Filteristic Soft Lattice Implication Algebras

Yi Liu, Yang Xu

Abstract—Applying the idea of soft set theory to lattice implication algebras, the novel concept of (implicative) filteristic soft lattice implication algebras which related to (implicative) filter(for short, (IF)-F-soft lattice implication algebras) are introduced. Basic properties of (IF)-F-soft lattice implication algebras are derived. Two kinds of fuzzy filters (i.e., \((\in, \in \vee \forall q_k)(\in, \in \vee \bigvee q_k)\))-fuzzy (implicative) filter of \(\mathcal{L}\) are introduced, which are generalizations of fuzzy (implicative) filters. Some characterizations for a soft set to be a (IF)-F-soft lattice implication algebra are provided. Analogously, this idea can be used in other types of filteristic lattice implication algebras (such as fantastical (positive implicative) filteristic soft lattice implication algebras).

Keywords—Soft set; (implicative) filteristic lattice implication algebras; fuzzy (implicative) filters; \((\in, \in \vee \forall q_k)(\in, \in \vee \bigvee q_k)\)-fuzzy (implicative) filters.

I. INTRODUCTION

In order to research the many-valued logical system whose propositional value is given in a lattice, in 1993, Xu[1] firstly established the lattice implication algebras by combining lattice and implication algebras, and investigated many useful structures[2], [3], [4]. This logical algebra has been extensively investigated by several researchers, and many elegant results are obtained, collected in the monograph[4].

Because of various uncertainties typical for complicated problems in economics, engineering and environment, they can’t be successfully solved by existing theories such as theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [12]. Molodtsov[12] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov(1999) introduced a novel concept called soft sets as a new mathematical tools for dealing with uncertainties. The soft set theory is free from many difficulties that has been troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Research works on soft sets are very active and progressing rapidly in these years. Maji[16] discussed the application of soft set theory to a decision-making problems. They also investigated some operations on the theory of soft sets. In 2001, Maji[14]et al. investigated the fuzzification of a soft set and obtained many useful results on fuzzy soft set. Aktas and Cagman[12] related soft sets to groups, they defined soft groups, derive some basic properties, and showed that soft groups extended fuzzy groups. Jun[18], [19] applied the soft set theory to the BCK-algebras, investigated soft BCK-subalgebras and soft ideals, introduced the notion of \(\in\)-soft set and \(q\)-soft set, and gave characterizations for subalgebras and ideals. Furthermore, Feng et al.[17] applied soft set theory to the study of semirings and initiated the notion called soft semirings. Zhan,et al.[22] applied soft set to BL-algebras, initiated the notion (implicative)filteristic soft BL-algebras.

The concept of fuzzy set was introduced by Zadeh[1965][5]. Since then this idea has been applied to other algebraic structures such as groups, semigroups, rings, modules, vector spaces and topologies. Some scholars[8], [20], [21] applied this fuzzification to the filter in lattice implication algebras, too. They further to introduce relative fuzzy filter such as fuzzy (positive) implicative filter, fuzzy fantastic filter and investigated some properties. The idea of fuzzy point and ‘belongingness’ and ‘quasi-coincidence’ with a fuzzy set were given by Pu et al.[6]. A new type of fuzzy subgroup (viz \((\in, \in \vee \forall q_k)(\in, \in \vee \bigvee q_k)\)-fuzzy subgroup) was introduced in[9]. In fact, \((\in, \in \vee \forall q_k)\)-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. The idea of fuzzy point and ‘belongingness’ and ‘quasi-coincidence’ with a fuzzy set have been applied some important algebraic system[10], [11]. Liu[7], [8] investigate the interval-valued \((\in, \in \vee q_k)\)-fuzzy lattice implication subalgebras and fuzzy filters, respectively.

The aim of this paper is to apply the idea of soft set theory to lattice implication algebras, and introduce the (implicative) filteristic soft lattice implication algebras which related to (implicative) filter(for short, (IF)-F-soft lattice implication algebras). Basic properties of (IF)-F-soft lattice implication algebras are investigated. We introduce the notion of \((\in, \in \vee \forall q_k)((\in, \in \vee \bigvee q_k))\)-fuzzy (implicative) filters, which are generalizations of fuzzy (implicative) filter. We provide characterizations for a soft set to be an (IF)-F-soft lattice implication algebra. Analogously, this idea can be used in other types lattice implication algebras such as fantastical filteristic lattice implication algebras, positive implicative filteristic soft lattice implication algebras. We hope that it will be of great use to provide theoretical foundation to design intelligent information processing systems.

