World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:12, 2011

Symmetries, conservation laws and reduction of wave and Gordon-type equations on Riemannian manifolds

Sameerah Jamal, Abdul Hamid Kara, and Ashfaque H. Bokhari

Abstract—Equations on curved manifolds display interesting properties in a number of ways. In particular, the symmetries and, therefore, the conservation laws reduce depending on *how curved* the manifold is. Of particular interest are the wave and Gordon-type equations; we study the symmetry properties and conservation laws of these equations on the Milne and Bianchi type III metrics. Properties of reduction procedures via symmetries, variational structures and conservation laws are more involved than on the well known flat (Minkowski) manifold.

Keywords—Bianchi metric, Conservation laws, Milne metric, Symmetries.

I. Introduction

A VAST amount of work has been published in the literature studying differential equations (DEs) in terms of the symmetry generators admitted by them [1]-[2]. These symmetries play an important role in finding exact analytic solutions of the nonlinear DEs. Other than Lie point symmetries, Noether symmetries are widely studied and are associated, in particular, with those DEs which possess Lagrangians. These symmetries represent physical features of DEs via the conservation laws they admit. The interesting link between symmetries and conservation laws in mathematical physics is provided in the classic work of Noether [3] showing that for every infinitesimal transformation admitted by the action integral of a system, there exists a conservation law. The Noether symmetries, which are symmetries of the Euler-Lagrange systems, have interesting applications in the study of properties of particles moving in under the influence of gravitational fields.

Recently, some published results were aimed at understanding Noether symmetries of Lagrangians that arise from certain pseudo-Riemannian metrics of interest [4]-[5]. More recently, Noether symmetries of the Euler-Lagrange equations on the Milne metric [6] were found and a discussion of the results were given via a comparison of Noether symmetries on the Milne metric with those of other conventional symmetries of the same space-time [7]. Concerning the pure wave equation (homogeneous), it is a priori clear that it will admit a maximal Noether symmetry group on a flat manifold. In that spirit, the work of Mahadi [7] gives limited information and needs further understanding. With this example in mind, we study Klein-Gordon [8] equations on the Milne metric and see how Noether symmetry structures change when classical wave equations are coupled with an inhomogeneous term and, for illustrative purposes we consider the wave equation on a Bianchi type III metric. For further alternative approaches to wave or Gordon-type equations on Riemannian manifolds, we refer the reader to [9]-[10]. The plan of the paper is as follows: Section II briefly discuss the procedure to obtain an expression representing a Noether symmetry and we describe the multiplier method. In section III, we list some Lie point symmetries of a class of Gordon-type equations and illustrate a few reductions. We also determine the Noether point symmetries of the Klein-Gordon equation and construct an associated conserved vector. In section IV, we determine the symmetry properties of a nonlinear wave equation on the Bianchi type III metric and list the associated conserved densities.

S. Jamal and A. H. Kara are with the School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa, e-mail: (sameerahj@hotmail.com & Abdul.Kara@wits.ac.za).

A. H. Bokhari is with the Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia, e-mail: (abokhari@kfupm.edu.sa).

II. Definitions and Notation

Consider an rth-order system of partial DEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$

$$G^\mu(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \quad \mu = 1, \dots, \tilde{m}, \qquad (1)$$

where $u_{(1)}, u_{(2)}, \dots, u_{(r)}$ denote the collections of all first, second, ..., rth-order partial derivatives.

Definition 1: The total differentiation operator with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n. \qquad (2)$$

Definition 2: A current $\Phi = (\Phi^1, \dots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \qquad (3)$$

along the solutions of (1).

It can be shown that every admitted conservation law arises from *multipliers* $Q_\mu(x, u, u_{(1)}, \dots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \qquad (4)$$

holds identically (i.e., off the solution space) for some current Φ^i. The conserved vector is then obtained by the homotopy operator [1]. Suppose $\mathcal{A}$ is the universal space of differential functions.

Definition 3: A Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1\ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \quad (5)$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ and the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} , \\ \zeta^\alpha_{i_1\ldots i_s} &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \quad s > 1. \end{aligned} \quad (6)$$

W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j . \quad (7)$$

Definition 4: A Lie symmetry generator of (1) is a one parameter Lie group transformation that leaves the given differential equation invariant under the transformation of all independent variables and dependent variables.
In this paper, we will assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u.

