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A new time discontinuous expanded mixed element method for convection-dominated diffusion equation

Jinfeng Wang[1], Yuanhong Bi[1], Hong Li[2*], Yang Liu[2*], Meng Zhao[2]

Abstract—In this paper, a new time discontinuous expanded mixed finite element method is proposed and analyzed for two-order convection-dominated diffusion problem. The proofs of the stability of the proposed scheme and the uniqueness of the discrete solution are given. Moreover, the error estimates of the scalar unknown, its gradient and its flux in the $L^\infty(\bar{J}, L^2(\Omega))$-norm are obtained.

Keywords—Convection-dominated diffusion equation; Expanded mixed method; Time discontinuous scheme; Stability; Error estimates.

I. Introduction

In this paper, we consider the following convection-dominated diffusion equation

$$\begin{cases} u_t - \nabla\cdot(a(x,t)\nabla u) + \mathbf{b}\cdot\nabla u(x,t) \\ \quad + cu(x,t) = f(x,t), \Omega\times J \\ u(x,t) = 0, \partial\Omega\times\bar{J}, \\ u(x,0) = u_0(x), \bar{\Omega}, \end{cases} \tag{1}$$

where Ω is a bounded convex polygonal domain in $R^d(d = 1,2,3)$ with $Lipschitz$ continuous boundary $\partial\Omega$, $J = (0,T]$ is the time interval with $0 < T < \infty$. $u_0(x)$ and $f(x,t)$ are given functions, coefficients $a = a(x,t)$ and $c = c(x,t)$ are two smooth and bounded functions, coefficient $\mathbf{b}(x) = (b_1(x),\cdots b_d(x))$ is a bounded vector, and $|\mathbf{b}| = (\sum\limits_{i=1}^{d} b_i^2)^{\frac{1}{2}} \le \frac{1}{2}$.

Convection-dominated diffusion equations are a class of important evolution partial differential equations, and have a lot of applications in many physical problems. Ref [1] proposed some numerical methods based on combining the method of characteristics with finite element or finite difference procedures for convection-dominated diffusion problems. Ware [2] studied a spectral Lagrange-Galerkin method for convection-dominated diffusion problems. Chen [3] talk a mixed element method for the convection-dominated diffusion problems with small parameter ε. Ref [4] proposed and analysed a nonconforming local projection stabilized method for the non-stationary convection diffusion problem. John *et al.* [5] studied a streamline-diffusion method of nonconforming finite element approximations for convection-diffusion problems. Ref [7] analysed a mixed time discontinuous space-time finite element method for convection diffusion equations.

In 1997, a expanded mixed finite element method was proposed and analysed by Arbogast *et al.* [11]. And some mathematical theories were given and proved by Chen [12] for second-order linear elliptic equation, [13] for second-order quasilinear elliptic equation and [14] for fourth-order elliptic problems. With the development of the expanded mixed finite element method, the method were applied to many evolution equations. In [15], some error estimates of the expanded mixed element for a kind of parabolic equation were given. Woodward and Dawson [16] studied the expanded mixed finite element method for nonlinear parabolic equation. In [17], a posteriori error estimator for expanded mixed hybrid methods was proposed. Chen *et al.* [18] studied a two-grid method for expanded mixed finite-element solution of semilinear reaction-diffusion equations. In [19], a two-grid method with expanded mixed method was studied for nonlinear racction-diffusion equations. Song and Yuan [20] proposed the expanded upwind-mixed multi-step method for the miscible displacement problem in three dimensions. Guo and Chen [21] developed and analysed an expanded characteristic-mixed finite element method for a convection-dominated transport problem. In 2010, Chen and Wang [22] proposed an H^1-Galerkin expanded mixed method for a nonlinear parabolic equation in porous medium flow and Liu and Li [23] studied the H^1-Galerkin expanded mixed method for pseudo-hyperbolic equation. Liu [24], studied the H^1-Galerkin expanded mixed method for RLW-Burgers equation and proved semi- and fully discrete optimal error estimates. Che *et al.* [28] studied the H^1-Galerkin expanded mixed method for nonlinear viscoelasticity-type equation. In [25] and [26], the expanded mixed covolume method was studied for the linear integro-differential equation of parabolic type and elliptic problems, respectively. Jiang and Li [27] studied an expanded mixed semidiscrete scheme for the problem of purely longitudinal motion of a homogeneous bar.