II. BASIC RESULTS ON LATTICE IMPLICATION ALGEBRAS

Definition 2.1: [1] Let \((L, \vee, \wedge, \prime, O, I)\) be a bounded lattice with an order-reversing involution \(\prime\), the greatest element \(I\) and the smallest element \(O\), and

\[\rightarrow: L \times L \rightarrow L\]

be a mapping. \(\mathcal{L} = (L, \vee, \wedge, \prime, \rightarrow, O, I)\) is called a lattice implication algebra if the following conditions hold for any \(x, y, z \in L\):

\[(I_1) \ x \rightarrow (y \rightarrow z) \Rightarrow y \rightarrow (x \rightarrow z),\]

\[(I_2) \ y \rightarrow z \Rightarrow x \rightarrow (y \rightarrow z),\]

\[(I_3) \ x \rightarrow I = x.\]
(I_2) x \rightarrow x = I,
(I_3) x \rightarrow y = y \rightarrow x \rightarrow I \text{ implies } x = y,
(I_4) x \rightarrow y = y \rightarrow x = I \text{ implies } x = y,
(I_5) (x \rightarrow y) \rightarrow z = (y \rightarrow z) \rightarrow (x \rightarrow z),
(I_6) (x \rightarrow y) \rightarrow z = (x \rightarrow z) \rightarrow (y \rightarrow z).

In this paper, denote \mathcal{L} as a lattice implication algebra.[4](L, \vee, \wedge, \rightarrow, O, I)

Definition 2.2: [4] A non-empty subset \( F \) of a lattice implication algebra \( \mathcal{L} \) is called a filter of \( \mathcal{L} \) if it satisfies

(F1) \( I \in F \).

(F2) \( \forall x \in F \)(\( \forall y \in L \)) (\( x \rightarrow y \in F \Rightarrow y \in F \)).

Definition 2.3: [4] A non-empty subset \( F \) of a lattice implication algebra \( \mathcal{L} \) is called an implicative filter of \( \mathcal{L} \) if it satisfies

(F1) \( I \in F \).

(F2) \( \forall (x, y, z \in L) \) (\( x \rightarrow y \in F \) and \( x \rightarrow y \in F \Rightarrow x \rightarrow z \in F \)).

A fuzzy subset of a nonempty set \( X \) is defined as a mapping from \( X \) to \([0, 1]\), where \([0, 1]\) is the usual interval of real numbers.

Definition 2.4: [2] A fuzzy subset \( \mu \) of \( \mathcal{L} \) is said to be a fuzzy filter if, for any \( x, y \in L \),

(1) \( \mu(I) \geq \mu(x) \),
(2) \( \mu(y) \geq \min \{ \mu(x), \mu(x \rightarrow y) \} \).

Definition 2.5: [2] A fuzzy subset \( \mu \) of \( \mathcal{L} \) is said to be a fuzzy implicative filter if, for any \( x, y \in L \),

(1) \( \mu(I) \geq \mu(x) \),
(2) \( \mu(x \rightarrow z) \geq \min \{ \mu(x \rightarrow y) \}, \mu(x \rightarrow y) \).

A fuzzy set \( \mu \) of a lattice implication algebra \( \mathcal{L} \) of the form: when \( y = x, \mu(y) = \ell \in \{0, 1\} \); otherwise, \( \mu(t) = 0 \).

This fuzzy set is said to be a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x \).

For a fuzzy point \( x \) and a fuzzy set \( \mu \) in \( \mathcal{L} \), \( Pu \) and \( Liu[6] \) gave meaning to the symbol \( x_1 \theta \), where \( \theta \in \left\{ \in, \in', \notin \right\} \).

For a fuzzy point \( x_1 \) is said to be belong to (resp. be quasi-coincident with) a fuzzy set \( A \), written as \( x_1 \in \mu \) (resp. \( x_1 \in \mu \)) if \( \mu(x) \geq t \) (resp. \( x_1 \in \mu \)).

III. (IF-) F-SOFT LATTICE IMPLICATION ALGEBRAS

Molodtsov [12] defined the soft set in the following way:

Let \( U \) be an initial universe set and \( F \) be a set of parameters.

Let \( P(U) \) denote the power set of \( U \) and \( A \subseteq E \).

Definition 3.1: [12] A pair \((F, A)\) is called a soft set over \( U \), where \( F \) is a mapping \( F : A \rightarrow P(U) \).

In other words, a soft set over \( U \) is a parameterized family of subsets of the universe \( U \). For any \( x \in A, F(x) \) may be considered as the set of \( x \)-approximate elements of the soft set \((F, A)\).

In 2003, Maji [14] defined operations and, \( \cap, \cup \) which were later termed as basic intersection, basic union, and union by D. Pei [24]. We are taking the following definitions from [24].

Definition 3.2: Let \((F, A)\) and \((G, B)\) be any two soft sets over a lattice implication algebra \( \mathcal{L} \).

1) The basic intersection of two soft sets \((F, A)\) and \((G, B)\) is defined as the soft set \((H, C) = (F, A) \cap (G, B)\), where \( C = A \times B \) and \( H(a, b) = F(a) \cap G(b) \) for any \((a, b) \in A \times B \).

2) The intersection of soft sets \((F, A)\) and \((G, B)\) over a common universe \( U \) is defined as the soft set \((H, C) = (F, A) \cap (G, B)\), where \( C = A \cap B \), and \( H(c) = F(c) \cap G(c) \) for any \( c \in C \).

3) The union \((H, C)\) of two soft sets \((F, A)\) and \((G, B)\) is defined as the soft set \((H, C) = (F, A) \cup (G, B)\), where \( C = A \cup B \), and \( H(c) = F(c) \cup G(c) \) when \( c \in A \cap B \).