Definition 5: The Euler-Lagrange equations, if they exist, associated with (1) is the system $\delta L/\delta u^\alpha = 0$, $\alpha = 1, \ldots, m$, where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by

$$\begin{aligned} &\tfrac{\delta}{\delta u^\alpha} = \tfrac{\partial}{\partial u^\alpha} + \textstyle\sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s} \tfrac{\partial}{\partial u^\alpha_{i_1\cdots i_s}} , \\ &\alpha = 1, \ldots, m. \end{aligned} \quad (8)$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_\Omega L(x, u, u_{(1)}, \ldots, u_{(r)})\text{x}$$

defined over Ω, where Ω is an open, connected subset of the Euclidean space with smooth boundary $\partial\Omega$.
A partial DE, for which there exists a Lagrangian, is said to be variational.

Definition 6: If we consider point dependent gauge terms $f_1, \ldots, f_n$, the Noether symmetries X are given by

$$X(L) + LD_i(\xi^i) = D_i(f_i). \quad (9)$$

Definition 7: The Noether operator associated with a Lie-Bäcklund operator X is given by

$$\begin{aligned} N^i = \; & \xi^i + W^\alpha \tfrac{\delta}{\delta u^\alpha_i} + \textstyle\sum_{s\geq 1} D_{i_1}\cdots D_{i_s}(W^\alpha) \\ & \tfrac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}} , \qquad i = 1, \ldots, n, \end{aligned} \quad (10)$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (8) by replacing u^α by the corresponding derivatives, e.g.,

$$\begin{aligned} \tfrac{\delta}{\delta u^\alpha_i} = \; & \tfrac{\partial}{\partial u^\alpha_i} + \textstyle\sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s} \tfrac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}} \\ & i = 1, \ldots, n, \quad \alpha = 1, \ldots, m . \end{aligned} \quad (11)$$

Theorem 8: For any Noether symmetry X corresponding to a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector $\Phi^i = (\Phi^1, \ldots, \Phi^n)$, $\quad \Phi^i \in \mathcal{A}$, defined by

$$\Phi^i = f_i - N^i(L) , \qquad i = 1, \ldots, n , \quad (12)$$

which is a conserved vector for the Euler-Lagrange equations.

III. GORDON-TYPE EQUATIONS IN MILNE SPACE-TIME

Consider the Milne metric [6]

$$ds^2 = -dt^2 + t^2(dx^2 + e^{2x}(dy^2 + dz^2)) \quad (13)$$

which represents an empty universe and is of interest in relativity for being a special case of a well known Friedmann $Lema\hat{i}tre$ Robertson Walker metric [8], [6]. The Gordon-type equation [8] on (13) is obtained by

$$u = \frac{1}{\sqrt{|-g|}} \frac{\partial}{\partial x^i}(\sqrt{|-g|} g^{ij} \frac{\partial}{\partial x^i} u) = k(u), \quad (14)$$

and takes the form

$$\begin{aligned} &u_{xx} - t^2 u_{tt} + e^{-2x} u_{yy} + e^{-2x} u_{zz} - 3tu_t + 2u_x \\ &-t^2 k(u) = 0. \end{aligned} \quad (15)$$

A. Lie Symmetries and Reduction of Order of Equation (15) - An Illustration

In order to find the Lie point symmetries of the above equation we restrict $k(u)$ to some special cases. These cases are assumed by keeping in mind the fact that we allow the inhomogeneous term, $k(u)$, to be taken as $\sin(u)$ and some powers of u. Note that the case $k(u) = \sin(u)$ is the sine-Gordon equation and $k(u) = u$ is the well known Klein-Gordon equation. The criteria that yields the Lie point symmetries is given by the invariance condition [2]

$$\begin{aligned} &X[u_{xx} - t^2 u_{tt} + e^{-2x} u_{yy} + e^{-2x} u_{zz} - 3tu_t + 2u_x \\ &-t^2 k(u)]|_{Eq.(15)=0} = 0, \end{aligned} \quad (16)$$

where X would be the prolonged symmetry generator in the jet space. Thus, the invariance of differential equations (15) leads to the Lie point symmetries possessed by (15). The procedure for finding Lie point symmetries is well known [2] and therefore will be given with out derivations. It turns out, from the symmetry study, that the polynomial cases of $k(u) = u^n, n \neq 0$ and $k(u) = \sin(u)$ yield many Lie point symmetry generators and, inter alia, the generators that are common to all of these cases are

$$\begin{aligned} X_1 &= y\partial_z - z\partial_y \\ X_2 &= -\partial_x + y\partial_y + z\partial_z. \end{aligned}$$

