

# Approximately Jordan maps and their stability

Nasrin Eghbali

*Abstract*—In this paper we consider the approximate Jordan maps and boundedness of these maps. Also we investigate the stability of approximate Jordan maps and prove some stability properties for approximate Jordan maps.

*Keywords*—Approximate Jordan map; Stability.

## I. INTRODUCTION AND PRELIMINARIES

At first, we recall two definitions for Jordan maps:

*Definition 1.1:* A linear map  $T : A \rightarrow B$  between two algebras  $A$  and  $B$  is called a Jordan map if

$$T(xy + yx) = T(x)T(y) + T(y)T(x)$$

for all  $x$  and  $y$  in  $A$ .

*Definition 1.2:* A linear map  $T : A \rightarrow B$  between two algebras  $A$  and  $B$  is called a Jordan map if

$$T(x^2) = T(x)^2$$

for all  $x \in A$ .

It is obvious that these definitions are equivalent.

In [4], Lee and Kim define the approximate Jordan maps:

*Definition 1.3:* A linear map  $T : A \rightarrow B$  between two algebras  $A$  and  $B$  is called an approximately Jordan map if there exists  $\delta \geq 0$  such that  $\|T(x^2) - T(x)^2\| \leq \delta\|x\|^2$ , for all  $x \in A$ .

Sometimes in the Definition 1.3 we call  $T$  is a  $\delta$ -Jordan map. If  $B$  is the complex field, then  $T$  is called the approximately Jordan functional. Note that if  $T$  is an approximately Jordan functional on a commutative Banach algebra, then  $T$  is an approximately multiplicative functional [3]. In this paper at first we define another definition for approximately Jordan mappings and then we show that two definitions of approximately Jordan mappings are equivalent.

It seems that the stability problem of functional equations had been first raised by Ulam [8]: For what metric groups  $G$  is it true that an approximate additive of  $G$  is necessarily near to a strict linear map?

From that time, different mathematicians have a wide research around this problem. For example Hyers [2] in 1941 and Th.M.Rassias [6] in 1978, have published some important results in this area.

In this note we consider the stability problem for approximately Jordan maps.

N. Eghbali is with the Young researchers club, Islamic Azad University, Ardabil branch, Ardabil, Iran e-mail: nasrineghbali@gmail.com.

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## II. APPROXIMATE JORDAN MAPPINGS

In the account of in work that follows, we suppose that all of algebras are noncommutative algebra. We start our work with this definition:

*Definition 2.1:* A linear map  $T : A \rightarrow B$  between two algebras  $A$  and  $B$  is called an approximately Jordan map if there exists  $\delta \geq 0$  such that

$$\|T(xy + yx) - T(x)T(y) - T(y)T(x)\| \leq 2\delta\|x\|\|y\|$$

for all  $x$  and  $y$  in  $A$ .

*Theorem 2.2:* The definitions (2.1) and (1.3) are equivalent.

*Proof:* By putting  $x = y$  in Definition (2.1) we can have Definition (1.3). Conversely, by definition (1.3) we have

$$\|T((x + y)^2) - T(x + y)^2\| \leq \delta\|x + y\|^2.$$

So,

$$\|T(xy + yx) - T(x)T(y) - T(y)T(x) + T(x^2) - T(x)^2 + T(y^2) - T(y)^2\| \leq \delta\|x + y\|^2.$$

Hence

$$\|T(xy + yx) - T(x)T(y) - T(y)T(x)\| \leq 2\delta(\|x\|^2 + \|y\|^2 + \|x\|\|y\|).$$

Putting  $\|x\| = \|y\| = 1$  we have

$$\|T(xy + yx) - T(x)T(y) - T(y)T(x)\| \leq 6\delta.$$

So

$$\|T(ab + ba) - T(a)T(b) - T(b)T(a)\| \leq 3\delta\|a\|\|b\|$$

for all  $a, b \in A$ . ■

*Definition 2.3:* [5] Let  $A$  and  $B$  be Banach algebras and  $\varphi : A \rightarrow B$  a linear map. We say that  $\varphi$  is a strongly approximate Jordan map if there exists an  $\delta \geq 0$  such that

$$\|\varphi(x^2) - \varphi(x)^2\| \leq \delta$$

for all  $x \in A$ .

