x(t), y(t), z(t)) \leq g(t)w(V(t, x(t), y(t), z(t)))$, where $(t, x, y, z) \in [\tau_{k-1}, \tau_k) \times R_H^n \times R_H^m \times R_H^p$ and the functions $g, w : R_+ \to R_+$ are locally integrable;
(iii) For all $k \in N$, $(x, y, z) \in R_H^n \times R_H^m \times R_H^p$, $V(\tau_k, x(\tau_k^-) + A_t(x) + B_t(y) + C_t(z), y(\tau_k^-) + D_t(x, y) + E_t(y, z) + F_t(z, x), z(\tau_k^-) + G_t(x, y, z)) \leq \phi(V(\tau_k^-, x(\tau_k^-), y(\tau_k^-), z(\tau_k^-)))$;
(iv) There exist a constant $A > 0$ such that $\int_{\tau_{k-1}}^{\tau_k} g(s)ds < A$ and $\int_{\mu}^{\phi^{-1}(\mu)} \frac{ds}{w(s)} \geq A$ for any $\mu > 0$ and $k \in N$.

Then the zero solution of system (1) is Ψ-uniformly eventually stable.

Proof: Let $\epsilon > 0$ and choose $\delta = \delta(\epsilon) > 0, \tau(\epsilon) > 0$ such that $\delta < a^{-1}(\phi(b(\epsilon))), t_0 \geq \tau(\epsilon)$. We are to prove that for $(x_0, y_0, z_0) \in R_H^n \times R_H^m \times R_H^p, \|\Psi_0 x_0 + \Psi_0 y_0 + \Psi_0 z_0\| < \delta$ implies

$$\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\| < \epsilon, t \geq t_0 \geq \tau(\epsilon).$$

Let $t_0 \in [\tau_{m-1}, \tau_m)$ for some $m \in N$. We firstly prove that

$$V(t, x, y, z) \leq \phi^{-1}(a(\delta)),\ t_0 \leq t < \tau_m. \tag{3}$$

Clearly,

$$V(t_0, x_0, y_0, z_0) \leq a(\|\Psi_0 x_0 + \Psi_0 y_0 + \Psi_0 z_0\|) \leq a(\delta) < \phi^{-1}(a(\delta)).$$

If (3) does not hold, then there is a $t_1 \in (t_0, \tau_m)$ such that

$$V(t_1, x(t_1), y(t_1), z(t_1)) > \phi^{-1}(a(\delta)) > a(\delta) \geq V(t_0, x_0, y_0, z_0).$$

From the continuity of $V(t, x, y, z)$ in $[\tau_{m-1}, \tau_m)$, there is an $s_1 \in (t_0, t_1)$ such that

$$\begin{aligned} V(s_1, x(s_1), y(s_1), z(s_1)) &= \phi^{-1}(a(\delta)), \\ V(t, x(t), y(t), z(t)) &> \phi^{-1}(a(\delta)),\ s_1 < t \leq t_1, \\ V(t, x(t), y(t), z(t)) &\leq \phi^{-1}(a(\delta)),\ t_0 \leq t \leq s_1, \end{aligned} \tag{4}$$

and there also exist an $s_2 \in (t_0, s_1)$ such that

$$\begin{aligned} V(s_2, x(s_2), y(s_2), z(s_2)) &= a(\delta), \\ V(t, x(t), y(t), z(t)) &\geq a(\delta),\ s_2 \leq t \leq s_1. \end{aligned} \tag{5}$$

Integrating the inequality given in (ii) within $[s_2, s_1]$ and by condition (iv), we get

$$\int_{V(s_2,x(s_2),y(s_2),z(s_2))}^{V(s_1,x(s_1),y(s_1),z(s_1))} \frac{ds}{w(s)} \leq \int_{s_2}^{s_1} g(t)dt \leq \int_{\tau_{m-1}}^{\tau_m} g(t)dt < A. \tag{6}$$

On the other hand, from the inequalities (4), (5) and condition (iv), we have

$$\int_{V(s_2,x(s_2),y(s_2),z(s_2))}^{V(s_1,x(s_1),y(s_1),z(s_1))} \frac{du}{w(u)} = \int_{a(\delta)}^{\phi^{-1}(a(\delta))} \frac{ds}{w(s)} \geq A, \tag{7}$$

which contradicts the inequality (6) and so the inequality (3) holds.