In this section we briefly show how the order of the (1+3) Gordon-type equation (15) can be reduced using its symmetries. We use a rotational and scaling symmetry - ultimately the equation with four independent variables is reduced to a partial DE that has two independent variables. The reduced equation may then be analyzed further using further Lie symmetry reductions or some appropriate alternative method. Since $[X_1, X_2] = 0$, where X_1 and X_2 appear above, we may begin reducing with either X_1 or X_2. Suppose we reduce (15) by X_1. The characteristic equations are

$$\frac{dx}{0} = \frac{dt}{0} = \frac{dy}{-z} = \frac{dz}{y} = \frac{du}{0}.$$

Integrating yields $\alpha = y^2 + z^2$ and (15) is reduced to

$$\begin{aligned} &\tfrac{1}{t^2} u_{xx} - u_{tt} + \tfrac{2}{t^2} e^{-2x}(\alpha u_{\alpha\alpha} + u_\alpha) - \tfrac{3}{t} u_t + \tfrac{2}{t^2} u_x \\ &-k(u) = 0 \end{aligned} \quad (17)$$

with $u = u(x, t, \alpha)$.

If we then reduce (17) by X_2 we obtain the scaling transformation $\bar{X} = -\partial_x + 2\alpha\partial_\alpha$. We now have the characteristic equations,

$$\frac{dt}{0} = \frac{dx}{-1} = \frac{d\alpha}{2\alpha} = \frac{du}{0}.$$

By integrating, we obtain $\beta = \ln\alpha + 2x$ and (17) reduces to

$$\frac{2}{t^2}u_{\beta\beta}(2 + e^{-\beta}) - u_{tt} + \frac{4}{t^2}u_\beta - \frac{3}{t}u_t - k(u) = 0 \quad (18)$$

with $u = u(t, \beta)$.

Equation (18) may be further analysed or reduced using the underlying symmetries. It turns out that the Lie point symmetries are cumbersome and involve special functions such as Bessel functions for the Klein-Gordon case, $k(u) = u$. For $k(u) = u^3$, (18) admits one symmetry

$$t\partial_t - u\partial_u.$$

However, for $k(u) = u^4$, the symmetries are

$$\begin{aligned} X_3 &= \tfrac{3}{4}t\partial_t - \tfrac{1}{2}u\partial_u, \\ X_4 &= 4t^2(1 + 2e^\beta)\partial_\beta + t^3(1 + 4e^\beta)\partial_t \\ &\quad -2t^2u(1 + 4e^\beta)\partial_u. \end{aligned} \quad (19)$$

Using X_4, (18) reduces to the ordinary DE

$$\gamma^4 F_{\gamma\gamma} + \gamma F_\gamma + 4F - 8\gamma^2 F^4 = 0, \quad (20)$$

where $\gamma = \dfrac{te^{-\frac{\beta}{4}}}{(1 + 2e^\beta)^{\frac{1}{4}}}$ and $F = ue^{\frac{\beta}{2}}(1 + 2e^\beta)^{\frac{1}{2}}$. We find that (20) admits the Lie point symmetry $G = -\frac{3}{2}\beta\partial_\beta + F\partial_F$ which leads to the first-order ode

$$q_p = \frac{2q + 24q^{\frac{2}{3}}p^{\frac{4}{3}} - 12q^{\frac{2}{3}}p^{\frac{1}{3}}}{2p + 3q^{\frac{1}{3}}p^{\frac{2}{3}}}, \quad (21)$$

where $p = \gamma^2 F^3$ and $q = \gamma^5 F'^3$.

There are no symmetries of (18) for $k(u) = u^n$, $n \neq 0, 1, 3, 4$ and $k(u) = \sin u$.

Also, one may consider reduction by studying the underlying conservation laws. This would require methods other than the variational one, i.e., Noether's theorem, since (18) is not variational.

