

An Iterative Algorithm to Compute the Generalized Inverse $A_{T,S}^{(2)}$ Under the Restricted Inner Product

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Abstract—Let T and S be a subspace of C^n and C^m , respectively. Then for $A \in C^{m \times n}$ satisfied $AT \oplus S = C^m$, the generalized inverse $A_{T,S}^{(2)}$ is given by $A_{T,S}^{(2)} = (P_{S^\perp}AP_T)^\dagger$. In this paper, a finite formulae is presented to compute generalized inverse $A_{T,S}^{(2)}$ under the concept of restricted inner product, which defined as $\langle A, B \rangle_{T,S} = \langle P_{S^\perp}AP_T, B \rangle$ for the $A, B \in C^{m \times n}$. By this iterative method, when taken the initial matrix $X_0 = P_TA^*P_{S^\perp}$, the generalized inverse $A_{T,S}^{(2)}$ can be obtained within at most mn iteration steps in absence of roundoff errors. Finally given numerical example is shown that the iterative formulae is quite efficient.

Keywords—Generalized inverse $A_{T,S}^{(2)}$, Restricted inner product, Iterative method, Orthogonal projection.

I. INTRODUCTION

As for our topic, we first recall the definition of generalized inverse $A_{T,S}^{(2)}$ of a matrix $A \in C^{m \times n}$, which has a range T and null space S .

Definition 1.1^[1]. Let $A \in C^{m \times n}$ be of rank r , T be a subspace of C^n of dimension $s \leq r$ and S be a subspace of C^m of dimension $m - s$. If X satisfies

$$XAX = X, R(X) = T, N(X) = S$$

then X is called the generalized inverse $A_{T,S}^{(2)}$ of A .

It is well-known that the common important generalized inverses are all the generalized inverse $A_{T,S}^{(2)}$, which have the prescribed range T and the null space S of $\{2\}$ -(or outer) inverse of A . These were introduced in [1,2].

The outer generalized inverse $A_{T,S}^{(2)}$ has been widely used in various fields, for instance, in statistics, control theory, power systems, nonlinear equations, optimization and numerical analysis, and so on. The applications and computations of the outer generalized inverse can be found in [1-12,14-16].

First using finite iterative method to solve the linear system can be seen in [17]. But using this method to compute the generalized inverse only can viewed in author's paper [13]. We could not directly use finite iterative formulae to compute the generalized inverse $A_{T,S}^{(2)}$, because its range and null space are not orthogonal. In this paper, we first define a restricted inner product. Then a restricted norm of a matrix A is generated by this inner product. In the end, a finite iterative formulae to compute the generalized inverse $A_{T,S}^{(2)}$ is devised.

Throughout the paper, let $C_r^{m \times n}$ denote the set of all $m \times n$ matrices with rank r over C . T and S is a subspace of C^n

and C^m , respectively, with dimension $s (s \leq r)$. For any $A \in C^{m \times n}$, we write $R(A)$ for its range, $N(A)$ for its null range. A^* and $r(A)$ stands for the conjugate transpose and the rank of A , respectively.

Let T and S be the subspace of C^n and C^m , respectively. The restricted conjugate transpose on T and S of $A_{T,S}^*$ for a complex matrix $A \in C^{m \times n}$ is defined as $A_{T,S}^* = (P_{S^\perp}AP_T)^* = P_TA^*P_{S^\perp}$. In the same way, in the space $C^{m \times n}$, a restricted inner product on subspace T and S is defined as $\langle A, B \rangle_{T,S} = \langle P_{S^\perp}AP_T, BP_T \rangle = \text{tr}(A^*P_{S^\perp}BP_T)$ for all $A, B \in C^{m \times n}$. Then the restricted norm on the subspace of T and S of a matrix A generated by this inner product is the Frobenius norm of matrix $P_{S^\perp}AP_T$ denoted by $\|A\|_{T,S}$.