From condition (iii), we have

$$\begin{aligned} & V(\tau_m, x(\tau_m), y(\tau_m), z(\tau_m)) \\ = \;& V(\tau_m, x(\tau_m^-) + A_t(x) + B_t(y) + C_t(z), y(\tau_m^-) + D_t(x, y) \\ & + E_t(y, z) + F_t(z, x), z(\tau_m^-) + G_t(x, y, z)) \\ \leq \;& \phi(V(\tau_m^-, x(\tau_m^-), y(\tau_m^-), z(\tau_m^-))) \\ \leq \;& \phi(\phi^{-1}(a(\delta))) = a(\delta). \end{aligned} \tag{8}$$

Now, we prove

$$V(t, x(t), y(t), z(t)) \leq \phi^{-1}(a(\delta)), \tau_m \leq t < \tau_{m+1}. \tag{9}$$

If the inequality (9) does not hold, then there exist $\hat{t} \in (\tau_m, \tau_{m+1})$ such that $V(\hat{t}, x(\hat{t}), y(\hat{t}), z(\hat{t})) > \phi^{-1}(a(\delta)) > a(\delta) \geq V(\tau_m, x(\tau_m), y(\tau_m), z(\tau_m))$.

From the continuity of $V(t,x,y,z)$ in $[\tau_m, \tau_{m+1})$, there is an $r_1 \in (\tau_m, \hat{t})$ such that

$$\begin{aligned} V(r_1, x(r_1), y(r_1), z(r_1)) &= \phi^{-1}(a(\delta)), \\ V(t, x(t), y(t), z(t)) &> \phi^{-1}(a(\delta)),\ r_1 < t \le \hat{t}, \\ V(t, x(t), y(t), z(t)) &\le \phi^{-1}(a(\delta)),\ t_0 \le t \le r_1, \end{aligned} \tag{10}$$

and there also exist an $r_2 \in (\tau_m, r_1)$ such that

$$\begin{aligned} V(r_2, x(r_2), y(r_2), z(r_2)) &= a(\delta), \\ V(t, x(t), y(t), z(t)) &\ge a(\delta), r_2 \le t \le r_1. \end{aligned} \tag{11}$$

Again integrating the inequality given in (ii) within $[r_2, r_1]$ and by similarly as above, we get a contradiction.
So the inequality (9) holds.
From (iii), we have

$$\begin{aligned} & V(\tau_{m+1}, x(\tau_{m+1}), y(\tau_{m+1}), z(\tau_{m+1})) \\ = \ & V(\tau_{m+1}, x(\tau_{m+1}^-) + A_t(x) + B_t(y) + C_t(z), y(\tau_{m+1}^-) \\ & + D_t(x,y) + E_t(y,z) + F_t(z,x), z(\tau_{m+1}^-) + G_t(x,y,z)) \\ \le \ & \phi(V(\tau_{m+1}^-, x(\tau_{m+1}^-), y(\tau_{m+1}^-), z(\tau_{m+1}^-))) \\ \le \ & \phi(\phi^{-1}(a(\delta))) = a(\delta). \end{aligned} \tag{12}$$

By induction, we can prove that in general

$$\begin{aligned} & V(t, x(t), y(t), z(t)) \le \phi^{-1}(a(\delta)), \tau_{m+i} \le t \le \tau_{m+i+1}, \\ & V(\tau_{m+i+1}, x(\tau_{m+i+1}), y(\tau_{m+i+1}), z(\tau_{m+i+1})) \le a(\delta), \end{aligned} \tag{13}$$

for $i = 0, 1, 2, ...$

As $a(\delta) < \phi^{-1}(a(\delta))$, it follows that form (3) and (13) that

$$V(t, x(t), y(t), z(t)) \le \phi^{-1}(a(\delta)) < b(\epsilon),\ t \ge t_0 \ge \tau(\epsilon). \tag{14}$$

Now by condition (i), we have $\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\| \le b^{-1}(V(t, x(t), y(t), z(t))) < b^{-1}(b(\epsilon)) = \epsilon,\ t \ge t_0 \ge \tau(\epsilon)$.
Thus the zero solution of (1) is Ψ-uniformly eventually stable.