For the Klein-Gordon case $k(u) = u$, it can be shown, for e.g., that a conserved vector of (18) is (Φ^β, Φ^t), where

$$\begin{aligned} \Phi^\beta &= 2(1 + 2e^\beta)\text{BesselJ}(1, t)u_\beta, \\ \Phi^t &= \tfrac{1}{2}e^\beta t[(t\text{BesselJ}(0, t) - 2\text{BesselJ}(1, t) \\ &\quad -t\text{BesselJ}(2, t))u - 2t\text{BesselJ}(1, t)u_t], \end{aligned}$$

such that $D_\beta\Phi^\beta + D_t\Phi^t = 0$ along the solutions of (18) and where BesselJ is the Bessel function of the first kind.

For $k(u) = u^3$ in (18), the components of the conserved vector are

$$\begin{aligned} \Phi^\beta &= (1 + 2e^\beta)t^2[u_t u_\beta - uu_{\beta t}], \\ \Phi^t &= -\tfrac{1}{4}t^2[e^\beta t^2u^4 + 2e^\beta t^2{u_t}^2 + 4u(e^\beta tu_t \\ &\quad -2e^\beta u_\beta - (1 + 2e^\beta)u_{\beta\beta})]. \end{aligned}$$

Similarly, for $k(u) = u^4$, the components of the conserved vector are

$$\begin{aligned} \Phi^\beta &= -\tfrac{1}{5}(1 + 2e^\beta)t^3(40e^\beta u^2 + 4e^\beta t^2u^5 \\ &\quad -5u_\beta((1 + 4e^\beta)tu_t + 4(1 + 2e^\beta)u_\beta) \\ &\quad +5tu(10e^\beta u_t + 2e^\beta tu_{tt} + (1 + 4e^\beta)u_{\beta t})), \\ \Phi^t &= -t^4(-2e^\beta(1 + 4e^\beta)u^2 + \tfrac{1}{5}e^\beta(1 + 4e^\beta)t^2u^5 \\ &\quad +\tfrac{1}{2}e^\beta tu_t((1 + 4e^\beta)tu_t + 4(1 + 2e^\beta)u_\beta) \\ &\quad -u(2e^\beta(3 + 8e^\beta)u_\beta + (1 + 2e^\beta)(2e^\beta tu_{\beta t} \\ &\quad +(1 + 4e^\beta)u_{\beta\beta}))). \end{aligned}$$

The cases $k(u) = \sin u$ and $k(u) = u^n$, $n \neq 1, 3, 4$, of equation (18) does not yield any conserved vectors.

B. Noether Symmetries of the Klein-Gordon Equation in Milne Space-time

Consider the wave equation (15) with $k(u) = u$ (Klein-Gordon), which has the Lagrangian,

$$L = \frac{1}{2}t^3e^{2x}u^2 + \frac{1}{2}te^{2x}u_x^2 + \frac{1}{2}tu_y^2 + \frac{1}{2}tu_z^2 - \frac{1}{2}t^3e^{2x}u_t^2. \quad (22)$$

We assume that

$$\begin{aligned} X &= \xi(t, x, y, z, u)\partial_x + \tau(t, x, y, z, u)\partial_t \\ &\quad +\eta(t, x, y, z, u)\partial_y + \gamma(t, x, y, z, u)\partial_z \\ &\quad +\phi(t, x, y, z, u)\partial_u \end{aligned} \quad (23)$$

be a Noether point operator that satisfies (9) with gauge vector $f_i, (i = 1, 2, 3, 4)$ dependent on (t, x, y, z, u). Then the Noether symmetry criterion (9) for the Lagrangian given by (22) takes the form,

$$\begin{aligned} &XL + L[D_t\tau + D_x\xi + D_y\eta + D_z\gamma] \\ &= D_tf_1 + D_xf_2 + D_yf_3 + D_zf_4. \end{aligned} \quad (24)$$