In this article, we will develop a new expanded mixed finite element method based time discontinuous finite element method [6], [7], [8], [9], [10], prove the stability and uniqueness for discrete scheme, and obtain the error estimates. In the near future, we will study the space-time discontinuous expanded mixed finite element method for some evolution equations.

II. Notations and definitions

In order to introduce the mixed time discontinuous space-time finite element method for equation(1), we discretize the

1. School of Statistics and Mathematics, Inner Mongolia Finance and Economics College, Hohhot 010051, China. E-mail: w45j85f@163.com(Jinfeng Wang); yuanhong918@163.com(Yuanhong Bi).

2. School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China. * Correspondence to: E-mail: smslh@imu.edu.cn (Hong Li); mathliuyang@yahoo.cn (Yang Liu).

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time interval $[0,T]$ by $0 = t^0 < t^1 < \cdots < t^N = T$ firstly. Let $I_n = (t^n, t^{n+1})$, time step $k_n = t^{n+1} - t^n, n = 0,1,2,\cdots,N-1$. T_h is the regular partition of Ω and the partition unit is τ. Define the space-time domain $Q := \Omega \times J$, the space-time slab $S^n := \Omega \times I_n$. Suppose $T_{h,n}$ is the regular partition of S^n and the partition unit is $K = \tau \times I_n$. Let $h_n = \max\limits_{K \in T_{h,n}} (h_K), n = 0,1,2..N-1$, $h = \max\limits_n h_n$. Define discrete approximate spaces

$$
\begin{aligned}
W_{h,n} &= \{v : v|_K \in P_k(\tau) \times P_k(I_n), \forall K \in T_{h,n}\},\\
Q_{h,n} &= \{v : v|_K \in P_m(\tau) \times P_m(I_n), \forall K \in T_{h,n}\},\\
\mathbf{V}_{h,n} &= \{\varphi : \varphi|_K \in (Q_{h,n})^d, \nabla \cdot \varphi|_K \in Q_{h,n}, \forall K \in T_{h,n}\},\\
\mathbf{\Lambda}_{h,n} &= \{\varphi : \varphi|_K \in (Q_{h,n})^d, \forall K \in T_{h,n}\},
\end{aligned}
$$

where P_k denotes the polynomial space of degree at most k . We introduce definitions and lemmas which will be used in this paper.

Definition 1: Define the inner product of space-time slab S^n by $(\omega, v)_n = (\omega, v)_{S^n} = \int_{I_n} (\omega, v) ds$, $where(\omega, v)$ is the inner product in Ω, and the corresponding norm is $||v||_n = (v,v)_n^{1/2} = (\int_{I_n} ||v||^2_\Omega ds)^{\frac{1}{2}}$.

Definition 2: At the time level $t = t^n$ $(n = 0,1,\cdots,N-1)$, define the inner product of L_2 by $\langle \omega, v\rangle_n = \langle \omega(\cdot,t^n), v(\cdot,t^n)\rangle_\Omega$and the corresponding norm is $|v|_n = \langle v, v\rangle_n^{1/2}$.

Definition 3: Define the left and right limits by $v_\pm(x,t) = \lim\limits_{s\to 0^\pm} v(x, t+s)$, and the jump term at discontinuous nodes in time by $[v] = v_+ - v_-$. Define norm $|||v|||^2 = \frac{1}{2}(|v|^2_N + |v|^2_0 + \sum\limits_{n=1}^{N-1} |[v]|^2_n)$.

Definition 4: Define the norm of space $L_2(J, L_2(\Omega))$ by $||v||^2_Q = \int_0^{t^N} ||v||^2_\Omega dt$.

