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Certain Estimates of Oscillatory Integrals and Extrapolation

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Abstract—In this paper we study the boundedness properties of certain oscillatory integrals with polynomial phase. We obtain sharp estimates for these oscillatory integrals. By the virtue of these estimates and extrapolation we obtain L^p boundedness for these oscillatory integrals under rather weak size conditions on the kernel function.

Keywords—Fourier transform, oscillatory integrals, Orlicz spaces, Block spaces, Extrapolation, L^p boundedness.

I. Introduction and statement of results

THROUGHOUT this paper, let $\mathbf{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma$. Also, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n \backslash\{0\}$ and p' denote the exponent conjugate to p, that is $1/p + 1/p' = 1$.

Let Ω be an integrable function on $\mathbf{S}^{n-1}$ and $\mathcal{P}(d)$ denote the set of all polynomials on $\mathbf{R}$ which have real coefficients and degrees not exceeding d. For $P \in \mathcal{P}(d)$ and $\Omega \in L^1(\mathbf{S}^{n-1})$, define the oscillatory integral $S_{\Omega,P}$ by

$$S_{\Omega,P}(\xi) = \int_{\mathbf{S}^{n-1}} e^{iP(\xi\cdot y)}\Omega(y)d\sigma(y),$$

where $\xi \in \mathbf{R}^n$. When $P(t) = t$ and $d\mu_{_\Omega} = \Omega d\sigma$, we notice that $S_{\Omega,P}(\xi) = \widehat{d\mu_{_\Omega}}(\xi)$ which is the Fourier transform of the measure $d\mu_{_\Omega}$ that is supported on the unit sphere with density Ω. The behavior of $S_{\Omega,P}(\xi)$ as $|\xi| \to \infty$ has been studied extensively in connection with various problems in harmonic analysis. For example, if Ω is sufficiently smooth, then $\widehat{d\mu_{_\Omega}}(\xi) = O(|\xi|^{-\frac{n-1}{2}})$ as $|\xi| \to \infty$. It turns out if the density Ω is merely in $L^q(\mathbf{S}^{n-1})$ for some $q > 1$, then there is still an average decrease of $\widehat{d\mu_{_\Omega}}(\xi)$ at infinity along any ray emanating from the origin. More precisely, the following result can be found in [7] and [16].

Theorem A. *Suppose $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then*

$$\left(\frac{1}{R}\int_0^R \left|\widehat{d\mu_{_\Omega}}(t\xi)\right|^2 dt\right)^{1/2} \leq C_{_\varepsilon}(R|\xi|)^{-\varepsilon}\|\Omega\|_{L^q(\mathbf{S}^{n-1})} \tag{1.1}$$

for all $R > 0$, $\xi \in \mathbf{R}^n$ and for any positive ε satisfying $\varepsilon < A(q) = \frac{1}{4}(1-q^{-1})$. The constant $C_{_\varepsilon}$ is independent of R and ξ.

Since $A(q) = 0$ when $q = 1$, the analogue of (1.1) is no longer meaningful. So it would be interesting to have a substitute for (1.1). The investigation of such a problem has attracted the attention of many authors. For relevant results one may consult [9], [10], [1], among others. We mention here that such estimates as in (1.1) above, (1.2) in [10], (2) in [9] and (1.3)–(1.4) in Theorem B below were instrumental in obtaining the L^2 boundedness of certain classes of singular integral operators and Marcinkiewicz integral operators (see [7], [13], [6], [1], [16]). Our main focus will be on the following result from [1].

Theorem B. *Let* $P \in \mathcal{P}(d)$ for some d and let $1 \leq \lambda < \infty$. *Let* Ω be a function on $\mathbf{S}^{n-1}$ satisfying

$$\int_{\mathbf{S}^{n-1}} \Omega(y)d\sigma(y). \tag{1.2}$$

Then there exists a constant C depends on n, p, d, but it is independent of Ω, *ξ and the coefficients of the polynomial P* such that

(a) If $\Omega \in L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})$, *then there exists a constant C independent such that*

$$\left(\int_0^\infty |S_{\Omega,P}(t\xi)|^\lambda \frac{dt}{t}\right)^{1/\lambda} \leq C\left(1 + \|\Omega\|_{L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})}\right), \tag{1.3}$$