Definition 3.3: Let \((F, A)\) be a nonempty soft set over a lattice implication algebra \( \mathcal{L} = (L, \vee, \wedge, \rightarrow, O, I) \).

Then \((F, A)\) is a called a F-soft lattice implication algebra if \( F(t) \) is a filter of \( \mathcal{L} \) for any \( t \in A \). For our convenience, the empty set \( \emptyset \) is regarded as a filter of \( \mathcal{L} \).

2) \((F, A)\) is a called a IF-soft lattice implication algebra if \( F(t) \) is an implicative filter of \( \mathcal{L} \) for any \( t \in A \). For our convenience, the empty set \( \emptyset \) is regarded as an implicative filter of \( \mathcal{L} \).

Example 3.1: Let \( L = \{0, a, b, c, d, I\} \), the Hasse diagram of \( L \) and its implication operator \( \rightarrow \) and negation operator \( ' \) be defined in EXAMPLE 2.1.4 in [4] Then \( \mathcal{L} = (L, \vee, \wedge, \rightarrow, O, I) \) is a lattice implication algebra.

1) Let \((F, A)\) be a soft set over \( \mathcal{L} \), where \( A = \{I, a, b\} \) and the set-valued function \( F : A \rightarrow P(L) \) defined by \( F(t) = \{x \in L | x \vee t = I\} \). Then \( F(I) = L, F(a) = \{I, b, c\}, F(b) = \{a, c\} \) are all filters of \( \mathcal{L} \). Therefore, \((F, A)\) is a \( F \)-soft lattice implication algebra over \( \mathcal{L} \).

2) Let \((F, A)\) be a soft set over \( \mathcal{L} \), where \( A = \{c, d\} \) and \( F : A \rightarrow P(L) \) the set-valued function defined by \( F(t) = \{y \in L | y \rightarrow t \in \{a, b\}\} \). Then \( F(c) = \{O, a, d\}, F(d) = \{O, c\} \) are all filters of \( \mathcal{L} \). Therefore, \((F, A)\) is a \( F \)-soft lattice implication algebra over \( \mathcal{L} \).

Example 3.2: Let \( L = \{O, a, b, I\} \), its implication operator \( \rightarrow \) and negation operator \( ' \) be defined as Table 2. Then \( \mathcal{L} = (L, \vee, \wedge, \rightarrow, O, I) \) is a lattice implication algebra.

Let \((F, A)\) be a soft set over \( \mathcal{L} \), where \( A = \{0, 1\} \) and \( F : A \rightarrow P(L) \) the set-valued function defined by \( F(t) = L \) when \( t \in (0, 0.5] \); \( F(t) = \{I, a\} \) when \( t \in (0.5, 0.9] \); \( F(t) = \emptyset \) when \( t \in (0.9, 1] \).

Then \( F(t) \) is an implicative filter of \( \mathcal{L} \) for \( t \in A \). Therefore, \((F, A)\) is an \( IF \)-soft lattice implication algebra over \( \mathcal{L} \).

Theorem 3.1: Let \((F, A)\) and \((G, B)\) be two \( F \)-soft lattice implication algebras over \( \mathcal{L} \), then \((F, A) \cap (G, B)\) is also a \( F \)-soft lattice implication algebra over \( \mathcal{L} \) if \( A \cap B \neq \emptyset \).

Proof: Let \((F, A) \cap (G, B) = (H, C)\), where \( C = A \cap B \neq \emptyset \) and \( H(c) = F(c) \cap G(c) \) for any \( c \in C \). We have \( F(c) \) and \( G(c) \) are two filters of \( \mathcal{L} \), hence \( H(c) = F(c) \cap G(c) \) is a filter of \( \mathcal{L} \) if \( H(c) = 0 \). That is, \((H, C) = (F, A) \cap (G, B)\) is a \( F \)-soft lattice implication algebra over \( \mathcal{L} \).

Theorem 3.2: Let \((F, A)\) and \((G, B)\) be two \( F \)-soft lattice implication algebras over \( \mathcal{L} \), then \((F, A) \cup (G, B)\) is also a
F-soft lattice implication algebra over $\mathcal{L}$ if $A \cap B = \emptyset$.

Proof: Let $(F, A) \cup (G, B) = (H, C)$ and $A \cap B = \emptyset$, where $C = A \cap B = \emptyset$. We have $c \in A \cup B$ or $c \in B \cup A$ for any $c \in C$. If $c \in A \cup B$, then $H(c) = F(c)$, it follows that $H(c)$ is a F-soft lattice implication algebra over $\mathcal{L}$. Similarly, we have $H(c)$ is a F-soft lattice implication algebra over $\mathcal{L}$. That is $(H, C)$ is a F-soft lattice implication algebra over $\mathcal{L}$.

Theorem 3.3: Let $(F, A)$ and $(G, B)$ be two F-soft lattice implication algebras over $\mathcal{L}$. Then $(F, A) \wedge (G, B)$ is also a F-soft lattice implication algebra over $\mathcal{L}$.