It has been proved in [5] that if  $f$  is a strongly approximate one-to-one Jordan functional on an algebra  $A$ , then  $f$  is a Jordan functional. Here we provide an easier proof of this result. Of course our result is stronger than Theorem of [5], because we have removed the one-to-one condition.

*Theorem 2.4:* If  $f$  is a strongly approximate Jordan functional on an algebra  $A$ , then  $f$  is a Jordan functional.

*Proof:* Suppose that  $a \in A$ . For all  $x \in A$  we have:

$$|f(x^2) - f(x)^2| \leq \delta.$$

Let  $x = 2^n a$ . We have:

$$|f(a^2) - (f(a))^2| \leq \delta 2^{-2n}$$

and therefore by letting  $n$  tend to infinity we have  $f(a^2) = f(a)^2$ . ■

**Definition 2.5:** [7] Let  $X$  and  $Y$  be Banach algebras, and  $\varphi : X \rightarrow Y$  a Jordan map. We define

$$\|\varphi\| = \{ \sup \|\varphi(x)\| : \|x\| = 1 \} \leq \infty.$$

The map  $\varphi$  is called bounded if  $\|\varphi\| < \infty$ .

**Theorem 2.6:** Let  $\varphi$  be a Jordan functional on a Banach algebra  $A$ . Then  $\|\varphi\| \leq 1 + \delta$ .

*Proof:* By definition,  $\|\varphi\| = \{ \sup |\varphi(a)| : \|a\| = 1 \} \leq \infty$ . By hypothesis,

$$|\varphi(a^2) - \varphi(a)^2| \leq \delta \|a\|^2$$

for all  $a \in A$ . Given  $\mu$  with  $0 < \mu < \sqrt{\delta}$ , choose  $a$  with  $\|a\| = 1$  such that  $\|\varphi\| - \mu < |\varphi(a)|$ . Then

$$(\|\varphi\| - \mu)^2 < |\varphi(a)|^2 \leq |\varphi(a^2)| + \delta \leq \|\varphi\| + \delta.$$

Letting  $\mu \rightarrow 0$  we have  $\|\varphi\|^2 - \|\varphi\| - \delta \leq 0$ . By solving this inequality we have

$$-\delta \leq \|\varphi\| \leq 1/2 + \frac{\sqrt{1+4\delta}}{2} \leq 1 + \delta. \quad \blacksquare$$

**Proposition 2.7:** If  $\varphi$  is an approximately Jordan map and  $\psi$  a bounded Jordan map, then  $\varphi + \psi$  is an approximately Jordan map. Also if  $\lambda \in C$ , then  $(1 + \lambda)\varphi$  is an approximately Jordan map.

*Proof:* It is straightforward. ■

**Remark 2.8:** If the algebra  $A$  does not have a unit,  $A$  can be extended by adjoining a unit to  $A$ . In this procedure, a linear functional  $\varphi$  may be extended to a linear functional  $\tilde{\varphi}$  by defining  $\tilde{\varphi}(1) = 1$ . It is easily shown that  $\varphi$  is an approximately Jordan if and only if  $\tilde{\varphi}$  is. In some cases, an algebra which lacks a unit may have an approximate identity. A net  $\{E_\alpha\}$  of elements  $E_\alpha \in A$  is called an approximate identity for  $A$  if for each  $x \in A$  there is a subset  $E_\beta$  such that  $\lim_{\beta} (E_\beta x) = x$ .