(b) If $\Omega \in B_q^{(0,\frac{1}{\lambda}-1)}\left(\mathbf{S}^{n-1}\right)$ *for some $q > 1$, then there exists a constant C independent Ω, ξ and the coefficients of the polynomial P such that*

$$\left(\int_0^\infty |S_{\Omega,P}(t\xi)|^\lambda \frac{dt}{t}\right)^{1/\lambda} \leq C\left(1 + \|\Omega\|_{B_q^{(0,\frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})}\right). \tag{1.4}$$

We notice that the constant C depends on the degree d of the polynomial P and it is independent of its coefficients. One of the main issues of concern in this paper is to determine the the optimal dependence of the constant C on the parameter d. Also, we present a unified approach different from the one employed in [1]. This approach will mainly rely on obtaining some a delicate estimate and the results in Theorem B are obtained by applying an extrapolation argument. Let us now state our main results.

Theorem 1.1. *Let $P \in \mathcal{P}(d)$ for some d and let $1 \leq \lambda < \infty$. Let Ω be a function on $\mathbf{S}^{n-1}$ satisfying (1.2) with $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq 2$. Then there exists a constant C independent Ω, ξ and the coefficients of the polynomial P such that*

$$\left(\int_0^\infty |S_{\Omega,P}(t\xi)|^\lambda \frac{dt}{t}\right)^{1/\lambda} \leq C(q-1)^{-\frac{1}{\lambda}}(\log d + 1)\|\Omega\|_q. \tag{1.5}$$

Moreover, for $P(t) = t$, the exponent $-\frac{1}{\lambda}$ is the best possible in the case $\lambda = 1$ or 2.

By the estimate *(1.5)* and extrapolation we get the following.

Theorem 1.2. *Let $P \in \mathcal{P}(d)$ for some d and let $1 \leq \lambda < \infty$. Let Ω be a function on $\mathbf{S}^{n-1}$ satisfying (1.2). Then there exists a constant C depends on n, p but it is independent of Ω, ξ and the coefficients of the polynomial P such that*

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(a) *If* $\Omega \in L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})$*, then there exists a constant* C *independent such that*

$$\left(\int_0^\infty |S_{\Omega,P}(t\xi)|^\lambda \frac{dt}{t}\right)^{1/\lambda} \leq C(\log d+1)\left(1+\|\Omega\|_{L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})}\right), \quad (1.6)$$

(b) *If* $\Omega \in B_q^{(0,\frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})$ *for some* $q>1$*,then there exists a constant* C *independent* Ω*,* ξ *and the coefficients of the polynomial* P *such that*

$$\left(\int_0^\infty |S_{\Omega,P}(t\xi)|^\lambda \frac{dt}{t}\right)^{1/\lambda} \leq C(\log d+1)\left(1+\|\Omega\|_{B_q^{(0,\frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})}\right). \quad (1.7)$$

It is worth mentioning that in addition to the condition $\Omega \in H^1(\mathbf{S}^{n-1})$, the conditions $\Omega \in B_q^{(0,\upsilon-1)}(\mathbf{S}^{n-1})$ and $\Omega \in L(\log^+ L)^\upsilon(\mathbf{S}^{n-1})$ (for $\upsilon>0$) had received the most amount of attention with respect to the study of the L^p mapping properties of singular integral operators, maximal integral operators and Marcinkiewicz integral operators.

We remark that on $\mathbf{S}^{n-1}$, for any $q>1$, the following inclusions hold and are proper:

$$L^q(\mathbf{S}^{n-1}) \subset L(\log L)(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}) \subset L^1(\mathbf{S}^{n-1}), \quad (1.8)$$

$$\bigcup_{r>1} L^r(\mathbf{S}^{n-1}) \subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) \text{ for any } -1<\upsilon, \quad (1.9)$$

$$L(\log L)^\beta(\mathbf{S}^{n-1}) \subset L(\log L)^\alpha(\mathbf{S}^{n-1}) \text{ if } 0<\alpha<\beta, \quad (1.10)$$

$$L(\log L)^\alpha(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}) \text{ for all } \alpha \geq 1, \text{ while} \quad (1.11)$$

$$L(\log^+ L)^\alpha(\mathbf{S}^{n-1}) \nsubseteq H^1(\mathbf{S}^{n-1}) \nsubseteq L(\log L)^\alpha(\mathbf{S}^{n-1}) \quad (1.12)$$

for all $0<\alpha<1$. With regard to the relationship between $B_q^{(0,\upsilon-1)}(\mathbf{S}^{n-1})$ and $L(\log^+ L)^\upsilon(\mathbf{S}^{n-1})$ (for $\upsilon>0$) remains open.