In this paper the following Lemmas are needed in what follows:

Lemma 1.1^[1] Let $A \in C^{m \times n}$ be of rank r , any two of the following three statements imply the third:

$$\begin{aligned} X &\in A\{1\} \\ X &\in A\{2\} \\ \text{rank } A &= \text{rank } X \end{aligned}$$

Lemma 1.2^[1] Let $A \in C^{m \times n}$ be of rank r , T be a subspace of C^n of dimension $s \leq r$ and S be a subspace of C^m of dimension $m - s$. Then A has a $\{2\}$ inverse X such that $R(X) = T$ and $N(X) = S$, if and only if

$$AT \oplus S = C^m$$

In which case X is unique and it is denoted by $A_{T,S}^{(2)}$.

Lemma 1.3^[1,2] Let A, T, S and G be the same as in Lemma 1.2. Then A has a $\{2\}$ inverse $A_{T,S}^{(2)}$, and

$$A_{T,S}^{(2)} = (P_{S^\perp}AP_T)^\dagger$$

Lemma 1.4^[1] (1) $P_{L,M}A = A$ if and only if $R(A) \subset L$,
(2) $AP_{L,M} = A$ if and only if $N(A) \supset M$

Throughout the paper, we assume that $AT \oplus S = C^m (A^*S^\perp \oplus T^\perp = C^n)$, in other words the generalized inverse $A_{T,S}^{(2)}$ of a matrix A is existed.

About the restricted inner product on subspace T and S , we have the following property.

Lemma 1.5 Let T and S be the subspace of C^n and C^m , $A, B \in C^{m \times n}$, then we have:

$$\begin{aligned} \langle A, B \rangle_{T,S} &= \langle A, P_{S^\perp}BP_T \rangle = \langle P_{S^\perp}AP_T, B \rangle \\ &= \overline{\langle B, A \rangle_{T,S}} = \langle B^*, P_TA^*P_{S^\perp} \rangle \end{aligned}$$

According to the definition and the properties of inner product, the above equalities are right.

II. ITERATIVE METHOD FOR COMPUTING $A_{T,S}^{(2)}$

In this section we first introduce an iterative method to obtain a solution of the matrix equation $P_{S^\perp}AXAP_T = P_{S^\perp}AP_T$, where $A \in C^{m \times n}$, $T \in C^n$ and $S \in C^m$ satisfied $AT \oplus S = C^m$. Then for any initial matrix X_0 with $R(X_0) \subset R(P_TA^*)$ and $N(X_0) \supset N(A^*P_{S^\perp})$, the matrix sequence $\{X_k\}$ generated by the iterative method converges to its a solution within at most mn iteration steps in absence of the roundoff errors. We also show that if let the initial matrix $X_0 = P_TA^*P_{S^\perp}$, then the solution X^* obtained by the iterative method is the generalized inverse $A_{T,S}^{(2)}$.

First we present the iteration method for solving the matrix equation $P_{S^\perp}AXAP_T = P_{S^\perp}AP_T$, the iteration method as follow:

Algorithm 2.1

1. In put matrices $A \in C^{m \times n}$, $P_{S^\perp} \in C^{m \times m}$, $P_T \in C^{n \times n}$ and $X_0 \in C^{n \times m}$ with $R(X_0) \subset R(P_TA^*)$ and $N(X_0) \supset N(A^*P_{S^\perp})$;

2. Calculate

$$\begin{aligned} R_0 &= A - AX_0A; \\ P_0 &= A(R_0)_{T,S}^*A; \\ k &:= 0; \end{aligned}$$

3. If $P_{S^\perp}R_kP_T = 0$, then stop; else, $k := k + 1$;

4. Calculate

$$\begin{aligned} X_k &= X_{k-1} + \frac{\|R_{k-1}\|_{T,S}^2}{\|P_{k-1}\|_{T,S}^2}(P_{k-1})_{T,S}^*; \\ R_k &= A - AX_kA \\ &= R_{k-1} - \frac{\|R_{k-1}\|_{T,S}^2}{\|P_{k-1}\|_{T,S}^2}A(P_{k-1})_{T,S}^*A; \\ P_k &= A(R_k)_{T,S}^*A + \frac{\|R_k\|_{T,S}^2}{\|R_{k-1}\|_{T,S}^2}P_{k-1}; \end{aligned}$$

5. Goto step3.

About Algorithm 2.1, we have the following basic properties.

