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Marangoni Convection in a Fluid Layer with Internal Heat Generation

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***Abstract*—In this paper we use classical linear stability theory to investigate the effects of uniform internal heat generation on the onset of Marangoni convection in a horizontal layer of fluid heated from below. We use a analytical technique to obtain the close form analytical expression for the onset of Marangoni convection when the lower boundary is conducting with free-slip condition. We show that the effect of increasing the internal heat generation is always to destabilize the layer.**

***Keywords*—Marangoni convection, heat generation, free-slip**

I. INTRODUCTION

THE effect of nonlinear temperature distribution arising due to heat generation is investigated on the stability of Marangoni convection with the objective of understanding the control of convective instability which is important in many engineering and applications in the heat and momentum transfer research. Theoretical analysis of Marangoni convection was started with the linear analysis by Pearson [1] who assumed an infinite fluid layer, a nondeformable case and zero gravity in the case of no-slip boundary conditions at the bottom. He showed that thermocapillary forces can cause convection when the Marangoni number exceeds a critical value in the absence of buoyancy forces.

In the above Marangoni instability analysis, the convective instability is induced by the temperature gradient which is decreasing linearly with fluid layer height. Sparrow et. al[2] and Roberts[3] analyze the thermal instability in a horizontal fluid layer with the nonlinear temperature distribution which is created by an internal heat generation. The effect of a quadratic basic state temperature profile caused by internal heat generation was first addressed by Char and Chiang [4] for Bénard-Marangoni convection. Later, Wilson [5] investigate the effect of the internal heat generation on the onset of Marangoni convection when the lower boundary is conducting and when it is insulating to temperature perturbations. He found that the effect of increasing the internal heat generation is always to destabilize the layer. Recently, Arifin and Bachok [6] have studied the onset of Marangoni convection in a fluid-solid layer system with internal heat generation and they found that the critical Marangoni number increases with the depth ratio or the thermal conductivity ratio. The effect of control on the onset of Marangoni convection in a horizontal fluid layer with heat generation have been investigated by Arifin and Bachok [7]. They have shown that the control can delay the onset of Marangoni convection.

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Theoretical analysis of Marangoni convection with free-slip at the bottom surface was started with the linear analysis by Boeck and Thess [8]. They obtained the analytical expression for the onset of Marangoni convection. Therefore, in this paper, we extend Boeck and Thess [8] analysis to include the internal heat generation on the onset of Marangoni convection. We first derive the analytical expressions for the critical Marangoni convection and next we demonstrate that the internal heat generation is a destabilizing factor.

II. MATHEMATICAL FORMULATION

Consider a horizontal fluid layer of depth d with a free upper surface heated from below subject to a uniform vertical temperature gradient. The fluid layer is bounded below by a horizontal solid boundary at constant temperature T_1 and above by a free surface at constant temperature T_2 which is in contact with a passive gas at constant pressure P_0 and constant temperature T_∞. We used Cartesian coordinates with two horizontal x- and y-axes located at the lower solid boundary and a positive z-axis is directed towards the free surface. The surface tension, τ is assumed to be a linear function of the temperature

$$\tau = \tau_0 - \gamma(T - T_0), \tag{1}$$

where τ_0 is the value of τ at temperature T_0 and the constant γ is positive for most fluids. The density of the fluid is given by

$$\rho = \rho_0\{1 - \alpha(T - T_0)\}, \tag{2}$$

where α is the positive coefficient of the thermal liquid expansion and ρ_0 is the value at the reference temperature T_0.

The fluid is assumed to be an incompressible Newtonian liquid satisfying the continuity equation together with the Navier-Stokes and the heat equations. These equations are, respectively

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu\Delta\mathbf{u}, \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)T = \kappa\nabla^2 T + q \tag{5}$$

where $\mathbf{u}$, T, p, ρ, ν, κ and q denote the velocity, temperature, pressure, density, kinematic viscosity, thermal diffusivity and uniformly distributed volumetric internal heat generation in the fluid layer, respectively. When motion occurs, the upper free surface of the layer will be deformable with its position at $z = d + f(x, y, t)$. At the free surface, we have the usual kinematic condition together with the conditions of continuity for the normal and tangential stresses. The temperature obeys the Newton's law of cooling, $k\partial T/\partial \mathbf{n} = h(T - T_\infty)$, where k and h are the thermal conductivity of the fluid and the heat transfer coefficient between the free surface and the air, respectively, and **n** is the outward unit normal to the free surface. The boundary conditions at the bottom wall, $z = 0$, are no-slip and conducting to the temperature perturbations.