Proof: Let $(H, C) = (F, A) \wedge (G, B)$, where $C = A \times B$ and $H(x_1, x_2) = F(x_1) \wedge F(x_2)$. $(x_1, x_2) \in A \times B$. Now, $F(x_1) \wedge F(x_2)$ are two filters of $\mathcal{L}$, so $F(x_1) \wedge F(x_2)$ is also a filter of $\mathcal{L}$. Hence $(H, C)$ is a F-soft lattice implication algebra over $\mathcal{L}$.

Definition 3.4: Let $(F, A)$ be a F-soft lattice implication algebra over $\mathcal{L}$.

1. $(F, A)$ is called the trivial F-soft lattice implication algebra over $\mathcal{L}$ if $F(x) = \{I\}$ for any $x \in A$.
2. $(F, A)$ is called the whole F-soft lattice implication algebra over $\mathcal{L}$ if $F(x) = \{I\}$ for any $x \in A$.

Example 3.3: In Example 3.1, let $(F, A)$ be a soft set over $\mathcal{L} = (L, \vee, \wedge, \rightarrow, O, I)$. $(1) A = \{I\}$ and $F: A \rightarrow P(L)$ the set-valued function defined by $F(x) = \{y \in L | x \leq y\}$, then $F(I) = \{I\}$ is a filter of $\mathcal{L}$. Hence $(F, A)$ is a trivial F-soft lattice implication algebra over $\mathcal{L}$.

$(2) A = \{I\}$ and $F: A \rightarrow P(L)$ the set-valued function defined by $F(x) = \{y \in L | x \leq y\}$, then $F(I) = \{I\}$ is a filter of $\mathcal{L}$. Hence $(F, A)$ is a whole F-soft lattice implication algebra over $\mathcal{L}$.

Let $\mathcal{L}_1 = (L_1, \vee, \wedge, \rightarrow, O, I), \mathcal{L}_2 = (L_2, \vee, \wedge, \rightarrow, O, I)$ be two lattice implication algebras and $f: \mathcal{L}_1 \rightarrow \mathcal{L}_2$ a mapping. If $(F_1, A)$ and $(F_2, B)$ are soft set over $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then $(f(F_1), A)$ is a soft set over $\mathcal{L}_2$, where $f(F_1): A \rightarrow P(L_2)$ defined by $f(F_1)(x) = f(F(x))$ for any $x \in A$. Let $f^{-1}(F_2)$, $B$ be a soft set over $\mathcal{L}_2$, where $f^{-1}(F_2): B \rightarrow P(L_1)$ is defined by $f^{-1}(F_2)(y) = f^{-1}(F_2)(y)$ for any $y \in B$.

Theorem 3.4: Let $\mathcal{L}_1, \mathcal{L}_2$ be lattice implication algebras and $f: \mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an implication algebra over $\mathcal{L}_2$, then $f^{-1}(F_2)(y)$ is a filter for any $y \in B$ and $f^{-1}(F_1)(x) = f^{-1}(F(x))$ are the greatest elements of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

Proof: (1) If $f(x) = D - Ker(f)$ for any $x \in A$, then $(f(F_1), A)$ is a trivial F-soft lattice implication algebra over $\mathcal{L}_2$.

(2) If $f$ is onto and $(F_1, A)$ is a whole F-soft lattice implication algebra over $\mathcal{L}_2$.

(3) If $G(y) = f(F_1)$ for any $y \in B$, then $(f^{-1}(G), B)$ is F-soft lattice implication algebra over $\mathcal{L}_2$.

(4) If $f$ is injective and $(G, B)$ is trivial, then $(f^{-1}(G), B)$ is a trivial F-soft lattice implication algebra over $\mathcal{L}_2$.

IV. (IF)-SOFT LATTICE IMPLICATION ALGEBRAS IN FUZZY CONTEXT

In this section, firstly, we discuss the relations between (IF)-F-soft lattice implication algebras and fuzzy (implicative) filters. Secondly, we generalized fuzzy filters of $\mathcal{L}$, initiating the notion of $(\xi, \mathfrak{v}, \mathfrak{q})$-fuzzy (implicative) filter $((\xi, \mathfrak{v}, \mathfrak{q})$-fuzzy (implicative) filter) of $\mathcal{L}$, their equivalent characterizations are derived. At last, we discuss the relations between (IF)-F-soft lattice implication algebras and $(\xi, \mathfrak{v}, \mathfrak{q})$-fuzzy (implicative) filter.

Given a fuzzy set $\mu$ in a lattice implication algebra $\mathcal{L} = (L, \vee, \wedge, \rightarrow, O, I)$ and $A \subseteq [0, 1]$, consider two set-valued functions $F: A \rightarrow P(L)$, defined by $F(t) = \{x \in L | x \leq t\}$ and $F_0: A \rightarrow P(L)$, defined by $F_0(t) = \{x \in L | x \leq t\}$. Then $(F, A)$ and $(F_0, A)$ are two soft set over $\mathcal{L}$. In fact, $(F, A)$ and $(F_0, A)$ is called $\mathcal{L}$-soft set and $\mathcal{L}$-soft set in [19], respectively.