We remark that when ($q>1$) the condition $\Omega \in L(\log L)^{1/\lambda}(\mathbf{S}^{n-1}) \cup B_q^{(0,\frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})$ is replaced the weaker condition $\Omega \in L^1(\mathbf{S}^{n-1})$, the above statements in (1.6) and (1.7) become false. Also, when $L(\log^+ L)^{1/\lambda}(\mathbf{S}^{n-1})$ (or $B_q^{(0,\frac{1}{\lambda}-1)}(\mathbf{S}^{n-1})$) (for a given $1 \leq \lambda < \infty$) is replaced by $L^q(\mathbf{S}^{n-1})$ for some $q>1$, it follows from (1.8)–(1.9) that the inequalities (1.6)–(1.7) remain valid.

Throughout the rest of the paper, we always use the letter C to denote a positive constant that may vary at each occurrence but it is independent of the essential variables.

II. Definitions and lemmas

Let $L(\log L)^\alpha(\mathbf{S}^{n-1})$ ($\alpha>0$) denote the class of all functions Ω which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(\mathbf{S}^{n-1})} = \int_{\mathbf{S}^{n-1}} |\Omega(x)| \log^\alpha(2+|\Omega(x)|)d\sigma(x) < \infty.$$

Now, let us recall the definition of the block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$. This space was introduced by Jiang and Lu (see [12]) and can be traced back to M. H. Taibleson and G. Weiss on their work on the convergence of the Fourier series in connection with the developments of the real Hardy spaces [17]. The space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ is defined as follows: A q-block on $\mathbf{S}^{n-1}$ is an L^q ($1<q\leq\infty$) function $b(x)$ that satisfies the following two conditions: (i) supp$(b) \subset I$; (ii) $\|b\|_{L^q} \leq |I|^{-1/q'}$, where $|I| = \sigma(I)$, and $I = B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : |x'-x_0'| < \theta_0\}$ is a cap on $\mathbf{S}^{n-1}$ for some $x_0' \in \mathbf{S}^{n-1}$ and $\theta_0 \in (0,1]$. The block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ is defined by

$$B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) = \left\{\Omega \in L^1(\mathbf{S}^{n-1})\colon \Omega = \sum_{\mu=1}^\infty \lambda_\mu b_\mu, M_q^{(0,\upsilon)}(\{\lambda_\mu\}) < \infty\right\},$$

where each λ_μ is a complex number; each b_μ is a $q-$block supported on a cap I_μ on $\mathbf{S}^{n-1}$, $\upsilon>-1$ and

$$M_q^{(0,\upsilon)}(\{\lambda_\mu\}) = \sum_{\mu=1}^\infty |\lambda_\mu| \left\{1+\log^{(\upsilon+1)}(|I_\mu|^{-1})\right\}.$$

Let $\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})} = \inf\{M_q^{(0,\upsilon)}(\{\lambda_\mu\})\colon \Omega = \sum_{\mu=1}^\infty \lambda_\mu b_\mu$ and each b_μ is a q-block function supported on a cap I_μ on $\mathbf{S}^{n-1}\}$. Then $\|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})}$ is a norm on the space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ and $(B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}), \|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})})$ is a Banach space.

We need the following result from [4].

Lemma 2.2. Let $h(t) = b_0 + b_1 t + \cdots + b_d t^d$ be a real polynomial of degree at most d and let $\psi \in C^1[a,b]$. Then for any j_0 with $1 \leq j_0 \leq d$, there exists a positive constant C independent of a, b, the coefficients of $b_0, \cdots, b_d$ and also independent of d such that

$$\left|\int_a^b e^{ih(t)}\psi(t)dt\right| \leq C|b_{j_0}|^{-\frac{1}{d}}\left\{\sup_{a\leq t\leq b}|\psi(t)| + \int_a^b |\psi'(t)|\,dt\right\}$$

holds for $0<a<b\leq 1$.

