

Wavelet Based Identification of Second Order Linear System

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Abstract—In this paper, a wavelet based method is proposed to identify the constant coefficients of a second order linear system and is compared with the least squares method. The proposed method shows improved accuracy of parameter estimation as compared to the least squares method. Additionally, it has the advantage of smaller data requirement and storage requirement as compared to the least squares method.

Keywords—Least squares method, linear system, system identification, wavelet transform.

I. INTRODUCTION

THE identification of second order linear time invariant system is of fundamental importance in many applications of science and engineering, including control systems, electrical networks, vibration of spring etc[1]. A second order time invariant linear system is defined by a second order linear differential equation with constant coefficients. The response of a second order linear system has two fundamental parameters, a time constant decided by the damping and a natural frequency of oscillation. The duration of the signal depends upon the time constant, while the maximum frequency of variation of the signal depends upon the natural frequency. In signal analysis, it is useful to approximate the signal as an expansion over some basis with only a few significant nonzero coefficients. This motivated us to use the wavelet transform for parameter estimation. The parameter estimation can be done using the least squares method, however, this would involve all the samples of the output signal over the given observation range. This requires a lot of data storage. On the other hand, we could choose an orthonormal basis for the signal space and consider estimating the coefficients appearing in the differential equation from knowledge of only the inner product of the output signal with this basis. The proper choice of the basis helps to get the information about the output signal in the inner products. Such a good basis can be chosen using the theory of wavelets, as the wavelet coefficients contain information about the resolution as well as the signal duration. Thus, if the parameter estimation is based on these few wavelet coefficients, then with lesser data storage, we could get improved estimates of the parameters. The wavelet and the least squares method have been used for system identification by different authors[2]-[4]. The parameters of second order system are estimated by matching the theoretical expression with the noisy measurement using the gradient search algorithm. The wavelet based identification

method requires smaller amount of data as compared to the least squares method and provides more accurate results as compared to the least squares method. This paper is organized as follows- Section II describes the parameter identification by the least squares method and present brief theory of wavelet transform and the identification of parameters by the wavelet transform method. Section III provides simulation results. Section IV presents conclusions.

II. METHODOLOGY

A. Identification of second order linear systems by the least squares method

The least squares method is widely used to estimate the numerical values of parameters and to characterize the statistical properties of parameters. In this parameter identification method, the unknown parameter is chosen to minimize the total squared error between the system output and the predicted value. It can be extended to more than one independent parameter estimation. In the geometrical framework, the least squares method can be interpreted as an orthogonal projection of the data vector onto the space defined by the independent variable. The projection error is orthogonal to the data on which the estimation is based.

The general second order linear differential equation with constant coefficients is given by:

$$\frac{d^2 y(t)}{dt^2} + a \frac{dy(t)}{dt} + by(t) = x(t) \quad (1)$$

where $x(t)$ is the input and $y(t)$ is the output. The problem of identifying such a system from input-output data taken over a duration $[0, T]$ can be solved by noting that if $y(0) = y'(0) = 0$, then the solution to this equation is given by:

$$y(t) = \int_0^t h(\tau, a, b)x(t - \tau)d\tau \quad (2)$$

where:

$$\begin{aligned} H(s, a, b) &= \int_0^\infty h(\tau, a, b)\exp(-s\tau)d\tau \\ &= \frac{1}{s^2 + as + b} = \frac{1}{(s + a/2)^2 + b - a^2/4} \end{aligned} \quad (3)$$

Setting:

$$\gamma = a/2, \omega = \sqrt{b - a^2/4} \quad (4)$$

gives on Laplace inversion:

$$h(t, a, b) = \frac{\sin(\omega t)}{\omega} \exp(-\gamma t)u(t) \quad (5)$$

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where $u(t)$ is the unit step function. The parameters a, b can be estimated when noisy measurements on $y(t)$ are taken, by minimizing:

$$F(\gamma, \omega) = \int_0^T \left(\int h(\tau, a, b)x(t - \tau)d\tau - d(t) \right)^2 dt \quad (6)$$

where:

$$d(t) = y(t) + w(t) \quad (7)$$

$w(t)$ is the noise. For example, if we apply an impulse input $x(t) = \delta(t)$, the problem boils down to minimizing:

$$F(\gamma, \omega) = \int_0^T (h(t, a, b) - d(t))^2 dt \quad (8)$$

The same method can be applied to identifying the parameters with zero input. The response is then decided by the initial conditions, i.e., we can write the general solution to the homogeneous differential equation:

$$y''(t) + ay'(t) + by(t) = 0 \quad (9)$$

as:

$$y(t) = Aexp(-\gamma t)sin(\omega t + \phi) \quad (10)$$

where the constants A, ϕ are determined from the initial conditions $y(0), y'(0)$. The parameters γ, ω and the initial conditions A, ϕ can then be estimated from noisy measurements on $y(t)$ by minimizing the function:

$$F(A, \phi, \gamma, \omega) = \int_0^T (Aexp(-\gamma t)sin(\omega t + \phi) - d(t))^2 dt \quad (11)$$

The minimization can be carried out using the gradient search algorithm. The practical implementation of this method involves discretizing the differential equation with a step size of Δ to get the difference equation:

$$\frac{d[n+2] - 2d[n+1] + d[n]}{\Delta^2} + a \frac{d[n+1] - d[n]}{\Delta} + bd[n] = 0 \quad (12)$$

The noise is assumed to be white Gaussian with zero mean and variance σ^2 , where σ is much smaller than the nominal value of $y[n]$. The nominal value of $y[n]$ is taken as $y[0]$ and the condition that the noise is small is expressed as $\sigma = y[0]/10$. We take the duration of simulation as ten times of the time constants. Simulations have been done using $\gamma = 1/10$ and $2\pi/\omega = 1$. The nominal duration of the signal is then 10 seconds and the nominal time period is one second which is one tenth of the duration. This means that in the nominal duration, we have approximately ten cycles. We take $y[0] = 1$ and $\sigma = 1/10$ and simulate the above difference equation. The time step Δ of discretization has been taken as $1/100$, which is one hundredth of the nominal time period. We let $N = 10/\Delta = 1000$ and simulate the above equation to obtain $y[n], n = 0, 1, \dots, N-1$. This is then matched to exact sampled response of the original continuous time system, i.e., to:

$$y(n\Delta) = Aexp(-\gamma\Delta n)sin(\omega\Delta n + \phi) \quad (13)$$

by minimizing:

$$F(A, \phi, \gamma, \omega) = \sum_{n=0}^{N-1} (y(n\Delta) - d[n])^2 = \sum_{n=0}^{N-1} (Aexp(-\gamma\Delta n)sin(\omega\Delta n + \phi) - d[n])^2 \quad (14)$$

with respect to A, ϕ, γ, ω . This minimization can be carried out either using a search method or the gradient method. The amount of data storage is $N = 1000$ samples. To carry out the gradient search we use the following partial derivatives:

$$\frac{\partial F}{\partial A} = 2 \sum_{n=0}^{N-1} [(Aexp(-\gamma\Delta n)sin(\omega\Delta n + \phi) - d[n]) exp(-\gamma\Delta n)sin(\omega\Delta n + \phi)] \quad (15)$$

$$\frac{\partial F}{\partial \phi} = 2 \sum_{n=0}^{N-1} (Aexp(-\gamma\Delta n)cos(\omega\Delta n + \phi) - d[n]) \quad (16)$$

$$\frac{\partial F}{\partial \gamma} = -2 \sum_{n=0}^{N-1} [(Aexp(-\gamma\Delta n)sin(\omega\Delta n + \phi) - d[n]) A\Delta n exp(-\gamma\Delta n)sin(\omega\Delta n + \phi)] \quad (17)$$

$$\frac{\partial F}{\partial \omega} = 2 \sum_{n=0}^{N-1} (Aexp(-\gamma\Delta n)sin(\omega\Delta n + \phi) - d[n]) A\Delta n exp(-\gamma\Delta n)cos(\omega\Delta n + \phi) \quad (18)$$