**Proposition 2.9:** suppose that  $\varphi$  is an approximately Jordan map with  $0 < \delta < \frac{1}{4k^2}$  for a given  $k > 0$  on a Banach algebra  $A$ . Let  $A$  have an approximate identity  $\{E_\alpha\}$  with  $\|E_\alpha\| \leq k$  for all  $\alpha$ . Then either  $\limsup_{\alpha} |1 - \varphi(E_\alpha)| < 2\delta k^2$  and  $\|\varphi\| > k^{-1} - 2k\delta$  or else  $\limsup_{\alpha} |\varphi(E_\alpha)| < 2\delta k^2$  and  $\frac{1 - \sqrt{1+4\delta k^2}}{2} \leq \|\varphi\| \leq \frac{1 + \sqrt{1+4\delta k^2}}{2}$ .

*Proof:* Let  $L$  be the set of points of accumulation of the set  $|\varphi(E_\alpha)|$  and  $l \in L$ . By hypothesis  $|\varphi(E_\alpha^2) - \varphi(E_\alpha)^2| \leq \delta \|E_\alpha\|^2 < \delta k^2$ . If we take limits on suitable subnets for  $\alpha$ , we get

$$|\varphi(E_\alpha) - l^2| < \delta k^2 \quad \text{and} \quad |l - l^2| < \delta k^2.$$

By solving the inequality  $\delta k^2 - |l - l^2| > 0$  we have

$$|l| < \frac{1 - \sqrt{1 - 4\delta k^2}}{2} \quad \text{or} \quad |l| > \frac{1 + \sqrt{1 - 4\delta k^2}}{2} > 1/2.$$

So we have

$$|l| < \frac{1 - \sqrt{1 - 4\delta k^2}}{2} < 2\delta k^2 \quad \text{or} \quad |1 - l| < 2\delta k^2.$$

The first case gives

$$\limsup_{\alpha} |1 - \varphi(E_\alpha)| < 2\delta k^2,$$

so that

$$k\|\varphi\| \geq \limsup_{\alpha} |\varphi(E_\alpha)| > 1 - 2\delta k^2.$$

The second case gives

$$\limsup_{\alpha} |\varphi(E_\alpha)| < 2\delta k^2.$$

In the inequality

$$|\varphi((E_\alpha b)^2) - \varphi(E_\alpha b)^2| \leq \delta \|E_\alpha b\|^2 \leq \delta k^2 \|b\|^2,$$

we take limit in a suitable subnet and obtain

$$|\varphi(bE_\alpha b) - \varphi(b)^2| \leq \delta k^2 \|b\|^2$$

and again by taking limit we deduce

$$|\varphi(b^2) - \varphi(b)^2| \leq \delta k^2 \|b\|^2,$$

for all  $b \in A$ . So

$$|\varphi(b)|^2 \leq (\delta k^2 + \|\varphi\|) \|b\|^2.$$

Hence

$$\|\varphi\|^2 - \|\varphi\| - \delta k^2 \leq 0.$$

So we have

$$\frac{1 - \sqrt{1 + 4\delta k^2}}{2} \leq \|\varphi\| \leq \frac{1 + \sqrt{1 + 4\delta k^2}}{2}. \quad \blacksquare$$

**corollary 2.10:** suppose that  $\varphi$  is an approximately Jordan map with  $0 < \delta < 1/4$  on a Banach algebra  $A$  and let  $A$  has a unit. Then either  $|1 - \varphi(1)| < 2\delta$  and  $\|\varphi\| > 1 - 2\delta$  or else  $\|\varphi\| \leq \frac{1 + \sqrt{1+4\delta}}{2}$ .

*Proof:* This corollary is a special case of proposition (2.9). ■

### III. STABILITY OF APPROXIMATELY JORDAN MAPPINGS

We start first point of this view by following Theorem about the stability of approximately Jordan mappings when we restrict  $\delta$  in the definition of approximately Jordan maps. this Theorem shows that for small  $\delta \geq 0$  we can find an approximately Jordan functional which are close to a  $\delta$ -Jordan functional.