Theorem 4.1: Let $\mu$ be a fuzzy set of $\mathcal{L}$ and $(F, (0, 1))$ be a soft set over $\mathcal{L}$. Then

(1) $(F, (0, 1))$ is a F-soft lattice implication algebra if and only if $\mu$ is a fuzzy filter of $\mathcal{L}$.

(2) $(F, (0, 1))$ is a IF-soft lattice implication algebra if and only if $\mu$ is a fuzzy implication lattice filter of $\mathcal{L}$.

Proof: (1) Suppose that $\mu$ is a fuzzy filter of $\mathcal{L}$ and let $x \in (0, 1]$. If $x \in F(t)$, then $x_1 \in \mu$, i.e. $I \in F(t)$. Let $x, x \rightarrow y \in F(t)$ for any $t \in (0, 1]$, then $x_1 \in \mu$ and $(x \rightarrow y) \in \mu$, i.e. $x \geq t$ and $\mu(x \rightarrow y) \geq t$. It follows that $\mu(y) \geq \min(\mu(x), \mu(y)) \geq t$, and so $y \in \mu$, i.e.
y ∈ F(t). Therefore F(t) is a filter of \( \mathcal{L} \) for any \( t \in (0, 1] \). Hence \((F, (0, 1])\) is a F-soft lattice implication algebra over \( \mathcal{L} \).

Conversely, assume that \((F, (0, 1])\) is a F-soft lattice implication algebra over \( \mathcal{L} \). If there exists \( a \in L \) such that \( \mu(I) < \mu(a) \), then we can choose \( t \in (0, 1] \) such that \( \mu(I) < t \leq \mu(a) \) and so \( I \notin \mu, \) i.e. \( I \notin F(t) \), contradiction. Thus \( \mu(I) = \mu(x) \) for any \( x \in L \). Suppose that there exist \( a, b \in L \) such that \( \mu(b) < \min(\mu(a \rightarrow b), \mu(a)) \), we choose \( t \in (0, 1] \) such that \( \mu(b) < t < \min(\mu(a \rightarrow b), \mu(a)) \). It follows that \( a \rightarrow b \in F(t) \) and \( b \notin F(t) \), which contradicts with \( F(t) \) is a filter of \( \mathcal{L} \). Hence \( \mu(y) \geq \min(\mu(x \rightarrow y), \mu(x)) \) for any \( x, y \in L \). Therefore \( \mu \) is a fuzzy filter of \( \mathcal{L} \).

(2) The case for (2) can be similarly proved.

**Theorem 4.2:** Let \( \mu \) be a fuzzy set of \( \mathcal{L} \) and \( (F_q, (0, 1]) \) be a soft set over \( \mathcal{L} \). Then

1. \( (F_q, (0, 1]) \) is a F-soft lattice implication algebra if and only if \( \mu \) is a fuzzy filter of \( \mathcal{L} \).
2. \( (F_q, (0, 1]) \) is a 1F-soft lattice implication algebra if and only if \( \mu \) is a fuzzy implicative filter of \( \mathcal{L} \).

**Proof:** (1) Assume that \( \mu \) is a fuzzy filter of \( \mathcal{L} \) and let \( t \in (0, 1] \) such that \( F_q(t) \neq \emptyset \). If \( I \notin F_q(t) \), i.e. \( I \notin \mu \) and so \( \mu(I) + t < 1 \). It follows that \( \mu(x) + t \leq \mu(I) + t < 1 \) for any \( x \in L \), so that \( F_q(t) = \emptyset \). This is a contradiction and so \( I \in F_q(t) \). Let \( x, y \in L \) be such that \( x \rightarrow y \in F_q(t) \) and \( x \in F_q(t) \), then \( (x \rightarrow y) \mu q u \) and \( x \mu q u, \mu \). Suppose that there exist \( a, b \in L \) such that \( \mu(a \rightarrow b) < \min(\mu(a), \mu(b)) \), then \( \mu(a) + s < 1 < \min(\mu(a) \rightarrow b), \mu(a)) \), with \( s \in (0, 1] \). It follows that \( \mu(x) + s \rightarrow b \mu q u \) and \( \mu(x) \rightarrow b \mu q u, \mu \). Suppose \( \mu(y) + t \rightarrow b \mu q u, \mu(x) \rightarrow b \mu q u, \mu \), with \( t \in F_q(t) \). So \( \mu(a) \rightarrow b \mu q u, \mu \) that is \( a \rightarrow b \mu q u, \mu \) and \( \mu(a) \rightarrow b \mu q u, \mu \). It follows from \( F_q(t) \) that \( F_q(t) \neq \emptyset \).