The gradient search algorithm then starts with an initial guess ($A_0, \phi_0, \gamma_0, \omega_0$) and performs the update:

$$A_{k+1} = A_k - \mu \frac{\partial F(A_k, \phi_k, \gamma_k, \omega_k)}{\partial A_k} \quad (19)$$

$$\phi_{k+1} = \phi_k - \mu \frac{\partial F(A_k, \phi_k, \gamma_k, \omega_k)}{\partial \phi_k} \quad (20)$$

$$\gamma_{k+1} = \gamma_k - \mu \frac{\partial F(A_k, \phi_k, \gamma_k, \omega_k)}{\partial \gamma_k} \quad (21)$$

$$\omega_{k+1} = \omega_k - \mu \frac{\partial F(A_k, \phi_k, \gamma_k, \omega_k)}{\partial \omega_k} \quad (22)$$

B. Wavelet Transform

Wavelets are families of functions generated from one single function called mother wavelet, by scaling and translating operations[5]. It cuts up data or functions into different frequency components and then studies each component with a resolution matched to its scale. A wavelet owns many attractive properties including the essential properties such as compact support, vanishing moments, dilating relation and other preferred properties such as smoothness and being a generator of an orthonormal basis of function spaces $L^2(R)$. Compact support guarantees the localization of wavelets; vanishing moments guarantee wavelets can distinguish essential features of a signal from non-essential features and dilating relation leads to fast wavelet algorithms. We used Daubechies

(D4) wavelet. Daubechies wavelets have good compression property for wavelet coefficients. They are efficient for compact representation of signal details[7]-[9].

In this paper the Daubechies mother wavelet function is constructed starting using the scaling function sequence $u(k)$ [6]. The scaling function is obtained by solving the scaling relation:

$$\begin{aligned} \phi(x) &= \sum_{k \in Z} u(k)\phi_{1,k}(x) \\ &= \sum_{k \in Z} u(k)\phi(2x - k) \end{aligned} \quad (23)$$

The Fourier transform of both sides of the scaling identity gives

$$\hat{\phi}_{1,k}(\xi) = \frac{1}{\sqrt{2}} e^{-ik\xi/2} \hat{\phi}(\xi/2) \quad (24)$$

where $\hat{\phi}(\xi)$ is the Fourier transform of $\phi(x)$. Interchanging the sum and the integral in the definition of the Fourier transform of the right side of equation (23)

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2}} \sum_{k \in Z} u(k) e^{-ik\xi/2} \hat{\phi}(\xi/2) \quad (25)$$

Defining:

$$m_o(\xi) = \frac{1}{\sqrt{2}} \sum_{k \in Z} u(k) e^{-ik\xi} \quad (26)$$

Then equation (24) gives:

$$\hat{\phi}(\xi) = m_o(\xi/2) \hat{\phi}(\xi/2) \quad (27)$$

The above equation can be iterated. Applying equation (26) with ξ replaced by $\xi/2$, we get:

$$\hat{\phi}(\xi/2) = m_o(\xi/4) \hat{\phi}(\xi/4) \quad (28)$$

One can continue this and obtain, for any $n \in N$:

$$\hat{\phi}(\xi) = m_o(\xi/2) m_o(\xi/4) \dots m_o(\xi/2^n) \hat{\phi}(\xi/2^n) \quad (29)$$

To generate the mother wavelet we used:

$$v(k) = (-1)^{k-1} \overline{u(1-k)} \quad (30)$$

where $\overline{u(1-k)}$ is the complex conjugate of $u(1-k)$. Defining the wavelet by:

$$\psi = \sum_{k \in Z} v(k)\phi_{1,k} \quad (31)$$

Taking the Fourier transform of both sides of the above equation:

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}} \sum_{k \in Z} v(k) e^{-ik\xi/2} \hat{\phi}(\xi/2) \quad (32)$$

Defining:

$$m_1(\xi) = \frac{1}{\sqrt{2}} \sum_{k \in Z} v(k) e^{-ik\xi} \quad (33)$$

we get:

$$\hat{\psi}(\xi) = m_1(\xi/2) \hat{\phi}(\xi/2) \quad (34)$$

Iterating equation (26) we get:

$$\hat{\psi}(\xi) = m_1(\xi/2) \prod_{j=2}^{\infty} m_o(\xi/2^j) \quad (35)$$

using the normalization that $\hat{\phi}(0) = 1$. Taking the inverse Fourier transform we obtained the mother wavelet function.

