World Academy of Science, Engineering and Technology
International Journal of Physical and Mathematical Sciences
Vol:5, No:3, 2011

Optimal Control of Piezo-Thermo-Elastic Beams

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Abstract—This paper presents the vibrations suppression of a thermoelastic beam subject to sudden heat input by a single piezoelectric actuator. An optimization problem is formulated as the minimization of a quadratic functional in terms of displacement and velocity at a given time and with the least control effort. The solution method is based on a combination of modal expansion and variational approaches. The modal expansion approach is used to convert the optimal control of distributed parameter system into the optimal control of lumped parameter system. By utilizing the variational approach, an explicit optimal control law is derived and the determination of the corresponding displacement and velocity is reduced to solving a set of ordinary differential equations.

Keywords—Modal expansion approach, optimal control, thermoelastic beam, variational approach.

I. Introduction

Many vibration suppressions are required to function across a variety of different temperatures, yet most ignores the effects of temperature changes on structures [1]. Indeed, if the temperature varies rapidly, vibration may occur, which can affect the dynamics and stability of the structures. Therefore, thermally induced vibration is an important concern for the design of theses structures. Active structural control provides an effective means of damping the vibrations of the structure subject to dynamic loads and external disturbances.

The change in temperature experienced by a spacecraft emerging from a shadow entering the sunlight is an example of a dynamic load experienced by a structure. The sudden exposure to heat leads to thermal vibrations, which need to be damped. Active control employed by dynamic actuators will lead to the desired damping to improve the safety and performance of the structure.

Active vibrations of control of structural elements subject to mechanical dynamic loads has been studied extensively, see for example [2], [3], [4]. In comparison to the vibration control of structures under mechanical loads, there have been few studies on structures subject to thermally induced vibrations, see for example [5]-[7].

The present study is aimed at solving the optimal control problem for an elastic beam undergoing thermally induced vibrations due to a time dependent heat input. The control objective is the suppression of vibrations and this is expressed by minimizing a performance index given in terms of displacement and velocity at a finite time. The vibrations are suppressed by piezoelectric actuator. The physical problem corresponds to the suppression of vibrations of a structure suddenly entering into the daylight zone.

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The solution method is based on a combination of modal expansion and variational approaches. The modal expansion approach is used to convert the optimal control of distributed parameter system (DPS) into the optimal control of lumped parameter system (LPS). By utilizing the variational approach, an explicit optimal control law is derived and the determination of the corresponding displacement and velocity is reduced to solving a set of ordinary differential equations.

II. Optimal Control Problem

The equation governing the motion of the nondimensional beam [5] (see Figure 1) is given by

$$B^4 w_{xxxx} + w_{tt} = c_0 v(t) \tfrac{d^2}{dx^2}\left[H(x-x_2) - H(x-x_1)\right], \quad x \in \Omega = (0,1). \tag{1}$$

where $w(x,t)$ represents the transverse displacement of the point (x,t), $v(t)$ is the applied voltage to the piezoelectic patch, c_0 represents the mechanical-electrical coupling coefficient between the actuator and the beam, $H(.)$ represents the unit heaviside function, (x_1, x_2) is the lower-left and upper-right coordinates of the actuator, and B is given in [5].

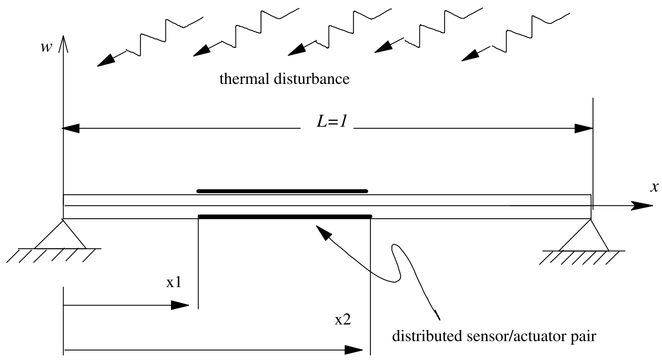

Fig. 1 A smart beam with actuators under the influence of thermal disturbances.

The boundary conditions are

$$\begin{aligned} w(0,t) = w(1,t) = 0 \\ w_{xx}(0,t) = w_{xx}(1,t) = -m_T(t) \end{aligned} \tag{2}$$

and the initial conditions

$$w(x,0) = w_t(x,0) = 0 \tag{3}$$

in which $m_T(t)$ is the thermal moment given by [5]

$$m_T(t) = \frac{1}{4}\left[\frac{\pi^4}{96} - \sum_{j=1,3,\cdots}^{\infty} \frac{e^{-j\pi^2 t}}{j^4}\right]. \tag{4}$$

Let the admissible control set be

$$A_{\text{ad}} = \left\{v(t) : v(t) \in L^2(\Omega_t)\right\}$$

where $\Omega_t = (0, t_f)$ in which t_f is a prescribed terminal time. Consider the performance index given by

$$J[v(t)] = \int_\Omega \left[\mu_1 w^2(x,t_f) + \mu_2 w_t^2(x,t_f)\right] d + \int_{\Omega_t} \mu_3 v^2(t) dt. \tag{5}$$

where μ_1, μ_2, and μ_3 are nonnegative weighting constants satisfying $\mu_1 + \mu_2 \geq 0$ and $\mu_3 > 0$. The last integral on the right hand side of (5) is penalty term on control energy.