III. Proof of main results

Proof of Theorem 1.1. Assume that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1<q\leq 2$ and satisfies (1.2). Before starting the proof we need some preparation. We may assume without loss of generality that $P(t)$ does not have a constant term. Write $P(t) = \sum_{s=1}^d a_s t^s$. Let Q be given by $Q(t) = \sum_{s=1}^{d/2} a_s t^s$. By a dilation in t we may assume, without loss of generality, that $\max_{\frac{d}{2}<j\leq d}|a_j| = 1$. Also, there is $\frac{d}{2} < j_0 \leq d$ so that $|a_{j_0}| = 1$. Let

$$A_d = A_d(\Omega,\xi,\lambda) = \sup_{\substack{0<\varepsilon<R,\\ P\in\mathcal{P}(d)}} |I_{\varepsilon,R}(P)|,$$

where

$$I_{\varepsilon,R}(\Omega,P,\lambda,\xi)=\left(\int_{\varepsilon}^{R}|S_{\Omega,P}(t\xi)|^{\lambda}\frac{dt}{t}\right)^{1/\lambda}.$$

We need to show that

$$A_d\leq C(\log d+1)(q-1)^{-1/\lambda}\|\Omega\|_{L^q(\mathbf{S}^{n-1})} \tag{3.1}$$

for some absolute positive constant C. We shall first prove (3.1) for the case $d=2^m$ for some integer $m\geq 0$ and then the general case will be an immediate consequence. Let $0<\varepsilon<R$ and $\xi\in\mathbf{R}^n$ be arbitrary. Without loss of generality we may assume that $\varepsilon<|\xi|^{-1}<R$. By the fact that $(a+b)^p\leq a^p+b^p$ if $a,b\geq 0$ and $0<p\leq 1$ we have

$$I_{\varepsilon,R}(\Omega,P,\lambda,\xi)\leq$$
$$\left(\int_{\varepsilon}^{|\xi|^{-1}}|S_{\Omega,P}(t\xi)|^{\lambda}\frac{dt}{t}\right)^{1/\lambda}+\left(\int_{|\xi|^{-1}}^{R}|S_{\Omega,P}(t\xi)|^{\lambda}\frac{dt}{t}\right)^{1/\lambda}$$
$$:=I^{(0)}_{\varepsilon,R}(\Omega,P,\lambda,\xi)+I^{(1)}_{\varepsilon,R}(\Omega,P,\lambda,\xi). \tag{3.2}$$

We start with $I^{(0)}_{\varepsilon,R}(\Omega,P,\lambda,\xi)$. By Minkowski's inequality we have

$$I^{(0)}_{\varepsilon,R}(\Omega,P,\lambda,\xi)\leq$$
$$\leq\left(\int_{\varepsilon}^{|\xi|^{-1}}|S_{\Omega,P}(t\xi)-S_{\Omega,Q}(t\xi)|^{\lambda}\frac{dt}{t}\right)^{1/\lambda}+I_0(\Omega,Q,\lambda,\xi).$$

Since $\deg(Q)\leq\frac{d}{2}$, by induction and generalized Minkowski's inequality we get

$$I^{(0)}_{\varepsilon,R}(\Omega,Q,\lambda,\xi)\leq$$
$$\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\left(\int_{\varepsilon}^{|\xi|^{-1}}\left(\sum_{\frac{d}{2}<j\leq d}|a_j|\,|\xi|^j\,t^j\right)^{\lambda}\frac{dt}{t}\right)^{1/\lambda}+A(\frac{d}{2})$$
$$\leq C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\sum_{\frac{d}{2}<j\leq d}|a_j|\,|\xi|^j\left(\int_0^{|\xi|^{-1}}t^{j\lambda-1}dt\right)^{1/\lambda}+A(\frac{d}{2})$$
$$\begin{aligned}&\leq C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\sum_{\frac{d}{2}<j\leq d}\frac{1}{j^{\lambda}}+A(\frac{d}{2})\\&\leq C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\sum_{\frac{d}{2}<j\leq d}\frac{1}{j}+A(\frac{d}{2})\\&\leq C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}+A(\frac{d}{2}).\end{aligned} \tag{3.3}$$