Theorem 3.2 Let all the conditions of Theorem 3.1 be satisfied except (iv), which is replaced by
(v) $r = \sup_{k \in Z}\{\tau_k - \tau_{k-1}\} < \infty$, $A = \sup_{t \ge 0} \int_t^{t+\gamma} g(s)ds < \infty$ *and* $B = \inf_{q>0} \int_{\phi(q)}^q \frac{ds}{w(s)} > A$.
Then the zero solution of system (1) is Ψ-uniformly asymptotically eventually stable.
Proof: If all the conditions of Theorem 3.2 holds, then all the conditions of Theorem 3.1 hold. Thus the zero solutions of system (1) is Ψ-uniformly stable.
Therefore, for given $q > 0$, for all $t_0 \in R_+$, we can choose $\delta > 0, \tau(q) > 0$ such that $a(\delta) = \phi(b(q))$, for all $(x_0, y_0, z_0) \in R_H^n \times R_H^m \times R_H^p$, such that $\|\Psi_0 x_0 + \Psi_0 y_0 + \Psi_0 z_0\| < \delta$ implies

$$\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\| \le q,\ t \ge t_0 \ge \tau(q).$$

Moreover, $V(t,x,y,z) \le a(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|) \le a(q),\ t \ge t_0 \ge \tau(q)$.
Now, let $\epsilon > 0$ be given, we can find $\tau(\epsilon) > 0$ such that $t_0 \ge \tau(\epsilon)$.
If $\tau(\epsilon) \le \tau(q)$, then $V(t,x,y,z) \le a(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|) \le a(q), t \ge t_0 \ge \tau(q) \ge \tau(\epsilon)$.
If $\tau(\epsilon) > \tau(q)$, then as $V(t,x,y,z) \le a(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|) \le a(q), t \ge t_0 \ge \tau(q)$, it is obvious that $V(t,x,y,z) \le a(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|) \le a(q), t \ge t_0 \ge \tau(\epsilon)$.
So in any case, we have $V(t,x,y,z) \le a(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|) \le a(q)$ holds for $t \ge t_0 \ge \tau(\epsilon)$.
In the following, we prove that for $T(\epsilon) > 0$ such that $\|\Psi_0 x_0 + \Psi_0 y_0 + \Psi_0 z_0\| < \delta$ implies $\|\Psi(t)x(t,t_0,x_0,y_0,z_0) + \Psi(t)y(t,t_0,x_0,y_0,z_0) + \Psi(t)z(t,t_0,x_0,y_0,z_0)\| < \epsilon$ for $t \ge t_0 + T(\epsilon), t_0 \ge \tau(\epsilon)$.
Now, let

$$M = M(\epsilon) = \sup\{\frac{1}{w(s)} : \phi(b(\epsilon)) \le s \le a(q)\}$$

and note that $0 < M < \infty$. For $b(\epsilon) \le p \le a(q)$, we have $\phi(b(\epsilon) \le \phi(p) < p \le a(q)$ and so $B \le \int_{\phi(p)}^p \frac{ds}{w(s)} \le M(p - \phi(p))$, from which we obtain $\phi(p) \le p - B/M < p - d$, where $d = d(\epsilon) > 0$ is chosen such that $d < \frac{B-A}{M}$.
Let $N = N(\epsilon)$ be the smallest positive integer for which $a(q) < b(\epsilon) + Nd$ and we define $T = T(\epsilon) = N\gamma$.
Given a solution $x = x(t,t_0,x_0,y_0,z_0), y = y(t,t_0,x_0,y_0,z_0), z = z(t,t_0,x_0,y_0,z_0)$ of system (1), where $t_0 \in [\tau_{l-1}, \tau_l)$ for some integer l, we will prove that if $\|\Psi_0 x_0 + \Psi_0 y_0 + \Psi_0 z_0\| < \delta$ then $\|\Psi(t)x(t,t_0,x_0,y_0,z_0) + \Psi(t)y(t,t_0,x_0,y_0,z_0) + \Psi(t)z(t,t_0,x_0,y_0,z_0)\| \le \epsilon,\ t \ge t_0 + T(\epsilon), t_0 \ge \tau(\epsilon)$.
Given $0 < D \le a(q)$ and $j \ge 1$, we will show that
(a) if $V(\tau_j, x(\tau_j), y(\tau_j), z(\tau_j)) \le D$ then $V(t, x(t), y(t), z(t)) \le D$ for $t \ge \tau_j$;
(b) if in addition $D \ge b(\epsilon)$, then $V(t, x(t), y(t), z(t)) \le D - d$ for $t \ge \tau_j$.
Firstly we prove (a).
If (a) does not holds, then there exist some $t \ge \tau_j$ such that $V(t, x(t), y(t), z(t)) > D$. Then let $t_1 = \inf\{t \ge \tau_j : V(t, x(t), y(t), z(t)) \ge D\}$. Thus $t_1 \in [\tau_k, \tau_{k+1})$ for some $k \ge j$. As $V(\tau_k, x(\tau_k), y(\tau_k), z(\tau_k)) \le \phi(V(\tau_k^-, x(\tau_k^-), y(\tau_k^-), z(\tau_k^-))) \le \phi(D) < D$, then $t_1 \in (\tau_k, \tau_{k+1})$. Moreover, $V(t_1, x(t_1), y(t_1), z(t_1)) = D$ and $V(t, x(t), y(t), z(t)) \le D$ for $t \in [\tau_j, t_1]$.
Let

