Equalities in a Variety of Multiple Algebras

Mona Taheri

***Abstract*—**The purpose of this research is to study the concepts of multiple Cartesian product, variety of multiple algebras and to present some examples. In the theory of multiple algebras, like other theories, deriving new things and concepts from the things and concepts available in the context is important. For example, the first were obtained from the quotient of a group modulo the equivalence relation defined by a subgroup of it. Gratzer showed that every multiple algebra can be obtained from the quotient of a universal algebra modulo a given equivalence relation.

The purpose of this study is examination of multiple algebras and basic relations defined on them as well as introduction to some algebraic structures derived from multiple algebras. Among the structures obtained from multiple algebras, this article studies sub-multiple algebras, quotients of multiple algebras and the Cartesian product of multiple algebras.

***Keywords*—**hypergroup, multiple algebras

I. INTRODUCTION

MANY studies have been conducted in Equalities in a variety of multiple algebras [1]-[4]. This article, also, aims at the same and helps to understanding better "variety of multiple algebras"

II. THE CARTESIAN PRODUCT OF MULTIPLE ALGEBRAS

Let $\left(\mathcal{A}_i \middle| i \in I\right)$ be a family of multiple algebras of type τ. the Cartesian product $\prod_{i\in I} A_i$ along with multiple operations that are defined as follows for every r < 0 (τ), form a multiple algebra of type τ.

$$f_r : \left(\prod_{i\in I} A_i\right)^{n_r} \to P^*\left(\prod_{i\in I} A_i\right)$$

$$f_r\left(\left(a_i^{\circ}\right)_{i\in I}, \ldots, \left(a_i^{n_r-1}\right)_{i\in I}\right) = \prod_{i\in I} f_r\left(a_i^{\circ}, \ldots, a_i^{n_r-1}\right)$$

For every $\left(a_i^{\circ}\right)_{i\in I}, \ldots, \left(a_i^{n_r-1}\right)_{i\in I} \in \prod_{i\in I} A_i$. We will denote this multiple algebras by $\prod_{i\in I} \mathcal{A}_i$.

If $\left(\mathcal{A}_i \middle| i \in I\right)$ and $\left(\mathcal{B}_i \middle| i \in I\right)$ are families of multiple algebras of the same type , such that for every $i \in I, \mathcal{B}_i$ is sub- multiple algebra of the multiple algebra $\mathcal{A}_i$, then by the definition of the sub-multiple algebra, We have $B_i \subseteq A_i, i \in I$. Also, for every $\left(b_i^{\circ}\right)_{i\in I}, \ldots, \left(b_i^{n_r-1}\right)_{i\in I} \in \prod_{i\in I} B_i$ We have

$$f_r\left(\left(b_i^{\circ}\right)_{i\in I}, \ldots, \left(b_i^{n_r-1}\right)_{i\in I}\right) = \prod_{i\in I} f_r\left(b_i^{\circ}, \ldots, b_i^{n_r-1}\right) \subseteq \prod_{i\in I} B_i.$$

Therefore $\prod_{i\in I} \mathcal{B}_i$ is a sub- multiple algebra of the multiple algebra $\prod_{i\in I} \mathcal{A}_i$. as a result, by definition of the sub-multiple algebra and the form of elements of the set $\prod_{i\in I} \mathcal{A}_i$, any sub-multiple algebra of the multiple algebra $\prod_{i\in I} \mathcal{A}_i$ is derives from the Cartesian product of sub-multiple algebras of the members of the family $\left(\mathcal{A}_i \middle| i \in I\right)$.

By the stated , if $\left(\mathcal{A}_i \middle| i \in I\right)$ is a family of multiple algebras of type τ and for every $i \in I, X_i$ is a non-empty subset of $\mathcal{A}_i$ Then

$$< \prod_{i\in I} X_i \geq \cap \left\{ B \in S\left(\prod_{i\in I} A_i\right) \middle| \prod_{i\in I} X_i \subseteq B \right\} = \prod_{i\in I} < X_i >.$$

One can easily verify that the canonical projections $e_j^I : \prod_{i\in I} A_i \to A_i$, Where $e_j^I\left(\left(a_i\right)_{i\in I}\right) = a_j \left(j \in I\right)$ is a homomorphism between multiple algebras $\prod_{i\in I} \mathcal{A}_i$ and $\mathcal{A}_i$.