Now we turn to estimate $I^{(1)}_{\varepsilon,R}(\Omega,P,\lambda,\xi)$. Let $\theta=2^{q'}$. By a change of variable we have

$$I^{(1)}_{\varepsilon,R}(\Omega,P,\lambda,\xi)=\left(\int_1^{R|\xi|}|S_{\Omega,P}(t\xi')|^{\lambda}\frac{dt}{t}\right)^{1/\lambda}.$$

Since $R|\xi|>1$, there exists a unique $k_0\in\mathbf{Z}_+$ such that $\theta^{k_0-1}\leq R|\xi|<\theta^{k_0}$. Hence

$$I^{(1)}_{\varepsilon,R}(\Omega,P,\lambda,\xi)\leq\sup_{k_0\in\mathbf{Z}_+}\left(\int_{\theta^{k_0-1}}^{\theta^{k_0}}|S_{\Omega,P}(t\xi')|^{\lambda}\frac{dt}{t}\right)^{1/\lambda}+$$
$$\sup_{k_0\in\mathbf{Z}_+}\sum_{k=k_0+1}^{\infty}\left(\int_{\theta^{k-1}}^{\theta^k}|S_{\Omega,P}(t\xi')|^{\lambda}\frac{dt}{t}\right)^{1/\lambda}=J_1+J_2. \tag{3.4}$$

It is easy to see that

$$J_1\leq(\log\theta)^{\frac{1}{\lambda}}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}. \tag{3.5}$$

Therefore, it remains to estimate J_2. To this end, we proceed as follows. We claim that there exists a positive constant C independent of d,ξ, and k_0 such that

$$\left(\int_{\theta^{k-1}}^{\theta^k}|S_{\Omega,P}(t\xi')|^{\lambda}\frac{dt}{t}\right)^{1/\lambda}\leq C\log\theta)^{1/\gamma}\left(\theta^{kj_0}\right)^{-\frac{1}{2\lambda dq'}}. \tag{3.6}$$

We start proving (3.6) for the case $\lambda=2$. By a change of variable we get

$$\left(\int_{\theta^{k-1}}^{\theta^k}|S_{\Omega,P}(t\xi)|^2\frac{dt}{t}\right)^{1/2}=E(\xi',k),$$

where

$$E(\xi',k)=\left(\int_{\theta^{-1}}^{1}\left|S_{\Omega,P}(t\theta^k\xi')\right|^2\frac{dt}{t}\right)^{1/2}.$$

Then

$$|E(\xi',k)|=$$
$$\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}X_k(x,y,\xi')\Omega(x)\overline{\Omega(y)}d\sigma(y)d\sigma(x),$$

where

$$\begin{aligned}&X_k(x,y,\xi')\\=&\int_{\theta^{-1}}^{1}e^{i\left(P(t\theta^k\xi'\cdot x)-P(t|\xi|^{-1}\theta^k\xi'\cdot y)\right)}\frac{dt}{t}.\end{aligned}$$

By invoking Lemma 2.1 we get

$$|X_k(x,y,\xi')|\leq C\theta\left|a_{j_0}\theta^{kj_0}\right|^{-\frac{1}{d}}\left|(\xi'\cdot x)^{j_0}-(\xi'\cdot y)^{j_0}\right|^{-\frac{1}{d}}.$$

By combining the last estimate with the trivial estimate

$$|X_k(x,y,\xi')|\leq C\log\theta$$

we get

$$|X_k(x,y,\xi')|\leq$$
$$C(\log\theta)\left|a_{j_0}\theta^{kj_0}\right|^{-\frac{1}{2dq'}}\left|(\xi'\cdot x)^{j_0}-(\xi'\cdot y)^{j_0}\right|^{-\frac{1}{2dq'}}. \tag{3.7}$$

Thus, by Hölder's, (3.7) and since $|a_{j_0}|=1$ we get

$$|E(\xi',k)|^2\leq C(\log\theta)\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}\left|\theta^{kj_0}\right|^{-\frac{1}{2dq'}}\times$$
$$\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\left|(\xi'\cdot x)^{j_0}-(\xi'\cdot y)^{j_0}\right|^{-\frac{1}{2d}}d\sigma(x)d\sigma(y)\right)^{\frac{1}{q'}}.$$