Theorem 2.2 In Algorithm 2.1, if we take the initial matrix $X_0 = A_{T,S}^*$, then the sequences $\{X_i\}$ and $\{P_i\}$ generalized by it such that

(1) $R(X_k) \subset R(P_TA^*)$, $N(X_k) \supset N(A^*P_{S^\perp})$ and $R(P_k) \subset R(AP_T)$, $N(P_k) \supset N(P_{S^\perp}A)$;

(2)if $AT \oplus S = C^m$ and $P_{S^\perp}R_kP_T = 0$, then $X_k = A_{T,S}^{(2)}$;

Proof (1)To prove the conclusion, we use the induction When $i = 0$, we have

$$\begin{aligned} X_0 &= A_{T,S}^* = P_TA^*P_{S^\perp} \\ P_0 &= A(R_0)_{T,S}^*A = AP_TR_0^*P_{S^\perp}A \end{aligned}$$

this implies the conclusion is right.

When $i = 1$, we have

$$\begin{aligned} X_1 &= X_0 + \frac{\|R_0\|_{T,S}^2}{\|P_0\|_{T,S}^2}(P_0)_{T,S}^* \\ &= P_TA^*(P_{S^\perp} + \frac{\|R_0\|_{T,S}^2}{\|P_0\|_{T,S}^2}P_{S^\perp}R_0P_TA^*P_{S^\perp}) \\ &= (P_T + \frac{\|R_0\|_{T,S}^2}{\|P_0\|_{T,S}^2}P_TA^*P_{S^\perp}R_0P_T)A^*P_{S^\perp} \end{aligned}$$

and

$$\begin{aligned} P_1 &= AP_TR_1^*P_{S^\perp}A + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2}P_0 \\ &= AP_T(P_TR_1^*P_{S^\perp}A + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2}P_TR_0^*P_{S^\perp}A) \\ &= (AP_TR_1^*P_{S^\perp} + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2}AP_TR_0^*P_{S^\perp})P_{S^\perp}A \end{aligned}$$

This shows when $i = 1$, the conclusion is also right.

Assume that conclusion holds for all $0 \leq i \leq s$ ($0 < s < k$). Then there exist matrices U , V , W , and Y such that

$$\begin{aligned} X_s &= P_TA^*U = VA^*P_{S^\perp} \\ P_s &= AP_TW = YP_{S^\perp}A \end{aligned}$$

Further, we have that

$$\begin{aligned} X_{s+1} &= X_s + \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2}(P_s)_{T,S}^* \\ &= P_TA^*U + \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2}P_TA^*P_{S^\perp}Y^*P_{S^\perp} \\ &= P_TA^*(U + \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2}P_{S^\perp}Y^*P_{S^\perp}) \\ &= VA^*P_{S^\perp} + \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2}P_TW^*P_TA^*P_{S^\perp} \\ &= (V + \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2}P_TW^*P_T)A^*P_{S^\perp} \end{aligned}$$

and

$$\begin{aligned} P_{s+1} &= AP_TR_{s+1}^*P_{S^\perp}A + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2}P_s \\ &= AP_T(R_{s+1}^*P_{S^\perp}A + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2}W) \\ &= (AP_TR_{s+1}^* + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2}Y)P_{S^\perp}A \end{aligned}$$

This implies that

$$R(X_{s+1}) \subset R(P_LA^*P_L) \quad \text{and} \quad N(X_{s+1}) \supset N(P_LA^*P_L)$$