To simplify the analysis, it is convenient to write the governing equations and the boundary conditions in a dimensionless form. In the dimensionless formulation, scales for length, velocity, time and temperature gradient are taken to be d, κ/d, d^2/κ and ΔT respectively. Furthermore, six dimensionless groups appearing in the problem are the Marangoni number, $M = \gamma\Delta T d/\rho_0\kappa\nu$, the Biot number, $B_{\rm i} = hd/k$, the Bond number, $B_{\rm o} = \rho_0 g d^2/\tau_0$, the Prandtl number, $P_{\rm r} = \nu/\kappa$, the Crispation number, $C_{\rm r} = \rho_0\nu\kappa/\tau_0 d$ and the internal heating, $Q = qd^2/2\kappa\Delta T$.

Standard methods of linear stability analysis are used to determine the effect of internal heat generation, Q, on the critical Marangoni number at the onset of convection with internal heat generation. We start with a linear stability analysis of the basic state in the usual manner by seeking perturbed solutions for any quantity $\Phi(x, y, z, t)$ in terms of normal modes in the form

$$\Phi(x, y, z, t) = \Phi_0(x, y, z) + \phi(z) \exp\left[i(\alpha_x x + \alpha_y y) + st\right], \tag{6}$$

where Φ_0 is the value of Φ in the basic state, $a = (\alpha_x^2 + \alpha_y^2)^{1/2}$ is the total horizontal wave number of the disturbance and s is a complex growth rate with the real part representing the growth rate of the instability and the imaginary part representing its frequency. At marginal stability, the growth rate s of perturbation is zero and the real part of s, Re(s)> 0 represents unstable modes while Re(s)< 0 represents stable modes.

Substituting equation (6) into equations (3)–(5), we obtain the corresponding linearized equations involving only the z-dependent parts of the perturbations to the temperature and the z-components of the velocity denoted by T and w respectively, namely

$$(D^2 - a^2)(D^2 - a^2 - sP_{\rm r}^{-1})w = 0, \tag{7}$$
$$(D^2 - a^2 - s)T + [1 - Q(1 - 2z)]w = 0. \tag{8}$$

The upper free surface is assumed to be deformable and conducting to temperature perturbations, and the boundary conditions are

$$sf - w = 0, \tag{9}$$
$$C_{\rm r}[(D^2 - 3a^2 - sP_{\rm r}^{-1})Dw] - a^2(a^2 + B_{\rm o})f = 0, \tag{10}$$
$$(D^2 + a^2)w + a^2 M(T - (1 + Q)f) = 0, \tag{11}$$
$$DT + B_{\rm i}(T - (1 + Q)f) = 0, \tag{12}$$
$$\tag{13}$$

evaluated on $z = 1$. The boundary conditions at the bottom are for rigid surface with conducting to temperature perturbation

$$w = 0, \tag{14}$$
$$D^2 w = 0, \tag{15}$$
$$T = 0, \tag{16}$$

evaluated on $z = 0$. The operator D=d/dz denotes differentiation with respect to the vertical coordinate z,

III. Results and discussion

By substituting the general solution of equations (7) and (8) into the boundary conditions (9)–(16) and requiring the existence of nontrivial solutions, we obtain the closed form analytical expression for M in terms of a, Q, $C_{\rm r}$, $B_{\rm i}$ and $B_{\rm o}$ on the marginal curve in the form

$$M = \frac{(a^2 + B_{\rm o})A_1}{(a^2 + B_{\rm o})[A_2 + QA_3] + C_{\rm r}(1 + Q)A_4}, \tag{17}$$

where

$$\begin{aligned} A_1 &= 8a^2(aC + B_{\rm i}S)S^2 \\ A_2 &= aCS^2 + a^2S - 2a^3C \\ A_3 &= aCS^2 + a^2S - 2S^3 \\ A_4 &= 8a^5C, \end{aligned}$$

where $S = \sinh a$ and $C = \cosh a$. We use the symbolic algebra package MAPLE 12 running on a Pentium PC to carry out much of the tedious algebraic manipulations to obtain equation (17). When we set $Q = 0$, the equation (17) reduces to the expression given by Boeck and Thess [8].

Marginal stability curve curves from Eq. 17 are shown in Fig. 1 and 2 for a range of values of $C_{\rm r} = 0$ in the case $B_{\rm o} = 0$ and $B_{\rm o} = 0.1$ respectively with $Q = 1$. Numerical calculation of (17) shows that M_c may occur at either $a_c = 0$ or $a_c \neq 0$ in the case $C_{\rm r} \neq 0$. From Fig. 2, it can be seen that for $B_{\rm o} > 0$, the critical Marangoni number no longer attains a zero value at $a = 0$.