$$\bar{t} = \sup\{t \in [\tau_k, t_1] : V(t, x(t), y(t), z(t)) \le \phi(D)\}.$$

As $V(t_1, x(t_1), y(t_1), z(t_1)) = D > \phi(D)$,
then $\bar{t} \in [\tau_k, t_1)$, $V(\bar{t}, x(\bar{t}), y(\bar{t}), z(\bar{t})) = \phi(D)$ and $V(t, x(t), y(t), z(t)) \ge \phi(D)$ for $t \in [\bar{t}, t_1]$
So integrating inequality $V'(t, x(t), y(t), z(t)) \le g(t)w(V(t,x,y,z))$ over $[\bar{t}, t_1]$, we get

$$\int_{V(\bar{t}, x(\bar{t}), y(\bar{t}), z(\bar{t}))}^{V(t_1, x(t_1), y(t_1), z(t_1))} \frac{ds}{w(s)} \le A.$$

Also $\int_{V(\bar{t}, x(\bar{t}), y(\bar{t}), z(\bar{t}))}^{V(t_1, x(t_1), y(t_1), z(t_1))} \frac{ds}{w(s)} = \int_{\phi(D)}^{D} \frac{ds}{w(s)} \ge B > A$.

This is a contradiction, so (a) holds.
Now we prove (b).
On the contrary, assume that there exist some $t \ge \tau_j$, such that $V(t, x(t), y(t), z(t)) > D - d$. Then define $r_1 = \inf\{t \ge \tau_j : V(t, x(t), y(t), z(t)) > D - d\}$

and let $k \geq j$ be chosen such that $r_1 \in [\tau_k, \tau_{k+1})$. As $b(\epsilon) \leq D \leq a(q)$, we have $\phi(D) < D - d$.
So from (a) and condition (iii),
$V(\tau_k, x(\tau_k), y(\tau_k), z(\tau_k)) \leq \phi(V(\tau_k^-, x(\tau_k^-), y(\tau_k^-), z(\tau_k^-)))$

$$\leq \phi(D) < D - d.$$

Thus $r_1 \in (\tau_k, \tau_{k+1})$.
Moreover, $V(r_1, x(r_1), y(r_1), z(r_1)) = D - d$ and for $t \in [\tau_k, r_1), V(t, x(t), y(t)) \leq D - d$.
Let $\bar{r} = \sup\{t \in [\tau_k, r_1), V(t, x(t), y(t), z(t)) \leq \phi(D)\}$.
As $V(r_1, x(r_1), y(r_1), z(r_1)) = D - d > \phi(D) \geq V(\tau_k, x(\tau_k), y(\tau_k), z(\tau_k))$, then $\bar{r} \in [\tau_k, r_1), V(\bar{r}, x(\bar{r}), y(\bar{r}), z(\bar{r})) = \phi(D)$ and $V(t, x(t), y(t), z(t)) \geq \phi(D)$ for $t \in [\bar{r}, r_1]$.
So integrating the inequality $V^{'}(t, x(t), y(t), z(t)) \leq g(t)w(V(t, x(t), y(t), z(t)))$ over $[\bar{r}, r_1]$, we have

$$\int_{V(\bar{r}, x(\bar{r}), y(\bar{r})), z(\bar{r})}^{V(r_1, x(r_1), y(r_1), z(r_1))} \frac{ds}{w(s)} \leq A.$$

Also

$$\int_{V(\bar{r}, x(\bar{r}), y(\bar{r}), z(\bar{r}))}^{V(r_1, x(r_1), y(r_1), z(r_1))} \frac{ds}{w(s)} = \int_{\phi(D)}^{D-d} \frac{ds}{w(s)}$$
$$= \int_{\phi(D)}^{D} \frac{ds}{w(s)} - \int_{D-d}^{D} \frac{ds}{w(s)}.$$

As $b(\epsilon) \leq D \leq a(q)$, we have
$\phi(b(\epsilon)) \leq \phi(D) < D - d < D \leq a(q)$.
Thus, $\frac{1}{w(s)} \leq M$ for $D - d \leq s \leq D$.