Conversely, assume that \((F_q, A)\) is a F-soft lattice implication algebra over \( \mathcal{L} \). If there exists \( a \in L \) such that \( \mu(I) < \mu(a) \), then \( \mu(I) + t \leq 1 < \mu(a) \), for some \( t \in (0, 1] \). Thus \( a \mu q u \) and \( a \mu q u, \emptyset \). We have \( F_q(t) \) is a filter of \( \mathcal{L} \) for any \( t \in (0, 1] \). Hence \( I \notin F_q(t) \) and \( I \notin \mu \) and so \( \mu(I) < \mu(x) \) for any \( x \in L \). Suppose that there exist \( a, b \in L \) such that \( \mu(a \rightarrow b) < \min(\mu(a), \mu(b)) \), then \( \mu(a) + s < 1 < \min(\mu(a) \rightarrow b), \mu(a)) \), with \( s \in (0, 1] \). It follows that \( \mu(x) + s \rightarrow b \mu q u \) and \( \mu(x) \rightarrow b \mu q u, \mu \). Suppose \( \mu(y) + t \rightarrow b \mu q u, \mu(x) \rightarrow b \mu q u, \mu \), with \( t \in F_q(t) \). So \( \mu(a) \rightarrow b \mu q u, \mu \) that is \( a \rightarrow b \mu q u, \mu \) and \( \mu(a) \rightarrow b \mu q u, \mu \). It follows from \( F_q(t) \) that \( F_q(t) \neq \emptyset \).

(2) The case for (2) can be similarly proved.

In what follows, let \( k \) denote an arbitrary element of \((0, 1] \) unless otherwise specified. To say \( x_i q u, \mu \), we mean \( \mu(x) + t + k > 1 \). To say \( x_i \in \mu \), we mean \( x_i \in \mu \) or \( x_i q u, \mu \). For \( \alpha \in (\epsilon, \in \mu q u) \), to say \( x_i \alpha q u, \mu \), we mean \( x_i \alpha q u, \mu \) doesn’t hold.

**Definition 4.1:** A fuzzy set \( \mu \) in \( \mathcal{L} \) is said to be an \((\epsilon, \in \mu q u)-\)fuzzy filter of \( \mathcal{L} \) if it satisfies the following:

1. \( x_i \in \mu \) implies \( I_i \notin \mu q u, \mu \), for any \( x_i, y \in L \) and \( t, r \in (0, 1] \).

**Definition 4.2:** A fuzzy set \( \mu \) in \( \mathcal{L} \) is said to be an \((\epsilon, \in \mu q u)-\)fuzzy implicative filter of \( \mathcal{L} \) if it satisfies the following:

1. \( x_i \in \mu \) implies \( I_i \notin \mu q u, \mu \), for any \( x_i, y \in L \) and \( t, r \in (0, 1] \).
2. \( x \rightarrow (y \rightarrow z) \subseteq \mu \), and \( (x \rightarrow y) \in \mu q u, \mu \), for any \( x, y \in L \) and \( t, r \in (0, 1] \).
3. \( \Gamma(1) \) in Example 3.1, we define a fuzzy set \( G \) of \( \mathcal{L} \) as following:

\[
G(I) = G(a) = 0.7, G(b) = 0.6, G(d) = 0.3, G(O) = G(c) = 0.2.
\]

It is routine to verify that \( \mu \) is an \((\epsilon, \in \mu q u, \mu)-\)fuzzy filter. But \( \mu \) is not a fuzzy filter nor an \((\epsilon, \in \mu q u)-\)fuzzy filter of \( \mathcal{L} \) since \( \mu(I) = 0.45 \mu q u(a) = 0.8, \) and \( a \leq I, \epsilon \in [0, 5] \), but \( I_{0.3} = 0.6 \).

**Example 4.1:** (1) In Example 3.1, we define a fuzzy set \( \mu \) of \( \mathcal{L} \) as following:

\[
\mu(I) = 0.45, \mu(a) = 0.8, \mu(b) = \mu(c) = \mu(d) = 0.3.
\]

Therefore, \( \mu \) is a fuzzy filter of \( \mathcal{L} \).
Therefore, we have

\[ L \in \text{alg over } x \rightarrow \text{alg over } x. \]

And so

\[ (\forall x \in F(t)) \rightarrow (\forall x \in F(t)), \quad \text{for any } \forall x \in F(t). \]

\[ \text{Proof:} \quad \text{It is similarly proved as Theorem 4.3.} \]

**Corollary 4.2:** (see [21]) Let \( \mu \) be a fuzzy subset of \( \mathcal{L} \).

Then \( \mu \) is an \((\in, \in \lor q)\)-fuzzy implicative filter of \( \mathcal{L} \) if and only if \( \mu \) satisfies

\[
\begin{align*}
(1) & \quad (\forall x \in L)(\mu(x) \geq \min \{\mu(x), 0.5\}), \\
(2) & \quad (\forall x, y \in L)(\mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5).
\end{align*}
\]

**Theorem 4.5:** Let \( \mathcal{L} \) be a fuzzy set of \( \mathcal{L} \) and \( (F, (0, \frac{1}{2})) \) be a soft set. Then \((F, (0, \frac{1}{2}))\) is a \( F \)-soft lattice implication algebra over \( \mathcal{L} \) and only if \( \mu \) is an \((\in, \in \lor q)\)-fuzzy filter of \( \mathcal{L} \).

