World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:3, No:7, 2009

Fuzzy Bi-ideals in Ternary Semirings

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Abstract—The purpose of the present paper is to study the concept of fuzzy bi-ideals in ternary semirings. We give some characterizations of fuzzy bi-ideals. Characterizations of regular ternary semirings are provided.

Keywords—Fuzzy ternary subsemiring, fuzzy quasi-ideal, fuzzy bi-ideal, regular ternary semiring

I. Introduction

TERNARY semirings are one of the generalized structures of semirings. The notion of ternary algebraic system was introduced by Lehmer [8]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. Dutta and Kar [1] introduced the notion of ternary semiring which is a generalization of the ternary ring introduced by Lister [9]. Good and Hughes [3] introduced the notion of bi-ideal and Steinfeld [11], [12] introduced the notion of quasi-ideal. In 2005, Kar [5] studied quasi-ideals and bi-ideals of ternary semirings.

Ternary semiring arises naturally, for instance, the ring of integers **Z** is a ternary semiring. The subset $\mathbf{Z}^+$ of all positive integers of **Z** forms an additive semigroup and which is closed under the ring product. Now, if we consider the subset $\mathbf{Z}^-$ of all negative integers of **Z**, then we see that $\mathbf{Z}^-$ is closed under the binary ring product; however, $\mathbf{Z}^-$ is not closed under the binary ring product, i.e., $\mathbf{Z}^-$ forms a ternary semiring. Thus, we see that in the ring of integers **Z**, $\mathbf{Z}^+$ forms a semiring whereas $\mathbf{Z}^-$ forms a ternary semiring. More generally; in an ordered ring, we can see that its positive cone forms a semiring whereas its negative cone forms a ternary semiring. Thus a ternary semiring may be considered as a counterpart of semiring in an ordered ring.

The theory of fuzzy sets was first inspired by Zadeh [14]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics. Rosenfeld [13] introduced fuzzy sets in the realm of group theory. Fuzzy ideals in rings were introduced by Liu [10] and it has been studied by several authors. Jun [4] and Kim and Park [7] have also studied fuzzy ideals in semirings. In 2007, [6] we have introduced the notions of fuzzy ideals and fuzzy quasi-ideals in ternary semirings.

Our main purpose in this paper is to introduce the notions of fuzzy bi-ideal in ternary semirings and study regular ternary semiring in terms of these two subsystems of fuzzy subsemirings. We give some characteriztions of fuzzy bi-ideals.

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II. Preliminaries

In this section, we review some definitions and some results which will be used in later sections.

Definition 2.1. A set R together with associative binary operations called addition and multiplication (denoted by + and . respectively) will be called a semiring provided:

(i) Addition is a commutative operation.
(ii) there exists 0$\in R$ such that a+ 0=a and a0=0a=0 for each $a\in R$,
(iii) multiplication distributes over addition both from the left and the right. i.e., $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$

Definition 2.2. A nonempty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S,+)$ is an additive commutative semigroup satisfying the following conditions:

(i) $(abc)de=a(bcd)e=ab(cde)$
(ii) $(a+b)cd=acd+bcd$
(iii) $a(b+c)d=abd+acd$
(iv) $ab(c+d)=abc+abd$, for all $a,b,c,d,e\in S$.

Definition 2.3. (i) Let S be a ternary semiring. An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3\in T$, for all $t_1,t_2,t_3\in T$.
(ii) Let S be a ternary semiring.If there exists an element 0$\in$ S such that 0+a $=$ a and 0ab $=$ a0b=ab0=0 for all $a,b\in S$, then "0" is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.
(iii) Let A,B,C be three subsets of ternary semiring S. Then by ABC, we mean the set of all finite sums of the form $\sum a_ib_jc_k$ with $a_i\in A, b_j\in B, c_k\in C$.
(iv) An additive subsemigroup I of S is called a left (resp., right, and lateral) ideal of S if s_1s_2i (resp.is_1s_2, s_1is_2)$\in I$, for all $s_1,s_2\in S$ and $i\in I$. If I is both left and right ideal of S, then I is called a two-sided ideal of S. If I is a left, a right and a lateral ideal of S, then I is called an ideal of S. An ideal I of S is called a proper ideal if $I\neq S$.

Definition 2.4. (i) An additive subsemigroup $(Q,+)$ of a ternary semiring S is called a quasi-ideal of S if $QSS\cap(SQS+SSQSS)\cap SSQ\subseteq Q$.
(ii) An additive subsemigroup $(Q,+)$ of a ternary semiring S is called a bi-ideal of S if $QSQSQ\subseteq Q$.

Now, we review the concept of fuzzy sets [10], [13], [14]). Let X be a non-empty set. A map $\mu: X\to[0,1]$ is called a *fuzzy set* in X, and the *complement* of a fuzzy set μ in X, denoted by $\overline{\mu}$, is the fuzzy set in X given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Let X and Y be two non-empty sets and $f : X \rightarrow Y$ a function, and let μ and ν be any fuzzy sets in X and Y respectively. The $image\ of\ \mu\ under f$, denoted by $f(\mu)$, is a fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for each $y \in Y$. The $preimage\ of\ \nu\ under f$, denoted by $f^{-1}(\nu)$, is a fuzzy set in X defined by $(f^{-1}(\nu))(x) = \nu(f(x))$ for each $x \in X$.