Definition 5: Define the norm of space $L_\infty(\bar{J}, L_2(\Omega))$ by $\max\limits_{t\in \bar{J}=[0,T]} \|\cdot\|_\Omega$, where $\|\cdot\|_\Omega$ denotes the corresponding norm of Sobolev space $L_2(\Omega)$.

Definition 6: $\mathbf{V} = \mathbf{H}(\mathbf{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^d | \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$, with norm $\| \mathbf{v} \|^2_V = \| \mathbf{v} \|^2 + \| \nabla\cdot\mathbf{v} \|^2, W = L^2(\Omega)$ or $W = \{w \in L^2(\Omega) | w_{|\partial\Omega} = 0\}, \mathbf{\Lambda} = (L^2(\Omega))^d$.

III. The existence, uniqueness and stability for Semi-discrete scheme

Introducing the two auxiliary variables $\lambda = -\nabla u$ and $\sigma = -a(x,t)\nabla u = a\lambda$, we obtain the following first-order system for (1)

$$
\begin{cases}
(a)\ u_t + \nabla\cdot\sigma - \mathbf{b}\cdot\lambda + cu = f, & (x,t) \in \Omega\times J,\\
(b)\ \lambda + \nabla u = 0, & (x,t) \in \Omega\times J,\\
(c)\ \sigma - a\lambda = 0, & (x,t) \in \Omega\times J,\\
(d)\ u(x,t) = 0, & (x,t) \in \partial\Omega\times \bar{J},\\
(e)\ u(x,0) = u_0(x), & x\in\Omega.
\end{cases} \tag{2}
$$

The new time discontinuous expanded mixed weak formulation of (2) is as follows

$$
\begin{cases}
(a)\ \int_0^{t^N} (u_t, w)dt + \int_0^{t^N} (\nabla\cdot\sigma, w)dt - \int_0^{t^N} (\mathbf{b}\cdot\lambda, w)dt\\
\quad + \sum\limits_{n=1}^{N-1} \langle [u], w_+\rangle_n + \langle u_+, w_+\rangle_0 + \int_0^{t^N} (cu, w)dt\\
\quad = \langle u, w_+\rangle_0 + \int_0^{t^N} (f, w)dt,\\
(b)\ \int_0^{t^N} (\lambda, \mathbf{v})dt - \int_0^{t^N} (u, \nabla\cdot\mathbf{v})dt = 0, \forall \mathbf{v}\in\mathbf{V}, t\in J,\\
(c)\ \int_0^{t^N} (a\lambda, \mu) - \int_0^{t^N} (\sigma, \mu) = 0, \forall\mu\in\lambda, t\in J.
\end{cases} \tag{3}
$$

That is in interval $I_n = (t^n, t^{n+1})$, it holds

$$
\begin{cases}
(a)\ (u_t, w)_n + (\nabla\cdot\sigma, w)_n - (\mathbf{b}\cdot\lambda, w)_n\\
\quad + \langle [u], w_+\rangle_n + (cu, w)_n = (f, w)_n,\\
(b)\ (\lambda, \mathbf{v})_n - (u, \nabla\cdot\mathbf{v})_n = 0,\\
(c)\ (a\lambda, \mu)_n - (\sigma, \mu)_n = 0.
\end{cases} \tag{4}
$$

Then, the semi-discrete mixed finite element scheme for (4) is to determine $(\sigma^h, \lambda^h, u^h) \in \mathbf{V}_{h,n} \times \mathbf{\Lambda}_{h,n} \times W_{h,n}$ such that

$$
\begin{cases}
(a)\ (u^h_t, \omega^h)_n + (\nabla\cdot\sigma^h, w^h)_n - (\mathbf{b}\cdot\lambda^h, w^h)_n\\
\quad + \langle [u^h], w^h_+\rangle_n + (cu^h, w^h)_n = (f, w^h)_n,\\
(b)\ (\lambda^h, \mathbf{v}^h)_n - (u^h, \nabla\cdot\mathbf{v}^h)_n = 0,\\
(c)\ (a\lambda^h, \mu^h)_n - (\sigma^h, \mu^h)_n = 0.
\end{cases} \tag{5}
$$

We will prove the stability for semi-discrete scheme (5).