and

$$R(P_{s+1}) \subset R(AP_L) \quad \text{and} \quad N(P_{s+1}) \supset N(P_LA)$$

By the principle of induction, the conclusion holds for all $i = 0, 1, \dots$

(2)From the lemma 1.4 and the results in (1) of the Theorem 2.2, we can easy to get $P_TX = XP_{S^\perp} = X$. According to Algorithm 2.1, we know that, if $P_{S^\perp}R_kP_T = 0$, then we have $X_k \in (P_{S^\perp}AP_T)\{1\}$. This implies $r(X_k) \geq r(P_{S^\perp}AP_T)$, then by the conclusion of (1) and $AT \oplus S = C^m$, we can easy get $r(X_k) = r(P_{S^\perp}AP_T)$. From Lemma 2.1 we know $X_k \in (P_{S^\perp}AP_T)\{1, 2\}$ with range $R(P_TA^*P_{S^\perp})$ and null space $N(P_TA^*P_{S^\perp})$. If $AT \oplus S = C^m$, then $R(P_TA^*P_{S^\perp}) = R(P_TA^*) = T$ and $N(P_TA^*P_{S^\perp}) = N(A^*P_{S^\perp}) = S$. By Lemma 1.3, it shows that $X_k = (P_{S^\perp}AP_T)^\dagger = A_{T,S}^{(2)}$.

Theorem 2.3 Let \tilde{X} be an solution of matrix equation $P_{S^\perp}AXAP_T = P_{S^\perp}AP_T$ with $R(X) \subset T$ and $N(X) \subset S$, then for any initial matrix X_0 with $R(X_0) \subset T$ and $N(X_0) \subset S$, the sequences $\{X_i\}$, $\{R_i\}$ and $\{P_i\}$ generalized by Algorithm 3.1 satisfy

$$\langle P_i, P_{S^\perp}(\tilde{X} - X_i)^*P_T \rangle_{T,S} = \|R_i\|_{T,S}^2, (i = 0, 1, 2, \dots).$$

Proof We prove the conclusion by induction. By Algorithm 2.1 and lemma 1.4, when $i = 0$, we have

$$\begin{aligned} & \langle P_0, P_{S^\perp}(\tilde{X} - X_0)^*P_T \rangle_{T,S} \\ &= \langle P_{S^\perp}P_0P_T, P_{S^\perp}(\tilde{X} - X_0)^*P_T \rangle \\ &= \langle P_0, P_{S^\perp}(\tilde{X} - X_0)^*P_T \rangle \\ &= \langle AP_TR_0^*P_{S^\perp}A, (\tilde{X} - X_0)^* \rangle \\ &= \langle P_TR_0^*P_{S^\perp}, A^*(\tilde{X} - X_0)^*A^* \rangle \\ &= \langle P_TR_0^*P_{S^\perp}, R_0^* \rangle = \|R_0\|_{T,S}^2 \end{aligned}$$

And when $i = 1$, we have

$$\begin{aligned} & \langle P_1, P_{S^\perp}(\tilde{X} - X_1)^*P_T \rangle_{T,S} \\ &= \langle P_{S^\perp}P_1P_T, P_{S^\perp}(\tilde{X} - X_1)^*P_T \rangle \\ &= \langle P_1, P_{S^\perp}(\tilde{X} - X_1)^*P_T \rangle \\ &= \langle P_1, (\tilde{X} - X_1)^* \rangle \quad \text{by Theorem 2.2} \\ &= \left\langle AP_TR_1^*P_{S^\perp}A + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2}P_0, (\tilde{X} - X_1)^* \right\rangle \\ &= \langle AP_TR_1^*P_{S^\perp}A, (\tilde{X} - X_1)^* \rangle \\ &\quad + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2} \langle P_0, (\tilde{X} - X_1)^* \rangle \\ &= \langle P_TR_1^*P_{S^\perp}, R_1^* \rangle + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2} \langle P_0, (\tilde{X} - X_0)^* \rangle \\ &\quad - \frac{\|R_1\|_{T,S}^2}{\|P_0\|_{T,S}^2} \langle P_0, (P_TP_0^*P_{S^\perp})^* \rangle \\ &= \|R_1\|_{T,S}^2 \end{aligned}$$