So we get

$$\int_{V(\bar{r}, x(\bar{r}), y(\bar{r}), z(\bar{r}))}^{V(r_1, x(r_1), y(r_1), z(r_1))} \frac{ds}{w(s)} \geq B - \int_{D-d}^{D} M ds = B - dM$$
$$> B + A - B = A,$$

which is a contradiction, so (b) holds.

We define the indices $k^{(i)}$ for $i = 1, 2, 3..., N$ as follows. Let $k^{(1)} = l$ and for $i = 2, ...N$, let $k^{(i)}$ be chosen such that $\tau_{k^{(i)}-1} < \tau_{k^{(i-1)}} < \tau_{k^{(i)}}$.
Then from condition (v), we have $\tau_{k^{(i)}} = \tau_l \leq t_0 + r$, and for $i = 1, 2, ..., N$,
$\tau_{k^{(i)}} \leq \tau_{k^{(i)}-1} + r < \tau_{k^{(i-1)}} + r$. Combining these inequalities, $\tau_{k^{(N)}} \leq t_0 + rN = t_0 + T(\epsilon)$.
We claim that for each $i = 1, 2, ..., N, V(t, x(t), y(t), z(t)) \leq a(q) - id$ for $t \geq \tau_{k^{(i)}}$.
By setting $D = a(q)$ in (b), by condition (iii) and $b(\epsilon) \leq a(q)$, we get $V(t, x(t), y(t), z(t)) \leq a(q) - d$ for $t \geq \tau_{k^{(1)}}$ as $V(t, x(t), y(t), z(t)) \leq a(q)$ for $t \in [t_0, \tau_{k^{(1)}})$, which establish the base case.
We now proceed by induction and assume that $V(t, x(t), y(t), z(t)) \leq a(q) - jd$ for $t \geq \tau_{k^{(j)}}$ for some $1 \leq j \leq N - 1$.
Let $D = a(q) - jd$. As $\tau_{k^{(j)}} \leq \tau_{k^{(j+1)}}, V(t, x(t), y(t), z(t)) \leq D$ for $t \geq \tau_{k^{(j+1)}}$ and so $V(t, x(t), y(t), z(t)) \leq D - d = a(q) - (j+1)d$ for $t \geq \tau_{k^{(j+1)}}$.
So we have proved our claim by induction.

When $j = N - 1$, we get

$$V(t, x(t), y(t), z(t)) \leq a(q) - Nd < b(\epsilon), t \geq \tau_{k^{(N)}}.$$

As $t_0 + T(\epsilon) \geq \tau_{k^{(N)}}$, by condition (i), we get $\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\| < \epsilon$ for $t \geq t_0 + T(\epsilon)$ and $t_0 \geq \tau(\epsilon)$.
Thus, the impulsive system (1) is Ψ-uniformly asymptotically eventually stable.

IV. EXAMPLE

In this section, we give an example to illustrate our theoretical result.
Consider the system

$$\begin{aligned}
\dot{x} &= cx(t) + dy(t) + ez(t), t \neq \tau_k, \\
&\quad x(\tau_k) = \alpha x(\tau_k^-) + \beta y(\tau_k^-) + \gamma z(\tau_k^-), \\
\dot{y} &= c_1 y(t) + d_1 z(t), t \neq \tau_k,\ y(\tau_k) = \alpha_1 y(\tau_k^-) + \beta_1 z(\tau_k^-), \\
\dot{z} &= e_1 z(t), t \neq \tau_k,\ z(\tau_k) = \gamma_1 z(\tau_k^-),\ k = 1, 2, 3, ..., \quad (15)
\end{aligned}$$