Theorem 3.1: The semi-discrete scheme (5) is stable and holds the following inequality

$$
\max_{0\le t\le T} (||u^h||_\Omega + ||\sigma^h||_\Omega + ||\lambda^h||_\Omega) \le M(||u_{h0}||_\Omega + ||f||_Q),
$$

where M is a constant independent of h_n and k_n.

Proof: Choosing $w^h = u^h, \mathbf{v}^h = \boldsymbol{\sigma}^h, \boldsymbol{\mu}^h = \boldsymbol{\lambda}_h$ in (5) and summing from $n = 1$ to N, we have

$$
\begin{aligned}
&\int_0^{t^N} (u^h_t, u^h)dt + \int_0^{t^N} (u^h, \nabla\cdot\boldsymbol{\sigma}^h)dt - \int_0^{t^N} (\mathbf{b}\cdot\lambda^h, u^h)dt\\
&+ \sum_{n=1}^{N-1} \langle [u^h], u^h_+\rangle_n + \langle u^h_+, u^h_+\rangle_0 + \int_0^{t^N} (cu^h, u^h)dt\\
&= \langle u^h, u^h_+\rangle_0 + \int_0^{t^N} (f, u^h)dt,\\
&\int_0^{t^N} (\lambda^h, \boldsymbol{\sigma}^h)dt - \int_0^{t^N} (u^h, \nabla\cdot\boldsymbol{\sigma}^h)dt = 0,\\
&\int_0^{t^N} (a\boldsymbol{\lambda}^h, \boldsymbol{\lambda}^h)dt - \int_0^{t^N} (\boldsymbol{\sigma}^h, \boldsymbol{\lambda}^h)dt = 0.
\end{aligned} \tag{6}
$$

Adding the three equations, we obtain

$$\begin{aligned}&\int_0^{t^N}(u_t^h,u^h)dt+\int_0^{t^N}(a\boldsymbol{\lambda}^h,\boldsymbol{\lambda}^h)dt-\int_0^{t^N}(\mathbf{b}\cdot\boldsymbol{\lambda}^h,u^h)dt\\&+\sum_{n=1}^{N-1}\langle[u^h],u_+^h\rangle_n+\langle u_+^h,u_+^h\rangle_0+\int_0^{t^N}(cu^h,u^h)dt\\=&\langle u^h,u_+^h\rangle_0+\int_0^{t^N}(f,u^h)dt.\end{aligned}\tag{7}$$

Integration by parts for the first term in (7), we have

$$\begin{aligned}&|||u^h|||^2+\int_0^{t^N}(a\boldsymbol{\lambda}^h,\boldsymbol{\lambda}^h)dt+\int_0^{t^N}(cu^h,u^h)dt\\=&\langle u^h,u_+^h\rangle_0+\int_0^{t^N}(f,u^h)dt+\int_0^{t^N}(\mathbf{b}\cdot\boldsymbol{\lambda}^h,u^h)dt.\end{aligned}\tag{8}$$

Using Cauchy-Schwarz inequality and Young inequality, we obtain

$$\begin{aligned}&|||u^h|||^2+\int_0^{t^N}||\boldsymbol{\lambda}^h||_\Omega^2dt+\int_0^{t^N}||u^h||_\Omega^2dt\\\le&||u_0^h||_\Omega^2+\int_0^{t^N}||f||_\Omega^2dt.\end{aligned}\tag{9}$$

Take $\boldsymbol{\mu}^h=\boldsymbol{\sigma}^h$ in (5c) and use Young inequality to get

$$\int_0^{t^N}||\boldsymbol{\sigma}^h||_\Omega^2dt\le C_0\int_0^{t^N}||\boldsymbol{\lambda}^h||_\Omega^2dt.\tag{10}$$

Combing (9) and (10) and taking the max norm with respect to time t, we get the conclusion for theorem 3.1. ■

Theorem 3.2: There exists a unique discrete solution to semi-discrete scheme (5).