Definition 2.5. A fuzzy ideal of a semiring R is a function $A : R \longrightarrow [0, 1]$ satisfying the following conditions:

(i) A is a fuzzy subsemigroup of (R,+); i.e., $A(x - y) \geq min\{A(x), A(y)\}$,
(ii) $A(xy) \geq max\{A(x), A(y)\}$, for all $x, y \in R$

Definition 2.6. Let A and B be any two subsets of S. Then $A \cap B$, $A \cup B$, A+B and $A \circ B$ are fuzzy subsets of S defined by

$$(A \cap B) = min\{A(x), B(x)\}$$

$$(A \cup B) = max\{A(x), B(x)\}$$

$$(A + B)(x) = \begin{cases} sup\{min\{A(y), A(z)\}, & \text{if } \ x = y + z, \\ 0 & \text{otherwise} \end{cases}$$

$$(A \circ B)(x) = \begin{cases} sup\{min\{A(y), A(z)\}, & \text{if } x = yz, \\ 0 & \text{otherwise} \end{cases}$$

For any $x \in S$ and $t \in (0, 1]$, define a fuzzy point x_t as

$$x_t(y) = \begin{cases} t, & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

If x_t is a fuzzy point and A is any fuzzy subset of S and $x_t \leq A$, then we write $x_t \in A$. Note that $x_t \in A$ if and only if $x \in A_t$ where A_t is a level subset of A. If x_r and y_s are fuzzy points, than $x_r y_s = (xy)_{min\{r,s\}}$.

Definition 2.7. [6]. A fuzzy subset A of a fuzzy subsemigroup of S is called a fuzzy ternary subsemiring of S if:

(i) $A(x - y) \geq min\{A(x), A(y)\}$, for all $x, y \in S$
(ii) $A(-x) = A(x)$
(iii) $A(xyz) \geq min\{A(x), A(y), A(z)\}$, for all $x, y, z \in S$.

Definition 2.8 [6]. A fuzzy subsemigroup A of a ternary semiring S called a fuzzy ideal of S if $A : S \longrightarrow [0, 1]$ satisfying the following conditions:

(i) $A(x - y) \geq min\{A(x), A(y)\}$, for all $x, y \in S$
(ii) $A(xyz) \geq A(z)$
(iii) $A(xyz) \geq A(x)$ and
(iv) $A(xyz) \geq A(y)$, for all $x, y, z \in S$

A fuzzy subset A with conditions (i) and (ii) is called a fuzzy left ideal of S. If A satisfies (i) and (iii), then it is called a fuzzy right ideal of S. Also if A satisfies (i) and (iv), then it is called a fuzzy lateral ideal of S. A fuzzy ideal is a ternary semiring of S, if A is a fuzzy left, a fuzzy right and a fuzzy lateral ideal of S. It is clear that A is a fuzzy ideal of a ternary semiring S if and only if $A(xyz) \geq max\{A(x), A(y), A(z)\}$ for all $x, y, z \in S$, and that every fuzzy left (right, lateral) ideal of S is a fuzzy ternary subsemiring of S.

Example 2.9 [6]. Let Z be a ring of integers and $S = \mathbf{Z}^-{}_0 \subset \mathbf{Z}$ be the set of all negative integers with zero. Then with the binary addition and ternary multiplication, $(\mathbf{Z}^-{}_0, +, .)$ forms a ternary semiring S with zero. Define a fuzzy subset $A : \mathbf{Z} \longrightarrow [0, 1]$, we have

$$A(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Z}^-{}_0 \\ 0, & \text{otherwise} \end{cases}$$

Then A is a fuzzy ternary subsemiring of S.

Example 2.10 [6]. Consider the set integer module 5, non-positive integer $\mathbf{Z}^-{}_5 = \{0, -1, -2, -3, -4\}$ with the usual addition and ternary multiplication, we have

+	0	-1	-2	-3	-4
0	0	-1	-2	-3	-4
-1	-1	-2	-3	-4	0
-2	-2	-3	-4	0	-1
-3	-3	-4	0	-1	-2
-4	-4	0	-1	-2	-3

·	0	-1	-2	-3	-4
0	0	0	0	0	0
-1	0	1	2	3	4
-2	0	2	4	1	3
-3	0	3	1	4	2
-4	0	4	3	2	1

·	0	1	2	3	4
0	0	0	0	0	0
-1	0	-1	-2	-3	-4
-2	0	-2	-4	-1	-3
-3	0	-3	-1	-4	-2
-4	0	-4	-3	-2	-1

Clearly $(\mathbf{Z}_5^-, +, .)$ is a ternary semiring. Let a fuzzy subset $A : \mathbf{Z}_5^- \longrightarrow [0, 1]$ be defined by $A(0) = t_0$ and $A(-1) = A(-2) = A(-3) = A(-4) = t_1$, where $t_0 \geq t_1$ and $t_0, t_1 \in [0, 1]$. Routine calculations show that A is a fuzzy ideal of $\mathbf{Z}^-{}_5$.