Assume that the conclusion holds for $i = s (s > 0)$, that $\langle P_s, P_{S^\perp}(\tilde{X} - X_s)^*P_T \rangle_{T,S} = \|R_s\|_{T,S}^2$, then $i = s + 1$, we have

$$\begin{aligned} & \langle P_{s+1}, P_{S^\perp}(\tilde{X} - X_{s+1})^*P_T \rangle_{T,S} \\ &= \langle P_{S^\perp}P_{s+1}P_T, P_{S^\perp}(\tilde{X} - X_{s+1})^*P_T \rangle \\ &= \langle P_{s+1}, P_{S^\perp}(\tilde{X} - X_{s+1})^*P_T \rangle \\ &= \langle P_{s+1}, (\tilde{X} - X_{s+1})^* \rangle \quad \text{by Theorem 2.2} \\ &= \left\langle AP_TR_{s+1}^*P_{S^\perp}A + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2}P_s, (\tilde{X} - X_{s+1})^* \right\rangle \\ &= \langle AP_TR_{s+1}^*P_{S^\perp}A, (\tilde{X} - X_{s+1})^* \rangle \\ &\quad + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2} \langle P_s, (\tilde{X} - X_{s+1})^* \rangle \\ &= \langle P_TR_{s+1}^*P_{S^\perp}, R_{s+1}^* \rangle + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2} \langle P_s, (\tilde{X} - X_s)^* \rangle \\ &\quad - \frac{\|R_{s+1}\|_{T,S}^2}{\|P_s\|_{T,S}^2} \langle P_s, P_{S^\perp}P_sP_T \rangle \\ &= \|R_{s+1}\|_{T,S}^2 \end{aligned}$$

By the principle of induction, the conclusion $\langle P_i, P_{S^\perp}(\tilde{X} - X_i)^*P_T \rangle_{T,S} = \|R_i\|_{T,S}^2$ holds for all $i = 0, 1, 2, \dots$

Remark 1 From Theorem 2.3 we know that if $P_{S^\perp}R_iP_T \neq 0$, then $P_{S^\perp}P_iP_T \neq 0$. This result shows that if $P_{S^\perp}R_iP_T \neq 0$, then Algorithm 3.1 can not be terminated.

Theorem 2.4 For the sequences $\{R_i\}$ and $\{P_i\}$ generated by Algorithm 2.1 with the $X_0 = P_TA^*P_{S^\perp}$, if there exists a positive number k such that $R_i \neq 0$ for all $i = 0, 1, 2, \dots, k$, then we have

$$\langle R_i, R_j \rangle_{T,S} = 0, \langle P_i, P_j \rangle_{T,S} = 0, (i \neq j, i, j = 0, 1, \dots, k)$$

Proof According to Lemma 1.5, we know that

$$\langle A, B \rangle_{T,S} = \overline{\langle B, A \rangle_{T,S}}$$

holds for all matrices A and B in $C^{m \times n}$, so we only need prove the conclusion hold for all $0 \leq i < j \leq k$. Using induction and two steps are required.

Step1. Show that $\langle R_i, R_{i+1} \rangle_{T,S} = 0$ and $\langle P_i, P_{i+1} \rangle_{T,S} = 0$ for all $i = 0, 1, 2, \dots, k$. To prove this conclusion, we also use induction. According to Lemma 1.5 and Algorithm 2.1, when $i = 0$, we have

$$\begin{aligned} & \langle R_0, R_1 \rangle_{T,S} = \langle P_{S^\perp}R_0P_T, R_1 \rangle \\ &= \left\langle P_{S^\perp}R_0P_T, R_0 - \frac{\|R_0\|_{T,S}^2}{\|P_0\|_{T,S}^2}AP_TP_0^*P_{S^\perp}A \right\rangle \\ &= \langle P_{S^\perp}R_0P_T, R_0 \rangle - \frac{\|R_0\|_{T,S}^2}{\|P_0\|_{T,S}^2} \langle P_{S^\perp}R_0P_T, AP_TP_0^*P_{S^\perp}A \rangle \\ &= \|R_0\|_{T,S}^2 - \frac{\|R_0\|_{T,S}^2}{\|P_0\|_{T,S}^2} \langle A^*P_{S^\perp}R_0P_TA^*, P_TP_0^*P_{S^\perp} \rangle \\ &= \|R_0\|_{T,S}^2 - \frac{\|R_0\|_{T,S}^2}{\|P_0\|_{T,S}^2} \langle P_0^*, P_TP_0^*P_{S^\perp} \rangle \\ &= \|R_0\|_{T,S}^2 - \frac{\|R_0\|_{T,S}^2}{\|P_0\|_{T,S}^2} \|P_0\|_{T,S}^2 = 0 \end{aligned}$$