Proof: In fact, since (5) is linear, it suffices to show that the associated homogeneous system

$$\begin{cases}(a)\ (u_t^h,w^h)_n+(\nabla\cdot\boldsymbol{\sigma}^h,w^h)_n-(\mathbf{b}\cdot\boldsymbol{\lambda}^h,w^h)_n\\\quad+\langle u_+^h,w_+^h\rangle_n+(cu^h,w^h)_n=0,\\(b)\ (\boldsymbol{\lambda}^h,\mathbf{v}^h)_n-(u^h,\nabla\cdot\mathbf{v}^h)_n=0,\\(c)\ (a\boldsymbol{\lambda}^h,\boldsymbol{\mu}^h)_n-(\boldsymbol{\sigma}^h,\boldsymbol{\mu}^h)_n=0,\\(d)\ (u^h(0),w^h)=0,\end{cases}\tag{11}$$

has only the zero trivial solution.

Taking $w^h=u^h,\mathbf{v}_h=\boldsymbol{\sigma}_h,\boldsymbol{\mu}_h=\boldsymbol{\lambda}_h$ in (11) and adding the three equations, we get

$$\begin{aligned}&|u_-^h|_{n+1}^2-|u_+^h|_n^2+2||a^{\frac12}\boldsymbol{\lambda}^h||_n^2\\&+2|u_+^h|_n^2+2||c^{\frac12}u^h||_n^2=(\mathbf{b}\cdot\boldsymbol{\lambda}^h,u^h)_n.\end{aligned}\tag{12}$$

Using Cauchy-Schwartz inequality, we have

$$\begin{aligned}&|u_-^h|_{n+1}^2+|u_+^h|_n^2+2(a_0-\varepsilon)||\boldsymbol{\lambda}^h||_n^2\\&+2(c_0-2\varepsilon_0)||u^h||_n^2\le 0\end{aligned}\tag{13}$$

So, we have $\boldsymbol{\lambda}^h=\mathbf{0}$ and $u^h=0$.

Choosing $\boldsymbol{\mu}^h=\boldsymbol{\sigma}_h$ in (11c), using Cauchy-Schwarz inequality and Young inequality, we have

$$||\boldsymbol{\sigma}^h||_n\le C||\boldsymbol{\lambda}^h||_n.\tag{14}$$

so, we have $\boldsymbol{\sigma}^h=\mathbf{0}$. ■

IV. Error estimates for $L^\infty(\bar J,L^2(\Omega))$ norm

In order to analyze the error estimates of the method, we first introduce the expanded mixed elliptic projection associated with our equations.

Lemma 4.1: Let $(\tilde u_h,\tilde{\boldsymbol{\lambda}}_h,\tilde{\boldsymbol{\sigma}}_h)\in W_{h,n}\times\mathbf{V}_{h,n}\times\boldsymbol{\Lambda}_{h,n}$ be given by the following mixed relations(see Refs [12], [13], [15])

$$\begin{cases}(a)\ \ (\nabla\cdot(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h),w_h)_n\\\quad+(\mathbf{b}\cdot(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}}_h),w_h)_n=0,\forall\omega_h\in W_{h,n},\\(b)\ \ (\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}}_h,\mathbf{v}_h)_n-(u-\tilde u_h,\nabla\cdot\mathbf{v}_h)_n=0,\forall\mathbf{v}_h\in\mathbf{V}_{h,n},\\(c)\ \ (a(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}}_h),\boldsymbol{\mu}_h)_n-(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h,\boldsymbol{\mu}_h)_n=0,\forall\boldsymbol{\mu}_h\in\boldsymbol{\Lambda}_{h,n}.\end{cases}\tag{15}$$