**Theorem 3.1:** Suppose that  $A$  has a unit element and  $\varphi$  be a  $\delta$ -Jordan functional on a Banach algebra  $A$  with  $0 < \delta < 1/4$  and  $\|\varphi\| > \frac{1 + \sqrt{1+4\delta}}{2}$ . Then for the functionals  $\psi_1 = \frac{\varphi}{\|\varphi\|}$  and  $\psi_2 = \frac{\varphi}{\varphi(1)}$  we have  $\|\psi_1\| = 1$ ,  $\psi_2(1) = 1$  and they are approximately Jordan functionals. Specially  $\|\varphi - \psi_1(\psi_2)\| \leq \frac{2\delta(1+\delta)}{1-2\delta}$ .

*Proof:* By hypothesis,  $\varphi$  satisfies the first alternative of Corollary (2.10), so that  $\|\varphi\| > 1 - 2\delta$ . Hence,  $\frac{1 - \|\varphi\|}{\|\varphi\|} < \frac{2\delta}{1 - 2\delta}$ . Now by using Theorem (2.6) we obtain

$$\|\psi_1 - \varphi\| \leq \|\varphi\| \frac{2\delta}{1 - 2\delta} \leq \frac{2\delta(1+\delta)}{1 - 2\delta}.$$

Next apply proposition (2.7) with  $\lambda + 1 = \|\varphi\|^{-1}$ , so that  $\lambda = \|\varphi\|^{-1} - 1 < \frac{2\delta}{1-2\delta}$ . By proposition (2.7),  $\psi_1$  is an approximately Jordan map.

The argument for  $\psi_2$  is similar. ■

Our next Theorem shows that near a given  $\delta$ -Jordan functional  $\varphi$ , there is an approximately Jordan functional  $\psi$  which satisfies both  $\|\psi\| = 1$  and  $\psi(1) = 1$ .

**Theorem 3.2:** Let the Banach algebra  $A$  has a unit,  $\|1\| = 1$  and  $0 < \delta < 1$ . Suppose that  $\varphi$  is a  $\delta$ -approximate Jordan bounded functional with  $\varphi(1) = 1$ . Then there exists a bounded linear functional  $\psi$  such that  $\|\psi\| = 1$ ,  $\psi(1) = 1$  and  $\|\varphi - \psi\| \leq 2 + \delta$ .

*Proof:* For any linear functional  $f$  we define

$$W = \{f(a); \|f\| = f(1)\}$$

and

$$D = \{f; \|f\| = f(1)\}.$$

Given  $a, b \in A$  and  $\mu \in R$ , suppose that  $Re z < \mu$  for all  $z \in W$  and  $\lambda \in C$  with  $Re \lambda > \mu$ . Take  $a \in A$  with  $\|a\| = 1$  and  $f$  a linear functional such that  $\|f\| = 1 = f(a)$ . Then  $g : c \mapsto f(ac)$  defines a linear functional such that  $\|g\| = g(1)$  and we have

$$\|a(\lambda 1 - b)\| \geq |f(a(\lambda 1 - b))| = |\lambda - g(b)| \geq$$

$$Re(\lambda - g(b)) \geq Re \lambda - \mu.$$

So

$$\begin{aligned} Re \lambda - \mu &\leq \|a(\lambda 1 - b)\| \\ &\leq \|a\|(\|\lambda 1 - b\|) \leq \|a\|(\|\lambda\| + \|b\|) \leq \|\lambda\| + \|b\|. \end{aligned}$$

We may take  $\lambda = \varphi(b)$ , and by using Theorem (2.6) we have

$$Re \varphi(b) - \mu \leq |\varphi(b)| + \|b\| \leq (\|\varphi\| + 1)\|b\| \leq (2 + \delta)\|b\|. \quad (1)$$

Obviously, the above inequality also holds when  $Re \varphi(b) \leq \mu$ .