The optimal control problem is now stated as follows:

Determine the optimal control function $\mathbf{v^*(t)} \in \mathbf{A}_{\text{ad}}$ so that

$$J[v^*(t)] = \min_{v(t)\in A_{\text{ad}}} J[v(t)] \tag{6}$$

and subject to equations (1)-(3).

III. Control Problem in Modal Space

First, we consider the transformation

$$w = u - \frac{1}{2} m_T(t)(x^2 - x) \tag{7}$$

the partial differential equation (1) becomes

$$B^4 u_{xxxx} + u_{tt} = \frac{1}{2}\ddot{m}_T(t)(x^2 - x) + c_0 v(t)\alpha(x, x_1, x_2) \tag{8}$$

where $\alpha(x, x_1, x_2) = H(x - x_2) - H(x - x_1)$. The new homogeneous boundary conditions are

$$\begin{array}{l} u(0,t) = y(1,t) = 0 \\ u_{xx}(0,t) = u_{xx}(1,t) = 0 \end{array} \tag{9}$$

and the new initial conditions are

$$\begin{array}{l} u(x,0) = 0 \\ u_t(x,0) = \frac{1}{2}\dot{m}_T(0^+)(x^2 - x) \end{array} \tag{10}$$

where $\dot{m}_T(0^+) = \lim_{\tau\to 0, \tau>0} \dot{m}_T(0^+)$.

The performance index becomes

$$J[v(t)] = \int_\Omega \left[\mu_1 u^2(x,t_f) + \mu_2 u_t^2(x,t_f)\right] dx + \int_{\Omega_t} \mu_3 v^2(t) dt \tag{11}$$

The distributed parameter system (8) can be transformed into a modal lumped parameter problem by using the expansion

$$u(x,t) = \sum_{n=1}^{N} z_n(t) \sin(n\pi x) \tag{12}$$

Using expansion (12), it can be shown that $z_n(t)$ satisfies

$$\begin{array}{l} \ddot{z}_n(t) + (n\pi B)^4 z_n(t) = 2\ddot{m}_T(t)\frac{(-1+(-1)^n)}{(n\pi)^3} \\ +2c_0 v(t)\pi\left[\cos(\pi x_2) - \cos(\pi x_1)\right] \end{array} \tag{13}$$

with initial conditions

$$\begin{array}{l} z_n(0) = 0 \\ z_n(0) = 2\ddot{m}_T(0)\frac{(-1+(-1)^n)}{(n\pi)^3} \end{array} . \tag{14}$$

The solution of system (13)-(14) is given by

$$\begin{array}{l} z_n(t) = \frac{2(-1+(-1)^n)\dot{m}_T(0^+)}{(n\pi)^3\lambda_n}\sin(\lambda_n t) \\ + \frac{2(-1+(-1)^n)}{(n\pi)^3\lambda_n}\int_0^t \sin\lambda_n(t-\tau)\ddot{m}_T(t)\ d\tau \\ + \frac{2\pi c_0}{\lambda_n}\left[\cos(\pi x_2) - \cos(\pi x_1)\right]\int_0^t \sin\lambda_n(t-\tau)v(\tau)\ d\tau \end{array} \tag{15}$$

In a simpler form, the solution (15) can be expressed in the form

$$z_n(t) = z^h(t) + z^p(t) \tag{16}$$

where

$$\begin{array}{l} z^h(t) = A_n \sin(\lambda_n t) + B_n \int_0^t \sin\lambda_n(t-\tau)\ddot{m}_T(t)\ d\tau, \\ z^p(t) = C_n \int_0^t \sin\lambda_n(t-\tau)v(\tau)\ d\tau, \end{array} \tag{17}$$

in which

$$\begin{array}{l} A_n = \dfrac{2\left(-1+(-1)^n\right)\dot{m}_T(0^+)}{(n\pi)^3\lambda_n}, \\ B_n = \dfrac{2\left(-1+(-1)^n\right)}{(n\pi)^3\lambda_n}, \\ C_n = \dfrac{2\pi c_0}{\lambda_n}\left[\cos(\pi x_2) - \cos(\pi x_1)\right]. \end{array} \tag{18}$$

By expansion (12), the performance index (11) becomes

$$J_N[v] = \sum_{n=1}^{N}\left[\mu_1 z_n^2(t_f) + \mu_2\left(\frac{d}{dt}z_n(t_f)\right)^2\right] + \int_{\Omega_t} \mu_3 v^2(t) dt. \tag{19}$$