and

$$\begin{aligned} & \langle P_0, P_1 \rangle_{T,S} = \langle P_{S^\perp}P_0P_T, P_1 \rangle \\ &= \left\langle P_{S^\perp}P_0P_T, AP_TR_1^*P_{S^\perp}A + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2}P_0 \right\rangle \\ &= \langle P_{S^\perp}P_0P_T, AP_TR_1^*P_{S^\perp}A \rangle + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2} \langle P_{S^\perp}P_0P_T, P_0 \rangle \\ &= \langle A^*P_{S^\perp}P_0P_TA^*, P_TR_1^*P_{S^\perp} \rangle + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2} \|P_0\|_{T,S}^2 \\ &= \frac{\|P_0\|_{T,S}^2}{\|R_0\|_{T,S}^2} \langle (R_0 - R_1)^*, P_TR_1^*P_{S^\perp} \rangle + \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2} \|P_0\|_{T,S}^2 \\ &= \frac{\|R_1\|_{T,S}^2}{\|R_0\|_{T,S}^2} \|P_0\|_{T,S}^2 - \frac{\|P_0\|_{T,S}^2}{\|R_0\|_{T,S}^2} \|R_1\|_{T,S}^2 = 0 \end{aligned}$$

Assume that conclusion holds for all $i \leq s (0 < s < k)$,

then

$$\begin{aligned}
 & \langle R_s, R_{s+1} \rangle_{T,S} = \langle P_{S^\perp} R_s P_T, R_{s+1} \rangle \\
 & = \left\langle P_{S^\perp} R_s P_T, R_s - \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2} A P_T P_s^* P_{S^\perp} A \right\rangle \\
 & = \langle P_{S^\perp} R_s P_T, R_s \rangle - \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2} \langle P_{S^\perp} R_s P_T, A P_T P_s^* P_{S^\perp} A \rangle \\
 & = \|R_s\|_{T,S}^2 - \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2} \langle A^* P_{S^\perp} R_s P_T A^*, P_T P_s^* P_{S^\perp} \rangle \\
 & = \|R_s\|_{T,S}^2 - \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2} \left\langle \left(P_s - \frac{\|R_{s-1}\|_{T,S}^2}{\|R_{s-1}\|_{T,S}^2} P_{s-1} \right)^*, P_T P_s^* P_{S^\perp} \right\rangle \\
 & = \|R_s\|_{T,S}^2 - \frac{\|R_s\|_{T,S}^2}{\|P_s\|_{T,S}^2} \|P_s\|_{T,S}^2 = 0
 \end{aligned}$$

and

$$\begin{aligned}
 & \langle P_s, P_{s+1} \rangle_{T,S} = \langle P_{S^\perp} P_s P_T, P_{s+1} \rangle \\
 & = \left\langle P_{S^\perp} P_s P_T, A P_T R_{s+1}^* P_{S^\perp} A + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2} P_s \right\rangle \\
 & = \langle A^* P_{S^\perp} P_s P_T A^*, P_T R_{s+1}^* P_{S^\perp} \rangle + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2} \langle P_{S^\perp} P_s P_T, P_s \rangle \\
 & = \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2} \langle (R_s - R_{s+1})^*, P_T R_{s+1}^* P_{S^\perp} \rangle + \frac{\|P_s\|_{T,S}^2}{\|R_s\|_{T,S}^2} \langle (R_s - R_{s+1})^*, P_T R_{s+1}^* P_{S^\perp} \rangle + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2} \|P_s\|_{T,S}^2 \\
 & = -\frac{\|P_s\|_{T,S}^2}{\|R_s\|_{T,S}^2} \|R_{s+1}\|_{T,S}^2 + \frac{\|R_{s+1}\|_{T,S}^2}{\|R_s\|_{T,S}^2} \|P_s\|_{T,S}^2 = 0
 \end{aligned}$$

By the principle of induction, $\langle R_i, R_{i+1} \rangle_{T,S} = 0$, and $\langle P_i, P_{i+1} \rangle_{T,S} = 0$, hold for all $i = 0, 1, \dots, k$.