Put  $\eta = 2 + \delta$  and let  $B^*$  denotes the closed unit ball in the set of linear functionals. To prove the Theorem, we need to show that the set  $D \cap (\varphi + \eta B^*)$  is not empty. Suppose on the contrary that this set is empty. Then by applying [1] (Proposition 4, sect. 5, No. 3, Chap II, P.84.) in the  $w^*$ -topology, we may find a hyperplane strongly separating the sets  $D$  and  $\varphi + \eta B^*$ , both of which are convex and compact in the  $w^*$ -topology. Therefore there exists an element  $a \in A$ , which we may assume  $\|a\| = 1$ , and a real number  $\mu$  such that  $Re f(a) < \mu$ , for all  $f \in D$  and  $Re f(a) > \mu$  for all  $f \in \varphi + \eta B^*$ . Now take  $g \in B^*$  with  $g(a) = 1$  and put  $f = \varphi - \eta g$ , so that  $Re \varphi(a) - \eta = Re f(a) > \mu$ . Hence  $Re \varphi(a) - \mu > \eta = (2 + \delta)\|a\|$ . This contradicts inequality (1), so we have shown that the set  $D \cap (\varphi + \eta B^*)$  is not empty. ■

**Theorem 3.3:** Let  $\delta \geq 0$ ,  $q$  an arbitrarily real number,  $A$  a Banach algebra, and  $f : A \rightarrow C$  be a linear complex-valued function such that

$$|f(a^2) - f(a)^2| \leq \delta \|a\|^{2q}$$

for all  $a \in A$ . Put  $\beta = \frac{1 + \sqrt{1 + 4\delta}}{2}$ . Then either  $|f(a)| \leq \beta \|a\|^q$  for all  $a \in A$  or  $f(a^2) = f(a)^2$  for all  $a \in A$ .

*Proof:* Observe that  $\beta^2 - \beta = \delta$  and  $\beta > 1$ . Assume that for some element  $a \in A$ ,  $|f(a)| > \beta \|a\|^q$ , so we can suppose  $|f(a)| > \beta$ , with  $\|a\| = 1$ . Then  $|f(a)| = \beta + p$ , where  $p > 0$ . We have

$$|f(a^2)| \geq |f(a)^2 - (f(a)^2 - f(a^2))| \geq |f(a)^2| - |f(a)^2 - f(a^2)|,$$

so that

$$|f(a^2)| > |f(a)|^2 - \delta = (\beta + p)^2 - \delta = 2\beta p + p^2 + \beta > \beta + 2p.$$

By induction on  $n$  we prove that  $|f(a^{2^n})| > \beta + (n + 1)p$ .

Suppose that  $|f(a^{2^n})| > \beta + (n + 1)p$  is true for all  $n \in N$ . Then

$$\begin{aligned} |f(a^{2^{n+1}})| &= |f(a^{2^n})f(a^{2^n}) - (f(a^{2^n})f(a^{2^n}) - f(a^{2^n} \cdot a^{2^n}))| \\ &\geq |f(a^{2^n})|^2 - \delta. \end{aligned}$$

So that

$$\begin{aligned} |f(a^{2^{n+1}})| &\geq (\beta + (n + 1)p)^2 + \beta - \beta^2 \\ &= (2n + 2)\beta p + \beta + (n + 1)^2 p^2 \geq \beta + (n + 2)p. \end{aligned}$$