where $0 \leq \tau_0 < \tau_1 < \tau_2 < ...$ and $\lim_{k \to \infty} \tau_k = \infty, c > 0,\ d > 0,\ e > 0,\ c_1 > 0,\ d_1 > 0,\ e_1 > 0,\ \alpha > 0,\ \beta > 0,\ \gamma > 0,\ \alpha_1 > 0,\ \beta_1 > 0,\ \gamma_1 > 0$ and the following conditions hold:
(1) $c > c_1,\ c > e_1,\ d > d_1,\ d > e,\ \alpha^2 > \beta^2 + \alpha_1^2,\ \alpha^2 > \beta_1^2 + \gamma^2 + \gamma_1^2$;
(2) $\tau_k - \tau_{k-1} < \frac{-ln(3\alpha^2 + \beta^2) + ln2}{2c + d}$.
Let $V(t, x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2),\ \phi(s) = (2\alpha^2 + \beta^2)s,\ w(s) = s,\ g(t) = 2c + d$.
Take $\Psi(t) = 1/2, a(x) = 4x^2,\ b(x) = x/2 \in K$.
Clearly,
$b(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|) \leq \frac{(x^2 + y^2 + z^2)}{2} = V(t, x, y, z) \leq a(\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\|)$.
Now,

$$\begin{aligned}
V^{'}(t, x, y, z) &= x\dot{x} + y\dot{y} + z\dot{z} \\
&= cx^2(t) + dx(t)y(t) + ex(t)z(t) + c_1 y^2(t) \\
&\quad + d_1 y(t)z(t) + e_1 z^2(t) \\
&\leq c(x^2(t) + y^2(t) + z^2(t)) + d\frac{x^2(t) + y^2(t) + z^2(t)}{2} \\
&\leq (2c + d)\frac{x^2(t) + y^2(t) + z^2(t)}{2} \\
&= g(t)w(V(t, x, y, z)).
\end{aligned}$$

Also,

$$\begin{aligned}
& V(\tau_k, x(\tau_k^-) + A_t(x) + B_t(y) + C_t(z), y(\tau_k^-) + D_t(x,y) \\
& +E_t(y,z) + F_t(z,x), z(\tau_k^-) + G_t(x,y,z)) \\
= \ & V(\tau_k, \alpha x(\tau_k^-) + \beta y(\tau_k^-) + \gamma z(\tau_k^-), \alpha_1 x(\tau_k^-) \\
& +\beta_1 y(\tau_k^-) + \gamma_1 z(\tau_k^-)) \\
= \ & \frac{1}{2}[(\alpha^2 x^2(\tau_k^-) + \beta^2 y^2(\tau_k^-) + \gamma^2 z^2(\tau_k^-) + 2\alpha\beta x(\tau_k^-) y(\tau_k^-) \\
& +2\beta\gamma y(\tau_k^-) z(\tau_k^-) + 2\alpha\gamma x(\tau_k^-) z(\tau_k^-) + \alpha_1^2 y^2(\tau_k^-) \\
& +\beta_1^2 z^2(\tau_k^-) + 2\alpha_1\beta_1 y(\tau_k^-) z(\tau_k^-) + \gamma_1^2 z^2] \\
\leq \ & \frac{1}{2}[\alpha^2(x^2(\tau_k^-) + y^2(\tau_k^-) + z^2(\tau_k^-)) + 2\alpha\beta(x^2(\tau_k^-) \\
& +y^2(\tau_k^-) + z^2(\tau_k^-))] \\
= \ & \frac{\alpha^2 + 2\alpha\beta}{2}[x^2(\tau_k^-) + y^2(\tau_k^-) + z^2(\tau_k^-)] \\
\leq \ & (2\alpha^2 + \beta^2)[x^2(\tau_k^-) + y^2(\tau_k^-) + z^2(\tau_k^-)] \\
= \ & \phi(V(\tau_k^-, x(\tau_k^-), y(\tau_k^-), z^2(\tau_k^-))).
\end{aligned}$$

Now, let $A = -ln\frac{2\alpha^2+\beta^2}{2}$, then $A > 0$ and $\int_{\tau_{k-1}}^{\tau_k} g(s)ds < (2c+d)\frac{(-ln2\alpha^2+\beta^2)+ln2}{2c+d} = -ln\frac{2\alpha^2+\beta^2}{2} = A.$

Lastly for any $\mu > 0$, $\int_{\mu}^{\phi^{-1}(\mu)} \frac{ds}{w(s)} = \int_{\mu}^{\frac{2}{(2\alpha^2+\beta^2)}\mu} \frac{ds}{s} = ln\frac{2}{2\alpha^2+\beta^2} = A.$

Therefore by Theorem 3.1, the zero solution of system (15) is Ψ-uniformly eventually stable.

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