Step2. Assume that $\langle R_i, R_{i+l} \rangle_{T,S} = 0$, and $\langle P_i, P_{i+l} \rangle_{T,S} = 0$, hold for all $0 \leq i \leq k$ and $1 \leq l < k$, show that $\langle R_i, R_{i+l+1} \rangle_{T,S} = 0$, and $\langle P_i, P_{i+l+1} \rangle_{T,S} = 0$.

$$\begin{aligned}
 & \langle R_i, R_{i+l+1} \rangle_{T,S} = \langle P_{S^\perp} R_i P_T, R_{i+l+1} \rangle \\
 & = \left\langle P_{S^\perp} R_i P_T, R_{i+l} - \frac{\|R_{i+l}\|_{T,S}^2}{\|P_{i+l}\|_{T,S}^2} A P_T P_{i+l}^* P_{S^\perp} A \right\rangle \\
 & = -\frac{\|R_{i+l}\|_{T,S}^2}{\|P_{i+l}\|_{T,S}^2} \langle P_{S^\perp} R_i P_T, A P_T P_{i+l}^* P_{S^\perp} A \rangle \\
 & = -\frac{\|R_{i+l}\|_{T,S}^2}{\|P_{i+l}\|_{T,S}^2} \langle A^* P_{S^\perp} R_i P_T A^*, P_T P_{i+l}^* P_{S^\perp} \rangle \\
 & = -\frac{\|R_{i+l}\|_{T,S}^2}{\|P_{i+l}\|_{T,S}^2} \left\langle P_i - \frac{\|R_i\|_{T,S}^2}{\|R_{i-1}\|_{T,S}^2} P_{i-1}, P_T P_{i+l}^* P_{S^\perp} \right\rangle \\
 & = 0
 \end{aligned}$$

and

$$\begin{aligned}
 & \langle P_i, P_{i+l+1} \rangle_{T,S} = \langle P_{S^\perp} P_i P_T, P_{i+l+1} \rangle \\
 & = \left\langle P_{S^\perp} P_i P_T, A P_T R_{i+l+1}^* P_{S^\perp} A + \frac{\|R_{i+l+1}\|_{T,S}^2}{\|R_{i+l}\|_{T,S}^2} P_{i+l} \right\rangle \\
 & = \langle P_{S^\perp} P_i P_T, A P_T R_{i+l+1}^* P_{S^\perp} A \rangle + \frac{\|R_{i+l+1}\|_{T,S}^2}{\|R_{i+l}\|_{T,S}^2} \langle P_{S^\perp} P_i P_T, P_{i+l} \rangle \\
 & = \langle A^* P_T P_i P_{S^\perp} A^*, P_{S^\perp} R_{i+l+1}^* P_T \rangle \\
 & = \frac{\|P_i\|_{T,S}^2}{\|R_i\|_{T,S}^2} \langle (R_{i-1} - R_i)^*, P_T R_{i+l+1}^* P_{S^\perp} \rangle = 0
 \end{aligned}$$

From step 1 and step 2, we have by principle induction that $\langle R_i, R_j \rangle_{T,S} = 0$, and $\langle P_i, P_j \rangle_{T,S} = 0$, hold for all $i, j = 0, 1, \dots, k, i \neq j$.

Remark 2 Theorem 2.4 implies that, for an initial matrix $X_0 = P_T A^* P_{S^\perp}$, since the R_0, R_1, \dots are orthogonal each other, based on restricted inner product on subspace T and S , in the finite dimension matrix space $C^{m \times n}$, it is certainly there exists a positive number $k \leq mn$ such that $\|R_k\|_{T,S} = 0$. Then by Theorem 2.2, the generalized inverse $A_{T,S}^{(2)}$ can be obtained within at most mn iteration steps.