We find that the Noether point symmetries of the Klein-Gordon equation for the Milne metric are given by,

$$\begin{aligned} X_1 &= e^{2x}t^3(\tfrac{e^x}{t}\partial_x - e^x\partial_t),\ f_i = 0, \\ X_2 &= e^{2x}t^3\partial_y,\ f_i = 0, \\ X_3 &= e^{2x}t^3(\tfrac{e^{-x}}{t}\partial_y - e^xy\partial_t + \tfrac{e^xy}{t}\partial_x),\ f_i = 0, \\ X_4 &= e^{2x}t^3\partial_z,\ f_i = 0, \\ X_5 &= e^{2x}t^3(\tfrac{e^{-x}}{t}\partial_z - e^xz\partial_t + \tfrac{e^xz}{t}\partial_x),\ f_i = 0, \\ X_6 &= e^{2x}t^3(y\partial_z - z\partial_y),\ f_i = 0, \\ X_7 &= e^{2x}t^3(-\partial_x + y\partial_y + z\partial_z),\ f_i = 0, \\ X_8 &= e^{2x}t^3(\tfrac{2e^{-x}y}{t}\partial_y + \tfrac{2e^{-x}z}{t}\partial_z + e^{-x}(-1- \\ &\quad e^{2x}(y^2 + z^2))\partial_t + \tfrac{e^{-x}(-1+e^{2x}(y^2+z^2))}{t}\partial_x), \\ &\quad f_i = 0, \\ X_9 &= e^{2x}t^3(2y\partial_x - 2yz\partial_z + (e^{-2x} - y^2 \\ &\quad +z^2)\partial_y), f_i = 0, \\ X_{10} &= e^{2x}t^3(-2yz\partial_y + 2z\partial_x + (e^{-2x} + y^2 \\ &\quad -z^2)\partial_z), f_i = 0. \end{aligned}$$

We may then obtain the corresponding conserved vectors for each $X_i\,(i = 1, \ldots, 10)$, for example, the symmetry X_6

has corresponding conserved vector,

$$\begin{aligned}\Phi_6^t = & -\tfrac{1}{2}e^{2x}t^3(yu_zu_t - zu_yu_t + u(-yu_{tz} \\ & +zu_{ty})) \\ \Phi_6^x = & \tfrac{1}{2}e^{2x}t(yu_zu_x - zu_yu_x + u(-yu_{xz} + zu_{xy})) \\ \Phi_6^y = & \tfrac{1}{2}t(e^{2x}t^2zu^2 + u_y(yu_z - zu_y) - u(u_z \\ & +zu_{zz} + yu_{yz} - 3e^{2x}tzu_t \\ & -e^{2x}t^2zu_{tt} + 2e^{2x}zu_x + e^{2x}zu_{xx})) \\ \Phi_6^z = & -\tfrac{1}{2}t(e^{2x}t^2yu^2 + u_z(-yu_z + zu_y) - u(u_y \\ & +zu_{yz} + y(u_{yy} + e^{2x}(-3tu_t - t^2u_{tt} \\ & +2u_x + u_{xx})))).\end{aligned}$$

Remark. One can determine *higher-order* variational symmetries and conservation laws for all the classes of (15), either using the multiplier method or recursion operators.

IV. THE WAVE EQUATION ON BIANCHI TYPE III MANIFOLD

Consider the Bianchi type III metric

$$ds^2 = -\beta^2dt^2 + t^{2L}(dx^2 + e^{\frac{-2ax}{N}}dy^2) + t^{\frac{2L}{m}}dz^2)) \qquad (25)$$

The wave equation on (25) takes the form

$$\begin{aligned}& -\tfrac{1}{\beta}(2L+1)t^{2L}e^{-\frac{a}{N}x}u_t - \tfrac{1}{\beta}t^{2L+1}e^{-\frac{a}{N}x}u_{tt} \\ & -\tfrac{a\beta}{N}te^{-\frac{a}{N}x}u_x + \beta te^{-\frac{a}{N}x}u_{xx} \\ & +\beta te^{\frac{a}{N}x}u_{yy} + \beta t^{2L+1-\frac{2L}{m}}e^{-\frac{a}{N}x}u_{zz} = 0.\end{aligned} \qquad (26)$$