Now for any natural number  $m, n$ :

$$\begin{aligned} &2|f(a^n \cdot a^m) - f(a^n)f(a^m)| \\ &\leq |f((a^n + a^m)^2) - f(a^n + a^m)^2| \\ &+ |f((a^n)^2) - f(a^n)^2| + |f((a^m)^2) - f(a^m)^2| \\ &\leq \delta(\|a^n + a^m\|^2 + \|a^n\|^2 + \|a^m\|^2) \\ &\leq \delta(\|a^n\| + \|a^m\|)^2 + \|a^n\|^2 + \|a^m\|^2 \\ &\leq 2\delta(\|a^n\|^2 + \|a^n\| \cdot \|a^m\| + \|a^m\|^2) \\ &\leq 2\delta(\|a\|^{2n} + \|a\|^{m+n} + \|a\|^{2m}) \leq 6\delta, \end{aligned}$$

so

$$|f(a^n \cdot a^m) - f(a^n)f(a^m)| \leq 3\delta.$$

Now we have

$$\begin{aligned} &|f(a^2) - f(a)^2| \\ &\leq \frac{1}{|f(a^{2^n})|} (|f(a^2)f(a^{2^n}) - f(a^2 \cdot a^{2^n})| \\ &+ |f(a^2 \cdot a^{2^n}) - f(a)f(a \cdot a^{2^n})| + |f(a)f(a \cdot a^{2^n}) - f(a)^2 f(a^{2^n})|) \\ &\leq \frac{1}{|f(a^{2^n})|} (3\delta + 3\delta + 3\delta |f(a)|) \\ &\leq \frac{3\delta(2 + |f(a)|)}{\beta + (n + 1)p}. \end{aligned}$$

By letting  $n$  tend to infinity we have  $f(a^2) = f(a)^2$ , for all  $a \in A$  with  $\|a\| = 1$ . Now for arbitrarily element  $x \in A$  put  $a = \frac{x}{\|x\|}$ . we have  $f(a^2) = f(a)^2$ . So

$$\begin{aligned} f(x^2) &= f(a^2\|x\|^2) = \|x\|^2 f(a^2) \\ &= \|x\|^2 f(a)^2 = (\|x\|f(a))^2 = f(\|x\|a)^2 = f(x)^2. \end{aligned}$$

■

**Theorem 3.4:** Let  $A$  be a Banach algebra and  $\varphi$  a functional on  $A$  such that

$$|\varphi(x^2) - \varphi(x)^2| \leq \delta\|x\|^2$$

and

$$|\varphi(x+y) - \varphi(x) - \varphi(y)| \leq \delta(\|x\| + \|y\|)$$

for some  $\delta \geq 0$ , then  $\varphi$  is an additive map, or

$$|\varphi(x)| \leq \frac{1 + \sqrt{1 + 4\delta}}{2} \|x\|, \quad (2)$$

for all  $x \in A$ .

*Proof:* Suppose that  $\varphi$  is not an additive map. So by hypothesis there exist  $a, b \in A$  such that  $\varphi(a+b) \neq \varphi(a) + \varphi(b)$ . For each  $x \in A$ , it follows from Theorem (2.6) that

$$\|\varphi(x)\| |\varphi(a+b) - \varphi(a) - \varphi(b)| \leq (1 + \delta)\delta(\|a\| + \|b\|).$$

Which implies that

$$|\varphi(x)| \leq \frac{(1 + \delta)\delta(\|a\| + \|b\|)}{|\varphi(a+b) - \varphi(a) - \varphi(b)|} \|x\|$$

for all  $x \in A - \{0\}$ .

We have  $|\varphi(x)| \leq k\|x\|$  for some constant  $0 \leq k < \infty$  for every  $x \in A - \{0\}$ . Set  $m = \sup_{x \in A - \{0\}} \frac{|\varphi(x)|}{\|x\|}$ . Then  $m \leq k < \infty$  by assumption. It follows that

$$|\varphi(x)|^2 \leq \delta\|x\|^2 + |\varphi(x^2)| \leq \delta\|x\|^2 + m\|x^2\| \leq (\delta + m)\|x\|,$$

for all  $x \in A$ .

This implies that  $m^2 \leq \delta + m$ . So by solving the inequality we have  $m \leq \frac{1 + \sqrt{1 + 4\delta}}{2}$  and the proof is complete. ■

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