Fig. 3 shows the critical Marangoni number at the onset of convection as a function of Q, for a range of values of $C_{\rm r}$ in the case $B_{\rm o} = 0.1$ and $B_{\rm i} = 0$. Fig. 4 shows the critical wave number a_c at the onset of convection as a function of Q, for a range of values of $C_{\rm r}$ in the case $B_{\rm o} = 0.1$ and $B_{\rm i} = 0$. Fig. 3 shows that M_c is a monotonically decreasing function of Q, while Fig 4 shows that a_c is a monotonically increasing function of Q and clearly shows how a_c jump discontinuously as the global minimum changes, the value M_c varies continuously as the other parameters are varied. This signifies a sudden changeover from convection occur at $a_c = 0$ to $a_c \neq 0$. The heat generation are clearly a destabilizing factor to make system more unstable.

IV. Conclusion

The effect of the feedback control on the onset of steady Marangoni convection instabilities in a fluid layer which is free above and rigid below with free-slip condition has been studied. The explicit analytical expressions for the critical

Marangoni number have been obtained. We have shown that the parameters Q, C_{r} and B_{o} are critically important to the onset of steady Marangoni convection. We showed that the effect of increasing the heat generation is always destabilize the fluid layer.

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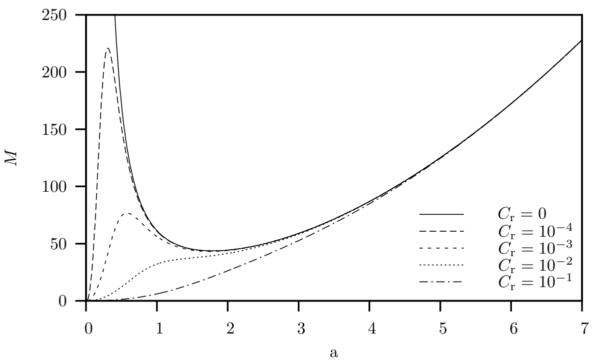

Fig. 1. Numerically calculated Marangoni number at the onset of convection as a function of wave number, a, for a range of values of Crispation number, C_{r}in the case $B_{\mathrm{i}} = 0$, $B_{\mathrm{o}} = 0$ and $Q = 1$.

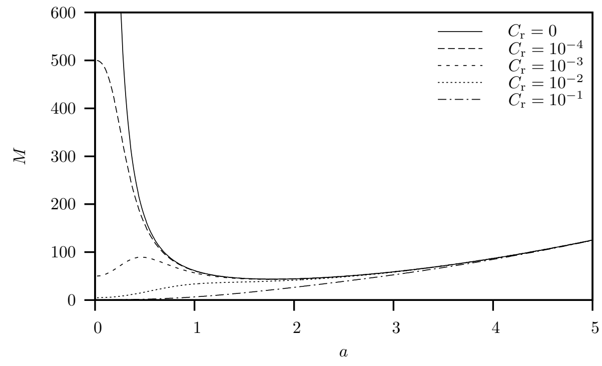

Fig. 2. Numerically calculated Marangoni number at the onset of convection as a function of wave number, a, for a range of values of Crispation number, C_{r}in the case $B_{\mathrm{i}} = 0$, $B_{\mathrm{o}} = 0.1$ and $Q = 1$.

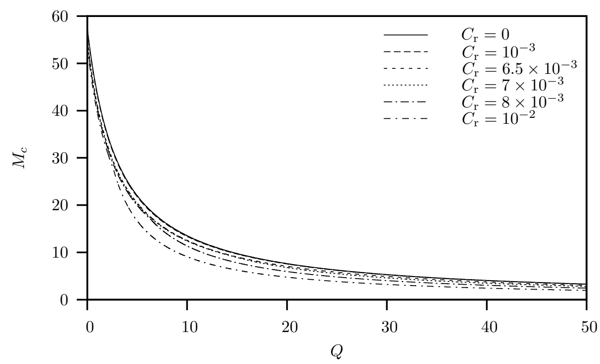

Fig. 3. The critical Marangoni number at the onset of convection as a function of Q, for a range of values of Crispation number, C_{r}in the case $B_{\mathrm{i}} = 0$ and $B_{\mathrm{o}} = 0.1$

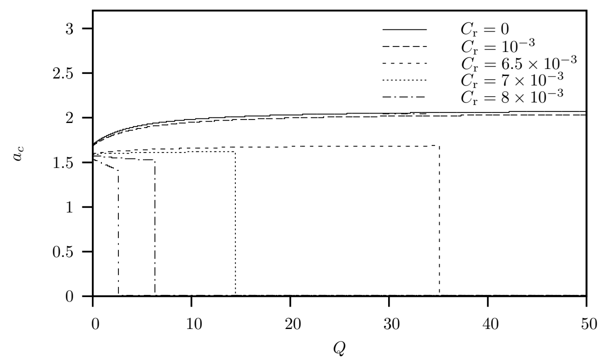

Fig. 4. The critical wave number at the onset of convection as a function of Q, for a range of values of $C_{\mathrm{r}} = 0$ in the case $B_{\mathrm{i}} = 0$ and $B_{\mathrm{o}} = 0.1$.