Since the last integral is finite we get

$$|E(\xi',k)|\leq C(\log\theta)^{\frac{1}{2}}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left|\theta^{kj_0}\right|^{-\frac{1}{4dq'}}$$

which proves (3.6) for the case $\lambda = 2$. Now if $1 < \lambda \leq 2$, by Hölder's inequality we have

$$\left(\int_{\theta^{k-1}}^{\theta^k} |S_{\Omega,P}(t\xi')|^{\lambda} \frac{dt}{t}\right)^{1/\lambda} \leq (\log\theta)^{\frac{2-\gamma}{2\gamma}} \left(\int_{\theta^{k-1}}^{\theta^k} |S_{\Omega,P}(t\xi')|^{2} \frac{dt}{t}\right)^{1/2}.$$

By the last inequality and (3.6) in the case $\lambda = 2$ we get (3.6) for the case $1 < \lambda < 2$. Finally we prove (3.6) for the case $\lambda > 1$. Since $|S_{\Omega,P}(t\xi)| \leq \|\Omega\|_{L^q(\mathbf{S}^{n-1})}$ we have

$$\left(\int_{\theta^{k-1}}^{\theta^k} |S_{\Omega,P}(t\xi')|^{\lambda} \frac{dt}{t}\right)^{1/\lambda} \leq \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{\frac{\gamma-2}{\gamma}} \left(\int_{\theta^{k-1}}^{\theta^k} |S_{\Omega,P}(t\xi')|^{2} \frac{dt}{t}\right)^{1/\lambda}.$$

By the last inequality and (3.6) for the case $\lambda = 2$ we get (3.6) for the case $\lambda > 1$. This completes the proof of (3.6). Now, by (3.6) we have

$$J_2 \leq C\log\theta)^{1/\lambda} \sup_{k_0\in\mathbf{Z}_+} \sum_{k=k_0+1}^{\infty} \left(\theta^{kj_0}\right)^{-\frac{1}{\lambda d q'}} \leq C\log\theta)^{1/\lambda}. \tag{3.8}$$

By (3.2)–(3.5) and (3.8) we obtain

$$A_d \leq C\,(q-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} + A_{d/2}. \tag{3.9}$$

Since $d = 2^m$, we get

$$A_{2^m} \leq C\,(q-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} + A_{2^{m-1}},$$

and hence by induction on m we have

$$A_{2^m} \leq Cm\,(q-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} + A_1. \tag{3.10}$$

Now, we need to estimate A_1. To this end, we notice that any $P \in \mathcal{P}(1)$ with a non constant term will be of the form $P(t) = at$ for some $a \in \mathbf{R}$. By a dilation in t we may assume, without loss of generality, that $|a| = 1$. By following a similar (but easier) argument as that employed in the proof of (3.9) we get

$$A_1 \leq C\,(q-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}. \tag{3.11}$$

Hence, by (3.10)–(3.11) we obtain

$$A_{2^m} \leq C(m+1)\,(q-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}. \tag{3.12}$$

The case now for the general d is easy. Choose a positive integer m so that $2^{m-1} < d \leq 2^m$. By definition of A_d and since $\mathcal{P}(n;d) \subset \mathcal{P}(n;2^m)$ we have

$$\begin{aligned} A_d &\leq A_{2^m} \leq C(m+1)\,(q-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \\ &\leq C(\log d+1)\,(q-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}, \end{aligned}$$

which completes the proof of Theorem 1.1.

Proof of Theorem 1.2 (a). We follow the extrapolation method of Yano (see [18] and [19]) and we follow a similar argument as in [3] and [15]. Let $\lambda \geq 1$ be fixed. Assume $\Omega \in L(\log L)^{1/\lambda}\left(\mathbf{S}^{n-1}\right)$ and satisfies (1.2). Let

$$T(\Omega) = \sup_{P\in\mathcal{P}(d)} \left(\int_0^{\infty} |S_{\Omega,P}(t\xi)|^{\lambda} \frac{dt}{t}\right)^{1/\lambda}.$$