**Proof:** Let \( \mu \) be an \((\in, \in \lor q)\)-fuzzy filter of \( \mathcal{L} \). For any \( \forall x \in F(t) \) is \( F \)-soft lattice implication algebra over \( \mathcal{L} \) and only if \( \mu \) is an \((\in, \in \lor q)\)-fuzzy filter of \( \mathcal{L} \).

\[ \text{Example 4.2:} \quad (x, y, z) \in (0, \frac{1}{2}) \text{ implies } x \lor y \lor z \text{ and } x \lor y \lor z \in (0, \frac{1}{2}). \]

**Corollary 4.3:** Let \( \mu \) be a fuzzy set of \( \mathcal{L} \) and \( (F, (0, \frac{1}{2})) \) be a soft set. Then \((F, (0, \frac{1}{2}))\) is a \( F \)-soft lattice implication algebra over \( \mathcal{L} \) if and only if \( \mu \) is an \((\in, \in \lor q)\)-fuzzy filter of \( \mathcal{L} \).

**Definition 4.3:** A fuzzy subset \( \mu \) on \( \mathcal{L} \) is said to be an \((\in, \in \lor q)\)-fuzzy filter if, and only if, for any \( x, y \in L, x \in (0, \frac{1}{2}) \):

\[ \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

\[ (1) L \in \mu \implies x \in \mathcal{L} \lor q \mu, \quad (2) \frac{\mu(x)}{\mu(y)} \lor q \mu \lor q \mu \lor q \mu. \]

**Definition 4.4:** A fuzzy subset \( \mu \) on \( \mathcal{L} \) is said to be an \((\in, \in \lor q)\)-fuzzy implicative filter if, and only if, for any \( x, y \in L, \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

\[ (1) L \in \mu \implies x \in \mathcal{L} \lor q \mu, \quad (2) \frac{\mu(x)}{\mu(y)} \lor q \mu \lor q \mu \lor q \mu. \]

**Example 4.2:** In Example 3.1, we define a fuzzy set \( \mu \) as follows:

\[ \mu(x) = 0.4, \quad \mu(0) = \mu(b) = \mu(c) = 0.9, \quad \mu(a) = \mu(d) = 0.3. \]

It is routine to verify that \( \mu \) is an \((\in, \in \lor q)\)-fuzzy (implicative) filter of \( \mathcal{L} \).

**Theorem 4.7:** Let \( \mu \) be a fuzzy subset of \( \mathcal{L} \), then \( \mu \) is an \((\in, \in \lor q)\)-fuzzy filter of \( \mathcal{L} \) if and only if for any \( x, y \in L, \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

\[ (1) \quad \frac{\mu(x)}{\mu(y)} \lor q \mu \lor q \mu \lor q \mu, \quad (2) \quad x \lor y \lor z \in (0, \frac{1}{2}). \]

**Proof:** Assume that there exist \( x, y \in L \), and \( \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

Assume that there exist \( x, y \in L \) such that \( \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

\[ \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

\[ (1) \quad \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

\[ (2) \quad \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

\[ (1) \quad \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]

\[ (2) \quad \mu(x) \lor \mu(y) \geq \min \{\mu(x), \mu(y)\}, x \lor y \geq 0.5. \]
Conversely, assume that there exist \( x \in L \) and \( t, r \in (\frac{1}{x+y}, 1] \) such that \( I_{x,r}(\mu) \), but \( x \not\in \mu \cup \{y\} \), then \( \mu(I) \) is defined as \( min\{\mu(x), \mu(y)\} \). This implies \( \mu(x) > \frac{1}{x+y} \). Thus \( \mu(I) \geq \frac{1}{x+y} \). It follows that \( \mu(x) > \frac{1}{x+y} \) and \( \mu(x) < \frac{1}{x+y} \). Hence \( \mu(I) \) is an \( (\in, \notin, \vee, \mu) \)-fuzzy filter of \( L \).

**Theorem 4.8:** Let \( \mu \) be a fuzzy subset of \( L \), then \( \mu \) is an \( (\in, \notin, \vee, \mu) \)-fuzzy lattice implication filter of \( L \) if and only if only for any \( x, y \in L \),

\[
\begin{align*}
(1) & \max\{\mu(I), \frac{1}{x+y}\} \geq \mu(x), \\
(2) & \max\{\mu(x), \mu(y)\} \geq \min\{\mu(x) \rightarrow (y \rightarrow z)\}, \mu(x \rightarrow y).
\end{align*}
\]

**Proof:** It is similarly proved as Theorem 4.7.

**Corollary 4.5:** Let \( \mu \) be a fuzzy subset of \( L \), then \( \mu \) is an \( (\in, \notin, \vee, \mu) \)-fuzzy lattice implication filter of \( L \) if and only if only for any \( x, y \in L \),

\[
\begin{align*}
(1) & \max\{\mu(I), 0.5\} \geq \mu(x), \\
(2) & \max\{\mu(x), 0.5\} \geq \min\{\mu(x) \rightarrow (y \rightarrow z)\}, \mu(x \rightarrow y).
\end{align*}
\]

**Theorem 4.9:** Let \( \mu \) be a fuzzy set of \( L \) and \( (F, (\frac{1}{x+y}, 1]) \) be a soft set. Then \( (F, (\frac{1}{x+y}, 1]) \) is a \( F \)-soft lattice implication algebra over \( L \) if and only if only if is an \( (\in, \notin, \vee, \mu) \)-fuzzy filter of \( L \).