Lemma 1

for every $n \in \mathbb{N}, \mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $\left(\left(a_i^{\circ}\right)_{i\in I}, \ldots, \left(a_i^{n-1}\right)_{i\in I}\right) \in I$

$$p\left(\left(a_i^{\circ}\right)_{i\in I}, \ldots, \left(a_i^{n-1}\right)_{i\in I}\right) = \prod_{i\in I} p\left(a_i^{\circ}, \ldots, a_i^{n-1}\right).$$

Proof:

if $\mathbf{p} = x_j$ where $j \in \{\circ, \ldots, n-1\}$ then

Mona Taheri, faculty of Mathematics Islamic Azad University , Yasooj bracch, Iran (corresponding author to provide phone: +989177410250; e-mail:m.taheri@iauyasooj.ac.ir).

$$p\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)=\left(a_i^{j}\right)_{i\in I}=\prod_{i\in I}a_i^{j}$$
$$=\prod_{i\in I}p\left(a_i^{\circ},\ldots,a_i^{n-1}\right)$$

If $\mathbf{p}\in\mathbf{P}^{(n)}(\tau)-\left\{X_j|j\in\{\circ,\ldots,n-1\}\right\}$ then $p=f_r\left(P_\circ,\ldots,p_{n_r-1}\right)$ where P_i are polynomial functions induced by $\mathbf{p}_i\in\mathbf{P}^{(n)}(\tau),\left(i\in\{\circ,\ldots,n-1\}\right)$ on $\prod_{i\in I}\mathcal{A}_i$. assume that for every $i\in\{\circ,\ldots,n-1\}$, p_i satisfies the result of lemma, then

$$p\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)$$
$$=f_r\left(p_\circ,\ldots,p_{n-1}\right)\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)$$
$$=f_r\left(p_\circ\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right),\ldots,p_{n_r-1}\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)\right)$$
$$=f_r\left(\prod_{i\in I}p_\circ\left(a_i^{\circ},\ldots,a_i^{n-1}\right),\ldots,\prod_{i\in I}p_{n_r-1}\left(a_i^{\circ},\ldots,a_i^{n-1}\right)\right)$$
$$=\left\{f_r\left(\left(b_i^{\circ}\right)_{i\in I},\ldots,\left(b_i^{n_r-1}\right)_{i\in I}\right)|\left(b_i^{j}\right)\in\prod_{i\in I}p_j\left(a_i^{\circ},\ldots,a_i^{n-1}\right)_{i\in I}\right\}$$
$$=\left\{\prod_{i\in I}f_r\left(b_i^{\circ},\ldots,b_i^{n_r-1}\right)|b_i^{j}\in p_j\left(a_i^{\circ},\ldots,a_i^{n_r-1}\right)\right\}$$
$$=\prod_{i\in I}f_r\left(p_\circ\left(a_i^{\circ},\ldots,a_i^{n-1}\right),\ldots,p_{n_r-1}\left(a_i^{\circ},\ldots,a_i^{n-1}\right)\right)$$
$$=\prod_{i\in I}f_r\left(p_\circ,\ldots,p_{n_r-1}\right)\left(a_i^{\circ},\ldots,a_i^{n-1}\right)$$
$$=\prod_{i\in I}p\left(a_i^{\circ},\ldots,a_i^{n-1}\right)$$

Lemma 2

let $\left(\mathcal{A}_i|i\in I\right)$ be a family of multiple algebras of type τ and $\mathbf{q},\mathbf{r}\in\mathbf{P}^{(n)}(\tau)$. if for every $i\in I,\mathbf{q}\cap\mathbf{r}\neq\varnothing$ is satisfied on $\mathcal{A}_i$ then the weak quality $\mathbf{q}\cap\mathbf{r}\neq\varnothing$ is satisfied on $\prod_{i\in I}\mathcal{A}_i$

Proof:

for every $\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\in\prod_{i\in I}A_i$,

$$q\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)\cap r\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)$$
$$=\prod_{i\in I}q\left(a_i^{\circ},\ldots,a_i^{n-1}\right)\cap\prod_{i\in I}r\left(a_i^{\circ},\ldots,a_i^{n-1}\right)$$

onthe other hand, for every $a_i^{\circ},\ldots,a_i^{n-1}\in A_i$

$$q\left(a_i^{\circ},\ldots,a_i^{n-1}\right)\cap r\left(a_i^{\circ},\ldots,a_i^{n-1}\right)\neq\varnothing$$

therefore, for every $i\in I$, there is an element $x_i\in A_i$ such that

$$\prod_{i\in I}x_i\in\prod_{i\in I}q\left(a_i^{\circ},\ldots,a_i^{n-1}\right)\cap\prod_{i\in I}r\left(a_i^{\circ},\ldots,a_i^{n-1}\right).$$

and therefore,

$$x_i\in q\left(a_i^{\circ},\ldots,a_i^{n-1}\right)\cap r\left(a_i^{\circ},\ldots,a_i^{n-1}\right).$$

consequently ,

$$q\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)\cap r\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)\neq\varnothing$$

Lemma 3

let $\left(\mathcal{A}_i|i\in I\right)$ be a family of multiple algebras of type τ and $\mathbf{q},\mathbf{r}\in\mathbf{P}^{(n)}(\tau)$. if for every $i\in I,\mathbf{q}=\mathbf{r}$ is satisfied on $\mathcal{A}_i$ then the strong equality $\mathbf{q}=\mathbf{r}$ holds on $\prod_{i\in I}\mathcal{A}_i$.