IV. Control Characterization

The optimal control $v^*(t) \in A_{\text{Ad}}$ is determined such that $J_N[v(t)]$ is minimum subject to the constraints of the modal equations of motions (13)-(14). We now proceed by taking the first variation of J_N with respect to $v(t)$. The necessary condition for the control $v(t)$ to be optimal is that $\delta_v J_N[v] = 0$, thus

$$\delta_v J_N[v] = \sum_{n=1}^{N}\left[2\mu_1 z_n(t_f)\ \delta z_n(t_f) + 2\mu_2 \dot{z}_n(t_f)\ \delta\dot{z}_n(t_f)\right] + 2\mu_3 \int_{\Omega_t} v(t)\ \delta v(t) dt = 0 \tag{20}$$

This implies that

$$\begin{array}{l} 2\mu_1\left[z_n^h(t_f) + z_n^p(t_f)\right]\ \sin\lambda_n(t_f-\tau) \\ +2\mu_2\left[\dot{z}_n^h(t_f) + \dot{z}_n^p(t_f)\right]\ \cos\lambda_n(t_f-\tau) \\ +2\mu_3 v(\tau) = 0. \end{array} \tag{21}$$

Thus the minimization of J_N leads to a degenerate system of integral equations (21) that can be solved for $v(t)$ in a closed form. The system of integral equations can be transformed into a system of linear algebraic equations of its twice size by multiplying equation (21) first by $\sin\lambda_n(t-\tau)$ and then by $\cos\lambda_n(t-\tau)$ and integrating over the time domain Ω_t to obtain the following linear algebraic equations

$$\begin{aligned}&2\mu_1\left[z_n^h(t_f)+C_n x\right]ss+2\mu_2\left[\dot z_n^h(t_f)+C_n\lambda_n y\right]cs\\&+2\mu_3 x=0\end{aligned}\tag{22}$$

and

$$\begin{aligned}&2\mu_1\left[z_n^h(t_f)+C_n x\right]cs+2\mu_2\left[\dot z_n^h(t_f)+C_n\lambda_n y\right]cc\\&+2\mu_3 y=0\end{aligned}\tag{23}$$

where

$$x=\int_0^{t_f}\sin\lambda_n(t-\tau)\ v(\tau)\ d\tau,$$
$$y=\int_0^{t_f}\cos\lambda_n(t-\tau)\ v(\tau)\ d\tau,$$
$$ss=\int_0^{t_f}\sin^2\lambda_n(t-\tau)\ \ d\tau,$$
$$cs=\int_0^{t_f}\sin\lambda_n(t-\tau)\ \cos\lambda_n(t-\tau)\ d\tau,$$
$$cc=\int_0^{t_f}\cos^2\lambda_n(t-\tau)\ \ d\tau.$$

Equations (22)-(23) can be written in the form

$$\begin{aligned}d_{11}x+d_{12}y&=e_1\\d_{21}x+d_{22}y&=e_2\end{aligned}\tag{24}$$

where

$$\begin{aligned}d_{11}&=2\mu_1C_n\ ss+2\mu_3,\\d_{12}&=2\mu_2C_n\ \lambda_n\ cs,\\d_{21}&=2\mu_1C_n\ cs,\\d_{22}&=2\mu_2C_n\lambda_n\ cc+2\mu_3,\\e1&=-2\mu_1z_n^h(t_f)\ ss-2\mu_2\dot z_n^h(t_f)\ cs,\\e_2&=-2\mu_1z_n^h(t_f)\ cs-2\mu_2\dot z_n^h(t_f)\ cc.\end{aligned}$$

Solving the linear system (24) for x and y, we obtain

$$y=\frac{d_{21}e_1-d_{11}e_2}{d_{21}d_{12}-d_{11}d_{22}}$$
$$x=\frac{d_{22}e_1-d_{12}e_2}{d_{21}d_{12}-d_{11}d_{22}}$$

Now an optimal control $v^*(t)$ is determined from (21) and is given by

$$v^*(t)=\frac{-1}{2\mu_3}\left[\begin{array}{c}2\mu_1\left[z_n^h(t_f)+C_n x\right]\sin\lambda_n(t_f-\tau)\\+2\mu_2\left[\dot z_n^h(t_f)+C_n\lambda_n y\right]\cos\lambda_n(t_f-\tau)\end{array}\right]\tag{25}$$

V. Conclusion

A simply supported beam with a piezoelectric actuator subject to a thermal disturbance has been presented. The piezoelectric actuator is implemented to actively suppress the motion caused by thermal disturbances. The basic problem of interest is to minimize a quadratic functional in terms of displacement and velocity within a prescribed time and with the least control effort. The solution of the problem is obtained by means of eigenfunction expansion and variational approach.

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