Remark 3 When $T = R(A^*)$ and $S = N(A^*)$, the Algorithm 2.1 is exact the finite iterative method to computed M-P inverse A^\dagger , which can be reviewed in [13]. Here we recalled it as following:

Algorithm 2.2^[13]

1. In put matrices $A \in C^{m \times n}$ and $X_0 \in C^{n \times m} = A^*$;
2. Calculate

$$\begin{aligned}
 R_0 &= A - A X_0 A; \\
 P_0 &= A R_0^* A; \\
 k &:= 0;
 \end{aligned}$$

3. If $R_k = 0$, then stop; else, $k := k + 1$;
4. Calculate

$$\begin{aligned}
 X_k &= X_{k-1} + \frac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2} P_{k-1}^*; \\
 R_k &= A - A X_k A \\
 &= R_{k-1} - \frac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2} A P_{k-1}^* A; \\
 P_k &= A R_k^* A + \frac{\|R_k\|^2}{\|R_{k-1}\|^2} P_{k-1};
 \end{aligned}$$

5. Goto step 3.

Remark 4 If $G \in C^{n \times m}$ satisfied $R(G) = T$ and $N(G) = S$ with m, n are large, we can use the Algorithm 2.2 to compute the G^\dagger , the $P_T = P_{R(G)} = G G^\dagger$ and $P_{S^\perp} = P_{N(G)^\perp} = G^\dagger G$.

III. NUMERICAL EXAMPLES

In this section, we will give a numerical example, which is taken from [11], to illustrate our results. All the tests are performed by MATLAB 6.1 and the initial iterative matrices are chosen as $X_0 = P_T A^* P_{S^\perp}$. Because of the influence of the error of roundoff, we regard the matrix $P_{S^\perp} A P_T$ as zero matrix if $\|A\|_{T,S} < 10^{-10}$.

Example 3.1 Take

$$A = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & -3 & 1 & 2 \end{pmatrix} \in R_3^{3 \times 4},$$

$$G = \begin{pmatrix} 3 & 1 & 0 \\ -2 & 4 & -2 \\ -5 & -4 & 1 \\ 0 & 7 & -3 \end{pmatrix} \in R_2^{4 \times 3}$$

We can easily show that $AR(G) \oplus N(G) = R^3$, then by Lemma 2.1 we know that $A_{R(G),N(G)}^{(2)}$ exists.

By directly computing,

$$P_{R(G)} = \frac{1}{59} \begin{pmatrix} 14 & -10 & -23 & -1 \\ -10 & 24 & 8 & 26 \\ -23 & 8 & 42 & -11 \\ -1 & 26 & -11 & 38 \end{pmatrix}$$

and

$$P_{N(G)^\perp} = P_{R(G^T)} = \frac{1}{59} \begin{pmatrix} 102 & 13 & 9 \\ 13 & 30 & -11 \\ 9 & -11 & 6 \end{pmatrix}$$

Using Algorithm 2.1 and iterate 11 steps, we have X_{11} as follow:

$$X_{11} = \begin{pmatrix} -0.27419354838934 & 0.32258064516088 & -0.17741935483885 \\ 0.09677419355105 & 0.70967741935533 & -0.29032258064499 \\ 0.50000000000316 & -0.99999999999942 & 0.50000000000020 \\ -0.12903225806276 & 1.38709677419387 & -0.61290322580634 \end{pmatrix}$$

with

$$\|R_{11}\|_{T,S}^2 = \|A - AX_{11}A\|_{T,S}^2 = 1.950379437652951 \times 10^{-21}$$

On other hand, by computing, we obtain that

$$A_{T,S}^{(2)} = \frac{1}{62} \begin{pmatrix} -17 & 20 & -11 \\ 6 & 44 & -18 \\ 31 & -62 & 31 \\ -8 & 86 & -38 \end{pmatrix}$$

Then from the above data, we can find that the iterative sequence $\{X_k\}$ converges to $A_{T,S}^{(2)}$.

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