Proof:

for every $\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\in\prod_{i\in I}A_i$,

$$q\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right)=\prod_{i\in I}q\left(a_i^{\circ},\ldots,a_i^{n-1}\right)$$
$$=\prod_{i\in I}r\left(a_i^{\circ},\ldots,a_i^{n-1}\right)$$
$$=r\left(\left(a_i^{\circ}\right)_{i\in I},\ldots,\left(a_i^{n-1}\right)_{i\in I}\right).$$

III. A VARIETY OF MULTIPLE ALGEBRAS

a set of multiple algebras that are closed relation to sub-multiple algebras, homomorphit images and Cartesian products of its elements, is called a variety of multiple algebras.

By definition of a variety of multiple algebras, if the set K is a variety of multiple algebras , then K induces $\bar{K}$.

Because, the basic algebra of any multiple algebras is its homomorphic image under the canonical mapping.

Remark 1

Let K be a variety of multiple algebras. Let Σ be set of weak and strong equalities. Let K_Σ be the set of all elements of K on which hold the equalites holding on Σ , then, according to lemass 2 and 3 K_Σ is a variety of multiple algebras of type τ .

By remark 1 , one could consider a set of super structures as a variety of multiple algebras.

Theorem 1

The Cartesian product of hypergroups is a hypergroup.

Proof:

Let $\left((H_i, o_i) | i \in I\right)$ be a family of hypergroup. Consider the set $\prod_{i\in I} H_i$ along with the following binary super operation:

$$(a_i)_{i\in I} \circ (b_i)_{i\in I} = \prod_{i\in I} (a_i \circ_i b_i)$$

For every $(a_i)_{i\in I}, (b_i)_{i\in I}, (c_i)_{i\in I} \in \prod_{i\in I} H_i$

$$\begin{aligned}
\left((a_i)_{i\in I} \circ (b_i)_{i\in I} \circ (c_i)_{i\in I}\right) &= \left(\prod_{i\in I} (a_i \circ_i b_i)\right) \circ \prod_{i\in I} (c_i) \\
&= \prod_{i\in I} \left((a_i \circ_i b_i) \circ_i c_i\right) \\
&= \prod_{i\in I} \left(a_i \circ_i (b_i \circ_i c_i)\right) \\
&= (a_i)_{i\in I} \circ (b_i \circ_i c_i)_{i\in I} \\
&= (a_i)_{i\in I} \circ \left((b_i)_{i\in I} \circ (c_i)_{i\in I}\right).
\end{aligned}$$

Also , for every $(a_i)_{i\in I} \in \prod_{i\in I} H_i$

$$(a_i)_{i\in I} \circ \prod_{i\in I} H_i = \prod_{i\in I} (a_i \circ_i H_i) = \prod_{i\in I} H_i .$$

Similarly it will be shown that $\prod_{i\in I} H_i \circ (a_i)_{i\in I} = \prod_{i\in I} H_i$. therefore, $\prod_{i\in I} H_i$ is a hypergroup.

Theorem 2

If (H, $\circ$) is a hypergroup , ($H', \circ'$) is a semi- hypergroup and $f : H \to H'$ is a homomorphism between H and H′ ,then the homomorphism image of H under f is a hypergroup.

Proof:

For every $x, y, z \in f(H)$ there are elements $a, b, c \in H$ such that $f(c) = z$, $f(b) = y$, $f(a) = x$. Therefore

$$\begin{aligned}
\left(x \circ' y\right) \circ' z &= \left(f(a) \circ' f(b) \circ' f(c)\right) \\
&= f(a \circ b) \circ' f(c) \\
&= f\left((a \circ b) \circ c\right) \\
&= f\left(a \circ (b \circ c)\right) \\
&= f(a) \circ' f(b \circ c) \\
&= f(a) \circ' \left(f(b) \circ' f(c)\right) \\
&= x \circ' \left(y \circ' z\right).
\end{aligned}$$

For every $x \in f(H)$ there is an element $a \in H$ such that $f(a) = x$. Therefore, for every $x \in f(H)$

$$\begin{aligned}
x \circ' f(H) = f(a) \circ' f(H) &= \bigcup_{b\in H} f(a \circ b) \\
&= f(a \circ H) = f(H).
\end{aligned}$$

Every hypergroup can be considered as a multiple algebra. By theorem 2 any sub-multiple algebra is a hypergroup, too. Therefore, a set of hypergroup forms a variety of multiple algebras. also any canonical hypergroup can be considered as the multiple algebra $(H, \circ, /, \backslash, e, ')$ where $(H, \circ, /, \backslash, e,)$ is hypergroup and e' are nullary and unary multiple operations satisfying the following equations.

$$a \circ b = b \circ a, \forall a, b \in H$$

$$e \circ a = a = a \circ e, \forall a \in A$$

$$a / b = (b / a)' \left(a \backslash b = (b \backslash a)'\right), \forall a, b \in H .$$

Therefore, by remark 1 the set of canonical hypergroup can be considered as a variety of multiple algebras .

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