the corresponding approximation properties hold

$$\parallel\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}}_h\parallel\le Ch^l(\parallel\boldsymbol{\lambda}\parallel_l+\parallel\boldsymbol{\sigma}\parallel_l),1\le l\le k+1,\tag{16}$$

$$\parallel\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h\parallel\le Ch^l(\parallel\boldsymbol{\lambda}\parallel_l+\parallel\boldsymbol{\sigma}\parallel_l),1\le l\le k+1,\tag{17}$$

$$\parallel u-\tilde u_h\parallel\le\begin{cases}Ch\parallel u\parallel_2,&k=1,\\Ch^l\parallel u\parallel_l,&2\le l\le k,k\ge2.\end{cases}\tag{18}$$

Theorem 4.2: Let $(\boldsymbol{\sigma},\boldsymbol{\lambda},u)$ and $(\boldsymbol{\sigma}^h,\boldsymbol{\lambda}^h,u^h)$ be the solution to (2) and (5), respectively. There exists a positive constant C independent of the spatial mesh parameters h_n and time-discretization parameter k_n such that

$$\begin{aligned}&\max_{t\in\bar J}(||\boldsymbol{\sigma}-\boldsymbol{\sigma}^h||_\Omega+||\boldsymbol{\lambda}-\boldsymbol{\lambda}^h||_\Omega)\\&\le\begin{cases}Ch(||u||_{2,Q}+||\boldsymbol{\sigma}||_{2,Q}+||\lambda||_{2,Q}),k=1,\\Ch^l(||u||_{l,Q}+||\boldsymbol{\sigma}||_{l,Q}+||\lambda||_{l,Q}),2\le l\le k,k\ge2,\end{cases}\end{aligned}$$

$$\max_{t\in\bar J}||u-u^h||_\Omega\le\begin{cases}Ch||u||_{2,Q},k=1,\\Ch^l||u||_{l,Q},2\le l\le k,k\ge2.\end{cases}$$

Proof: Let

$$\begin{aligned}u-u^h&=u-\tilde u^h+\tilde u^h-u^h=\zeta_1+\zeta_2,\\\boldsymbol{\sigma}-\boldsymbol{\sigma}^h&=\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}^h+\tilde{\boldsymbol{\sigma}}^h-\boldsymbol{\sigma}^h=\boldsymbol{\eta}_1+\boldsymbol{\eta}_2,\\\boldsymbol{\lambda}-\boldsymbol{\lambda}^h&=\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}}^h+\tilde{\boldsymbol{\lambda}}^h-\boldsymbol{\lambda}^h=\boldsymbol{\theta}_1+\boldsymbol{\theta}_2.\end{aligned}$$

Using (3)-(5) and Lemma, we have

$$\begin{aligned}&\int_0^{t^N}(u_t-u_t^h,w^h)dt+\int_0^{t^N}(\nabla\cdot\boldsymbol{\eta}_2,w^h)dt\\&-\int_0^{t^N}(\mathbf{b}\cdot\boldsymbol{\theta}_2,w^h)dt+\sum_{n=1}^{N-1}\left\langle[u-u^h],\omega_+^h\right\rangle_n\\&+\left\langle(u_+-u_+^h),w_+^h\right\rangle_0+\int_0^{t^N}(c(u-u^h),w^h)dt\\&=\left\langle u-u^h,w_+^h\right\rangle_0,\\&\int_0^{t^N}(\boldsymbol{\theta}_2,\mathbf{v}^h)dt-\int_0^{t^N}(\zeta_2,\nabla\cdot\mathbf{v}^h)dt=0,\\&\int_0^{t^N}(a\boldsymbol{\theta}_2,\boldsymbol{\mu}^h)dt-\int_0^{t^N}(\boldsymbol{\eta}_2,\boldsymbol{\mu}^h)dt=0.\end{aligned}\tag{19}$$