Then we have $T(\Omega_1 + \Omega_2) \leq T(\Omega_1) + T(\Omega_2)$. Now, we decompose Ω as follows: For $m \in \mathbf{N}$, let $\mathbf{E}_m = \{x \in \mathbf{S}^{n-1} : 2^m \leq |\Omega(x)| < 2^{m+1}\}$. For $m \in \mathbf{N}$, set $b_m = \Omega\chi_{\mathbf{E}_m}$, where $\chi_{\mathbf{A}}$ is the characteristic function of a set A. Set $E(\Omega) = \left\{m \in \mathbf{N} : \|b_m\|_1 \geq 2^{-4m}\right\}$ and define the sequence of functions $\{\Omega_m\}_{m\in\mathbf{E}(\mathbf{\Omega})\cup\{0\}}$ by

$$\Omega_m(x) = \|b_m\|_1^{-1}\left(b_m(x) - \int_{\mathbf{S}^{n-1}} b_m(x)d\sigma(x)\right)$$

for $m \in E(\Omega)$, and

$$\Omega_0(x) = \Omega(x) - \sum_{m\in\mathbf{E}(\mathbf{\Omega})} \|b_m\|_1 \;\; \Omega_m(x).$$

It is easy to verify that the following hold for all $m \in E(\Omega) \cup \{0\}$ and for some positive constant C:

$$\sum_{m\in\mathbf{E}(\mathbf{\Omega})} m^{1/\lambda} \|b_m\|_1 \leq C\,\|\Omega\|_{\mathbf{L}(\log\mathbf{L})^{1/\lambda}(\mathbf{S}^{n-1})}; \tag{3.13}$$

$$\int_{\mathbf{S}^{n-1}} \Omega_m(u)\,d\sigma(u) = 0; \tag{3.14}$$

$$\|\Omega_m\|_{1+\frac{1}{m}} \leq 2^6 \text{ for } m \in E(\Omega) \text{ and } \|\Omega_0\|_2 \leq 2^2. \tag{3.15}$$

Therefore,

$$\begin{aligned} T(\Omega) &\leq T(\Omega_0) + \sum_{m\in\mathbf{E}(\mathbf{\Omega})} \|b_m\|_1\, T(\Omega_m) \\ &\leq C\,(\log d+1) \times \left(\|\Omega_0\|_{L^2(\mathbf{S}^{n-1})} + \sum_{m\in\mathbf{E}(\mathbf{\Omega})} m^{1/\lambda} \|b_m\|_1 \|\Omega_m\|_{1+\frac{1}{m}}\right) \\ &\leq C\,(\log d+1)\left(1 + \|\Omega\|_{L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})}\right). \end{aligned}$$

Proof of Theorem 1.2 (b). Assume that $\Omega \in B_q^{(0,1/\lambda-1)}(\mathbf{S}^{n-1})$ for some $q > 1$ and satisfies (1.2). Without loss of generality we may assume $1 < q \leq 2$. Since $\Omega \in B_q^{(0,1/\lambda-1)}(\mathbf{S}^{n-1})$, we can write Ω as $\Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu$, where $\lambda_\mu \in \mathbf{C}$, b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$ and $M_q^{(0,1/\lambda-1)}\left(\{\lambda_\mu\}\right) < \infty$. To each block function $b_\mu(\cdot)$, let $\tilde{\Omega}_\mu(\cdot)$ be a function defined by

$$\tilde{\Omega}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u)d\sigma(u).$$

Let $\mathbf{K} = \left\{\mu \in \mathbf{N} : |I_\mu| < e^{-(q-1)^{-1}}\right\}$ and let $\tilde{\Omega}_0 = \Omega - \sum_{\mu\in\mathbf{K}}^{\infty} \lambda_\mu \tilde{\Omega}_\mu$. Also, for $\mu \in \mathbf{K}$ we let $\alpha_\mu = \log(|I_\mu|^{-1})$ and

$\beta = \sum_{\mu=1}^{\infty} \left|\lambda_\mu\right|$. Then it is easy to see that

$$\int_{\mathbf{S}^{n-1}} \tilde{\Omega}_\mu (u)\, d\sigma(u) = 0 \text{ for all } \mu \in \mathbf{K} \cup \{0\}; \quad (3.16)$$

$$\left\|\tilde{\Omega}_0\right\|_q \leq \beta e^{\frac{1}{q}}; \quad (3.17)$$

$$\left\|\tilde{\Omega}_\mu\right\|_{1+\frac{1}{\alpha_\mu}} \leq 4 \text{ for all } \mu \in \mathbf{K}. \quad (3.18)$$