**Proof:** Let \( (F, (\frac{1}{x+y}, 1]) \) is a \( F \)-soft lattice implication algebra over \( L \) for any \( t \in (\frac{1}{x+y}, 1] \). If there exists \( \alpha \in L \) such that \( \mu(a) \geq \max\{\mu(I), (\frac{1}{x+y})\}, \mu(a) \geq \max\{\mu(I), (\frac{1}{x+y})\} \) for some \( t \in (\frac{1}{x+y}, 1] \) and \( \mu(I) < x \), i.e. \( I \notin F(t) \), contradiction. Hence \( \mu(I) \geq (\frac{1}{x+y}) \) for any \( x \in L \). If there exist \( a, b \in F(t) \) such that \( \mu(a > b), \mu(a) \geq \max\{\mu(b), (\frac{1}{x+y})\} \) for some \( t \in (\frac{1}{x+y}, 1] \). Thus \( a > b \) in \( F(t) \) and \( a \in F(t) \), so that \( b \notin F(t) \), contradiction. Therefore \( \max\{\mu(y), (\frac{1}{x+y})\} \geq \min\{\mu(x) \rightarrow (y \rightarrow z)\}, \mu(x \rightarrow y) \) for any \( x, y \in L \). That is \( \mu \) is an \( (\in, \notin, \vee, \mu) \)-fuzzy lattice implication filter of \( L \).

Conversely, let \( \mu \) be an \( (\in, \notin, \vee, \mu) \)-fuzzy filter of \( L \). For any \( t \in (\frac{1}{x+y}, 1] \), we have \( \mu(x) \geq (\frac{1}{x+y}) \) for \( x \in F(t) \). Therefore \( \mu(I) \geq (\frac{1}{x+y}) \) and \( \mu(I) \) is defined as \( min\{\mu(x), \mu(y)\} \). This implies \( \mu(x) > (\frac{1}{x+y}) \). Thus \( \mu(I) \geq (\frac{1}{x+y}) \). It follows that \( \mu(I) \) is an \( (\in, \notin, \vee, \mu) \)-fuzzy lattice implication filter of \( L \).

**Corollary 4.6:** Let \( \mu \) be a fuzzy set of \( L \) and \( (F, (\frac{1}{x+y}, 1]) \) be a soft set. Then \( (F, (\frac{1}{x+y}, 1]) \) is a \( F \)-soft lattice implication algebra over \( L \) if and only if \( \mu \) is an \( (\in, \notin, \vee, \mu) \)-fuzzy lattice implication filter of \( L \).

**Theorem 4.10:** Let \( \mu \) be a fuzzy set of \( L \) and \( (F, (\frac{1}{x+y}, 1]) \) be a soft set. Then \( (F, (\frac{1}{x+y}, 1]) \) is an \( F \)-soft lattice implication algebra over \( L \) if and only if \( \mu \) is an \( (\in, \notin, \vee, \mu) \)-fuzzy lattice implication filter of \( L \).

**Proof:** It is similarly proved as Theorem 4.9.

**Corollary 4.7:** Let \( \mu \) be a fuzzy set of \( L \) and \( (F, (\frac{1}{x+y}, 1]) \) be a soft set. Then \( (F, (\frac{1}{x+y}, 1]) \) is an \( F \)-soft lattice implication algebra over \( L \) if and only if \( \mu \) is an \( (\in, \notin, \vee, \mu) \)-fuzzy lattice implication filter of \( L \).

V. CONCLUSION

Soft sets are related to fuzzy sets and rough sets. It applied to some algebraic structures. In this paper, we introduce the (implicative) filteristic soft lattice implication algebras which related with (implicative) filter (or soft, \( (IF-I\) )-soft lattice implication algebras). Basic properties of \( (IF-I)\)-soft lattice implication algebras are investigated. We introduce the notion of \( (\in, \notin, \vee, q) \)-fuzzy lattice implication algebras, which are generalizations of fuzzy lattice implication filters. Provide the information for a soft set to be a \( (IF-I)\)-soft lattice implication algebra. Analogously, this idea can be applied in other structures (such as positive implicative filters, ultra-filter, fantastic filter, and so on), analogously. It will be of great use to provide theoretical foundation to design intelligent information processing systems.

ACKNOWLEDGMENT

This work is supported by National Natural Science Foundation of P.R.China (Grant no. 60875034).

REFERENCES


Yi Liu He received Master in 2007 in Sichuan Normal University. Now, he is a lecturer in Neijiang Normal University, at the same time, he is also a PHD in Southwest Jiaotong University. His research interest includes: logical algebras, algebraic logic and automatic reasoning.

Yang Xu He is a professor in Southwest Jiaotong University, at the same time, also a PHD supervisor. His research interest is intelligent information processing.