Take $w^h = \zeta_2, \mathbf{v}^h = \boldsymbol{\eta}_2, \boldsymbol{\mu}^h = \boldsymbol{\theta}_2$ in (19) and add the three equations to obtain

$$\begin{aligned}&\int_0^{t^N} (u_t - u_t^h, \zeta_2)dt - \int_0^{t^N} (\mathbf{b}\cdot\boldsymbol{\theta}_2, \zeta_2)dt \\ &+ \sum_{n=1}^{N-1} \left\langle [u-u^h], \zeta_{2+}\right\rangle_n + \left\langle (u_+ - u_+^h), \zeta_{2+}\right\rangle_0 \\ &+ \int_0^{t^N} (c(u-u^h), \zeta_2)dt + \int_0^{t^N} (a\boldsymbol{\theta}_2, \boldsymbol{\theta}_2)dt \\ &= \left\langle (u-u^h), \zeta_{2+}\right\rangle_0 .\end{aligned} \tag{20}$$

Using the definition of $|||v|||$, the above equation can be written as

$$\begin{aligned}&|||\zeta_2|||^2 + \int_0^{t^N} (\zeta_{1t}, \zeta_2)dt + \int_0^{t^N} (c\zeta_2, \zeta_2)dt \\ &+ \int_0^{t^N} (c\zeta_1, \zeta_2)dt - \int_0^{t^N} (\mathbf{b}\cdot\boldsymbol{\theta}_2, \zeta_2)dt \\ &+ \sum_{n=1}^{N-1} \langle [\zeta_1], \zeta_{2+}\rangle_n + \langle \zeta_{1+}, \zeta_{2+}\rangle_0 \\ &+ \int_0^{t^N} (a\boldsymbol{\theta}_2, \boldsymbol{\theta}_2)dt = \left\langle (u-u^h), \zeta_{2+}\right\rangle_0 .\end{aligned} \tag{21}$$

Integration by parts for the second term in the left hand side of (21)

$$\begin{aligned}&|||\zeta_2|||^2 + \int_0^{t^N} (c\zeta_2, \zeta_2)dt + \int_0^{t^N} (a\boldsymbol{\theta}_2, \boldsymbol{\theta}_2)dt \\ &= \langle \zeta_1 + \zeta_2, \zeta_{2+}\rangle_0 + \int_0^{t^N} (\zeta_{2,t}, \zeta_1)dt \\ &+ \sum_{n=1}^{N-1} \langle \zeta_{1-}, [\zeta_2]\rangle_n - \langle \zeta_{1-}, \zeta_{2-}\rangle_N \\ &- \int_0^{t^N} (c\zeta_1, \zeta_2)dt + \int_0^{t^N} (\mathbf{b}\cdot\boldsymbol{\theta}_2, \zeta_2)dt.\end{aligned} \tag{22}$$

Using Cauchy-Schwarz inequality, Young inequality and the definition of $|||v|||$, we have

$$\begin{aligned}&|||\zeta_2|||^2 + \frac{1}{2}\int_0^{t^N} ||\zeta_2||_\Omega^2 dt + \frac{1}{2}\int_0^{t^N} ||\boldsymbol{\theta}_2||_\Omega^2 dt \\ \le &\frac{1}{2}|||\zeta_2|||^2 + K(\sum_{n=1}^{N} |\zeta_{1-}|^2 + ||\zeta_1||_Q^2).\end{aligned} \tag{23}$$

Using the inequality (16)-(18), we obtain

$$|||\zeta_2||| + ||\boldsymbol{\theta}_2||_Q + ||\zeta_2||_Q \le Ch^l ||u||_{l,Q}. \tag{24}$$

Take $\boldsymbol{\mu}^h = \boldsymbol{\eta}_2$ in (19) and use Cauchy-schwarz inequality to get

$$||\boldsymbol{\eta}_2||_Q \le C||\boldsymbol{\theta}_2||_Q. \tag{25}$$

Apply (24), (25), (16)-(18) and the triangle inequality to get the conclusion of theorem 4.2. ■

V. Concluding Remarks

In this article, we propose a new time discontinuous expanded mixed finite element scheme for convection-dominated diffusion equation. In the near further, the proposed method will be applied to others evolution equations such as evolution integro-differential equations.

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