C. Identification of second order linear systems using wavelet transform method

Different parametric and non-parametric methods have been used for system identification purpose[10]-[14]. Wavelet transform can be used to identify the above second order system. Suppose that the mother wavelet $\psi(t)$ is concentrated over the interval $[p, q]$. Then $\psi_{n,k}(t) = 2^{n/2} \psi(2^n t - k)$ is concentrated over the interval $[(k+p)/2^n, (k+q)/2^n]$. Let N_1 and N_2 be respectively the minimum and the maximum resolution indices of the wavelets. These must be chosen so that the durations of support $(q-p)/2^{N_1}$ and $(q-p)/2^{N_2}$ of the corresponding wavelets are respectively of the order of the signal duration and the reciprocal of the maximum signal frequency, i.e., $(q-p)/2^{N_1} = 10$, $(q-p)/2^{N_2} = 1$. For a given n , we allow k to range from $k_1(n) = -p$ to $k_2(n) = 2^n 10 - q$. The number of wavelet coefficients needed are:

$$\sum_{n=N_1}^{N_2} (2^n 10 - q + p)$$

which is much smaller than 1000. There exist nonzero wavelet coefficients for $n \leq N_1$ and $n \geq N_2$ but these will not be significant. N_1 and N_2 have been chosen in accordance with the signal duration and maximum frequency and hence only if the resolution falls in the range (N_1, N_2) , the wavelet coefficients will be significant. By the wavelet method we will be able to identify the system with lesser data storage as compared to the least squares method. The first step is to compute the wavelet coefficients:

$$c(n, k), N_1 \leq n \leq N_2, k_1(n) \leq k \leq k_2(n) \quad (36)$$

using the formula:

$$\begin{aligned} c(n, k) &= \int_0^T y(t) \psi_{n,k}(t) dt \approx \sum_{m=0}^{N-1} y(m\Delta) \psi_{n,k}(m\Delta) \Delta \\ &= \sum_{m=0}^{N-1} A \exp(-\gamma \Delta m) \sin(\omega \Delta m + \phi) \psi_{n,k}(m\Delta) \Delta \end{aligned} \quad (37)$$

This computation must be carried out for different values of the parameters A, ϕ, γ, ω over the range where these parameters are likely to fall. To emphasize the fact that these coefficients will be dependent upon the values of these parameters, we denote these by $c(n, k|A, \phi, \gamma, \omega)$. Having done so, we simulate

the noisy system and for the given realization, estimate the wavelet coefficients:

$$\hat{c}(n, k) = \sum_{m=0}^{N-1} d[m] \psi_{n,k}(m\Delta) \Delta \quad (38)$$

The values of the parameters for the simulated process are assumed to be unknown and to be estimated. This is done by minimizing:

$$F(A, \phi, \gamma, \omega) = \sum_{(n,k) \in E} (c(n, k|A, \phi, \gamma, \omega) - \hat{c}(n, k))^2 \quad (39)$$

where:

$$E = \{(n, k) : N_1 \leq n \leq N_2, k_1(n) \leq k \leq k_2(n)\} \quad (40)$$

The minimization is carried out by using a search from the look up table that gives the wavelet coefficients $c(n, k|A, \phi, \gamma, \omega)$ as a function of A, ϕ, γ, ω .

III. SIMULATIONS

The proposed methods are experimented on second order linear time invariant system. Numerical simulation has been done using MATLAB. For simulation, white Gaussian noise with zero mean and small variance is used. To find the minimum value of $F(A, \phi, \gamma, \omega)$ the look up table is performed using the range of values where they are likely to fall. The wavelet method is performed using 256 samples and the least squares method is performed using 1000 samples. The signal is reconstructed using the values of $(A, \phi, \gamma, \omega)$ obtained by both methods that corresponds to minimum value of $F(A, \phi, \gamma, \omega)$. Fig. 1-4 show the error in estimating the parameters. These figures show that the result of the wavelet-based method is closer to the theoretical one, while the result of the least squares method has greater estimation error.