By (3.16)–(3.18) and invoking Theorem 1.1 we get

$$\begin{aligned} T(\Omega) &\leq T(\tilde{\Omega}_0) + \sum_{\mu\in\mathbf{K}} \left|\lambda_\mu\right| T(\tilde{\Omega}_\mu) \\ &\leq C(\log d + 1) \times \Bigg((q-1)^{-\frac{1}{\lambda}} \left\|\tilde{\Omega}_0\right\|_q + \\ &\quad \sum_{\mu\in\mathbf{K}} \left|\lambda_\mu\right| \left(\log \left|I_\mu\right|^{-1}\right)^{1/\lambda} \left\|\tilde{\Omega}_\mu\right\|_{1+\frac{1}{\alpha_\mu}} \Bigg) \\ &\leq C(\log d + 1) \times \\ &\quad \left(\beta e^{\frac{1}{q}} (q-1)^{-\frac{1}{\lambda}} + 4 \sum_{\mu\in\mathbf{K}} \left|\lambda_\mu\right| \left(\log \left|I_\mu\right|^{-1}\right)^{1/\lambda} \right) \\ &\leq C(\log d + 1) \left(1 + \|\Omega\|_{B_q^{(0,1/\lambda-1)}(\mathbf{S}^{n-1})}\right). \end{aligned}$$

REFERENCES

[1] H. M. Al-Qassem, Oscillatory integrals on the unit sphere, Southeast Asian Bull. Math. 30 (2006), no. 1, 1–21.
[2] H. M. Al-Qassem and A. Al-Salman and Y. Pan, Singular integrals associated to homogeneous mappings with rough kernels, Hokkaido Math. J. **33** (2004), 551–569.
[3] H. Al-Qassem and Y. Pan, On certain estimates for Marcinkiewicz integrals and extrapolation, Collect. Math. 60 (2009), no. 2, 123–145.
[4] H. M. Al-Qassem, L. Cheng and Y. Pan, Rough oscillatory singular integrals on $\mathbf{R}^n$, preprint.
[5] A. Al-Salman and Y. Pan, Singular integrals with rough kernels in $L\log^+ L(\mathbf{S}^{n-1})$, J. London Math. Soc. (2) 66 (2002), 153-174.
[6] Yong Ding, Dashan Fan and Yibiao Pan, On the L^p boundedness of Marcinkiewicz integrals, Michigan Math. J. **50**, 1 (2002), 17-26.
[7] J. Duoandikoetxea and J. L. Rubio de Francia, Maximal functions and singular integral operators via Fourier transform estimates, Invent. Math. 84 (1986), 541-561.
[8] D. Fan, A singular integral on $L^2(\mathbf{R}^n)$, Canad. Math. Bull. 37 (2) (1994), 197-201.
[9] D. Fan, Restriction theorems related to atoms, Illinois J. Math. 40 (1) (1996), 13-20.
[10] D. Fan and Y. Pan, Oscillatory integrals and atoms on the unit sphere, Manusripta Math. 89 (1996), 179-192.
[11] M. Keitoku and E. Sato, Block spaces on the unit sphere in $\mathbf{R}^n$, Proc. Amer. Math. Soc. 119 (1993), 453-455.
[12] S. Lu, M. Taibleson and G. Weiss, "Spaces Generated by Blocks", Beijing Normal University Press, 1989, Beijing.
[13] Y. Pan, Singular integrals with rough kernels, Proc. of the conference "Singular integrals and related topics, III", Oska, Japan (2001), 39-45.
[14] F. Ricci and E.M. Stein, Harmonic analysis on nilpotent groups and singular integrals I: Oscillatory integrals, Jour. Func. Anal. 73 (1987), 179-194.
[15] S. Sato, Estimates for singular integrals and extrapolation. Studia Math. 192 (2009), no. 3, 219–233.
[16] E. M. Stein, Harmonic analysis real-variable methods, orthogonality and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.
[17] M. H. Taibleson and G. Weiss, Certain function spaces associated with a.e. convergence of Fourier series, Univ. of Chicago Conf. in honor of Zygmund, Woodsworth, 83.
[18] S.Yano, An extrapolation theorem, J. Math. Soc. Japan 3 (1951), 296-305.
[19] A. Zygmund, Trigonometric series 2nd ed., Cambridgew Univ. Press, Cambridge, London, New York and Melbourne, 1977.