The standard procedure for determining Lie and Noether symmetry generators is much more cumbersome so that reduction or obtaining conservation laws via Noether's theorem is virtually impossible. To this end, we consider the multiplier method to determine some of the conserved densities. That is, since the Euler-Lagrange operator annihilates total divergences, we get

$$\frac{\delta}{\delta u}[Q * \text{LHS of}\,(26)] = 0, \qquad (27)$$

where $Q = Q(x, y, z, t, u_x, u_t, u_y, u_z)$. The lengthy calculations lead to the set of multipliers

$$\begin{aligned}Q = & \ C_1u_y + C_2u_z + e^{\frac{ax}{2N}} \\ & (C_3e^{\frac{x\sqrt{a^2\beta^2+4N^2d_1}}{2N\beta}} + C_4e^{\frac{-x\sqrt{a^2\beta^2+4N^2d_1}}{2N\beta}})Y(t),\end{aligned} \qquad (28)$$

where d_1 and the $C_i \quad (i = 1, 2, 3, 4)$ are arbitrary constants and $Y(t)$ is the solution of, $-t^{-2L+1}d_1Y(t) + Y'(t)(2L + 1)t^{-1} + Y''(t) = 0$. If we let $d_1 = 0$, we solve $Y''(t) + \frac{(2L+1)}{t}Y'(t) = 0$ and obtain $Y(t) = -\frac{t^{-2L}}{2L}u + v$, where u, v are arbitrary constants. We thus obtain the multipliers,

$$\begin{aligned}Q_1 &= \partial_z, \\ Q_2 &= \partial_y, \\ Q_3 &= \left(-p\frac{t^{-2L}}{2L}e^{\frac{a}{N}x} - q\frac{t^{-2L}}{2L} + se^{\frac{a}{N}x} + r\right)\partial_u,\end{aligned}$$

where, p, q, r, s are arbitrary constants. These lead to conserved densities

$$\begin{aligned}\Phi_1^t &= \tfrac{1}{2\beta}e^{-\frac{ax}{N}}t^{1+2L}(-u_zu_t + uu_{tz}), \\ \Phi_2^t &= \tfrac{1}{2\beta}e^{-\frac{ax}{N}}t^{1+2L}(-u_yu_t + uu_{ty}), \\ \Phi_3^t &= \tfrac{1}{2L\beta}e^{-\frac{ax}{N}}(2L(e^{\frac{ax}{N}}p + q)u + t(q - 2Lrt^{2L} \\ & \quad +e^{\frac{ax}{N}}(p - 2Lst^{2L}))u_t).\end{aligned}$$

V. CONCLUSION

This paper investigated a class of (1+3)-dimensional wave and Gordon-type equations in Milne and Bianchi type III space-times. In particular, we derived some symmetries and conservation laws for these equations, and included some reductions. It is hoped that an analysis of the nonlinear wave and Klein-Gordon equation in a genuinely curved space-time will provide interesting insight from the point of view of conserved quantities.

REFERENCES

[1] S. Anco and G. Bluman, "Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications," *European J. of Appl. Maths*, vol. 13, 2002, pp. 545-566.
[2] P. Olver, *Application of Lie Groups to Differential Equations*, Springer, New York, 1993.
[3] E. Noether, Invariante Variationsprobleme, *Transport Theory and Statistical Physics*, vol. 1(3), 1971, pp. 186-207, original paper in, *Nachrichten der Akademie der Wissenschaften in Göttingen, Mathematisch-Physikalische Klasse*, vol. 2, 1918, pp. 235-257.
[4] A. H. Bokhari and A. H. Kara, "Noether versus Killing symmetry of conformally flat Friedmann metric," *Gen. Relativ. Gravit.*, vol. 39, 2007, pp. 2053-2059.
[5] A. H. Bokhari, A. H. Kara, A. R. Kashif and F. D. Zaman, "Noether Symmetries Versus Killing Vectors and Isometries of Spacetimes," *Int J Th. Physics*, vol. 45(6), 2006, pp. 1029-1039.
[6] A. D. Rendall, "Applications of the theory of evolution equations to general relativity," in *General Relativity - Proc. of the 16th International Conference*, Ed.: N. T. Bishop and S. D. Maharaj, Singapore, World Scientific, 2002.
[7] A. Mahadi, "PhD Thesis," King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia, 2011, unpublished.
[8] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitaion*, San Francisco: W. H. Freeman, 1973.
[9] K. Yagdijian and A Galstian, "Fundamental Solutions for the Klein-Gordon Equation in de Sitter Spacetime", *Comm. Math. Phys.*, vol. 285, 2009, pp. 293-344.
[10] J. Marzuola, J. Metcalfe, D. Tataru, and M. Tohaneanu, "Strichartz estimates on Schwarzschild black hole backgrounds," *Comm. Math. Phys.*, vol. 293(1), 2010, pp. 3783.