The Signal to Noise ratio obtained by both methods is given below:

$$SNR = \frac{\sum |\hat{y}(n)|^2}{\sum |(\hat{d}(n) - \hat{y}(n))|^2} \quad (41)$$

where $\hat{y}(n)$ is obtained using true values of a, b and $\hat{d}(n)$ is obtained using estimated values of a, b .

Table 1 Signal to Noise Ratio using the least squares method and the wavelet method

Least squares method	Wavelet method
0.315	2.0

IV. CONCLUSIONS

This paper presents a wavelet based method for identification of second order linear system. The wavelet method gives larger value of signal to noise ratio as compared to the least squares method. Additionally, it gives very much smaller value of $F(A, \phi, \gamma, \omega)$ as compared to the least squares method. This is due to compact representation of signal details using Daubechies wavelet. In the least squares method, samples are used, which causes information loss contained in the signal. In

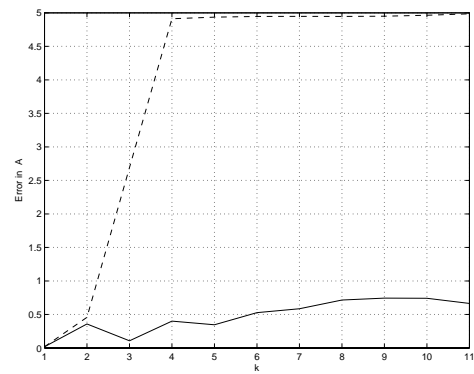


Fig. 1. Plot of error in numerical value of ϕ with respect to theoretical value. The dashed line shows the least squares method and the solid line shows the wavelet method.

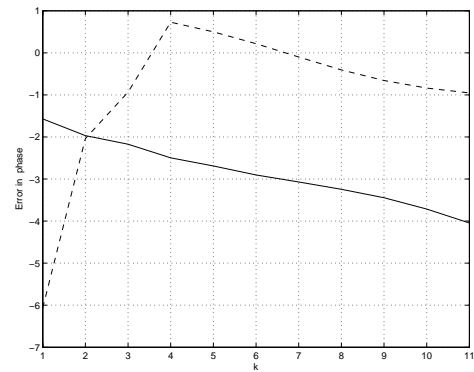


Fig. 2. Plot of error in numerical value of ϕ with respect to theoretical value. The dashed line shows the least squares method and the solid line shows the wavelet method.

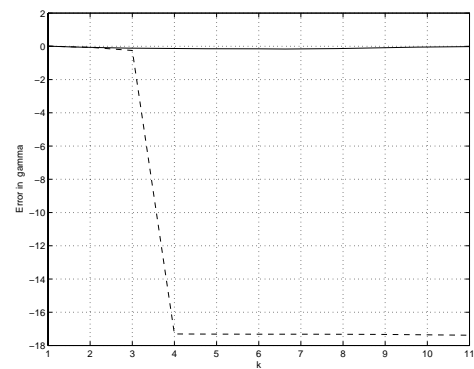


Fig. 3. Plot of error in numerical value of γ with respect to theoretical value. The dashed line shows the least squares method and the solid line shows the wavelet method.

the wavelet method, by choosing a sufficient high resolution level, we can retain most of the signal features(local as well as global). To obtain better signal to noise ratio large amount of data is required by the least squares method. The wavelet method requires fewer samples as compared to the least squares method because the inherent time frequency resolution of the signal structure is taken into account.

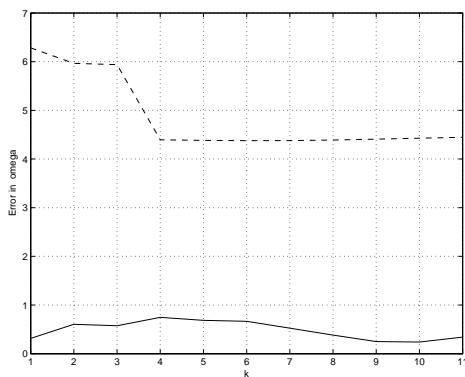


Fig. 4. Plot of error in numerical value of ω with respect to theoretical value. The dashed line shows the least squares method and the solid line shows the wavelet method.

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