Definition 2.11 [6] Let A be a fuzzy subset of ternary semiring S. We define
$SAS + SSASS(z)$

$$= \begin{cases} sup\{min\{A(a), A(b)\}, & \text{if } z = x(a + xby)y, \\ 0, & \text{otherwise} \end{cases}$$

for all $x, y, a, b \in S$

III. Fuzzy Bi-ideal of Ternary Semiring

Definition 3.1. A fuzzy subsemigroup μ of a ternary semiring S is called a fuzzy quasi-ideal of S [6] if

(*FQI1*)$\mu\mathbf{SS} \cap \mathbf{S}\mu\mathbf{S} \cap \mathbf{SS}\mu \leq \mu$
(*FQI2*)$\mu\mathbf{SS} \cap \mathbf{SS}\mu\mathbf{SS} \cap \mathbf{SS}\mu \leq \mu$

i.e., $\mu(x) \geq min\{(\mu\mathbf{SS})(x), (\mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS})(x), (\mathbf{SS}\mu)(x)\}$

To strengthen the above definition, we present the following example.

Example 3.2. Consider the ternary semiring ($\mathbf{Z}_5^-$,+, .) as defined in Example 2.10 in this paper. Let $A = \{0, -2, -3\}$. Then $SSA = \{-2, -3, -4\}$, $(SAS + SSASS) = \{0, -1, -2, -3\}$ and $ASS = \{-1, -2, -3\}$. Therefore $ASS \cap (SAS + SSASS) \cap SSA = \{-2, -3\} \subseteq A$. Hence A is a quasi-ideal of $\mathbf{Z}_5^-$. Define a fuzzy subset $A : \mathbf{Z}_5^- \longrightarrow [0, 1]$ by $A(0) = A(-2) = A(-3) = 1$ and $A(-1) = A(-4) = 0$. Clearly A is a fuzzy quasi-ideal of $\mathbf{Z}_5^-$.

Definition 3.3. A fuzzy ternary subsemiring μ of S is called a *fuzzy bi-ideal* of S if

$$\mu\mathbf{S}\mu\mathbf{S}\mu \leq \mu$$

i.e., $\mu(xs_1ys_2z) \geq min\{\mu(x), \mu(y), \mu(z)\}$ $\forall$ $x, y, z, w, v \in S$

Example 3.4 Let $\mathbf{Z}^-$=**S** be the set of all negative integers. Then $\mathbf{Z}^-$ is a ternary semiring under usual addition and ternary multiplication. Let B =2**S** Thus $B\mathbf{S}B\mathbf{S}B = 2\mathbf{SS}2\mathbf{SS}2\mathbf{S} = 6\mathbf{(SSS)SS} = 6(\mathbf{SSS}) = 6\mathbf{S} \subseteq 2\mathbf{S} = B$. Hence B is a bi-ideal of $\mathbf{Z}^-$.

Define $\mu : S \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} t, & \text{if } x \in 2\mathbf{S} \\ 0, & \text{otherwise} \end{cases}$$

For any $t \in [0, 1]$, $\mu_t = \{2\mathbf{S}\}$, since $\{2\mathbf{S}\}$ is a bi-ideal in $\mathbf{Z}^-$, μ_t is the bi-ideal in $\mathbf{Z}^-$ for all t. Hence μ is a fuzzy bi-ideal of $\mathbf{Z}^-$.

Lemma 3.5. Let μ be a fuzzy subset of S. If μ is a fuzzy left ideal, fuzzy right ideal and lateral ideal of ternary semiring of S, then μ is a fuzzy quasi-ideal of S.

Proof: Let μ be a fuzzy left ideal, fuzzy right ideal and fuzzy lateral ideal of S.Let $x = as_1s_2 = s_1(b_1 + s_1cs_2)s_2 = s_1s_2d$ where $a, b, c, d, s_1, s_2 \in S$.
Consider $(\mu\mathbf{SS} \cap (\mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS}) \cap \mathbf{SS}\mu)(x)$

$$= min\Big\{(\mu\mathbf{SS})(x), (\mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS})(x), (\mathbf{SS}\mu)(x)\Big\}$$

$$= min\Big\{\sup_{x=as_1s_2}\{\mu(a)\}, \sup_{x=s_1(b+s_1cs_2)s_2}\{\mu(b), \mu(c)\}, \sup_{x=s_1s_2d}\{\mu(d)\}\Big\}$$

$$\leq min\Big\{1, \sup_{x=s_1(b+s_1cs_2)s_2}\{\mu(s_1(b + s_1cs_2)s_2)\}, 1\Big\}$$

(as μ is a fuzzy left, fuzzy right and fuzzy lateral ideal,

$$\mu\Big\{s_1(b + s_1cs_2)s_2\Big\} \geq min\{\mu(b), \mu(c)\}$$

$= \mu(b)$ if $\mu(b) < \mu(c)$, $(= \mu(c)$ if $\mu(b) > \mu(c)))$ we get,

$$(\mu\mathbf{SS} \cap (\mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS}) \cap \mathbf{SS}\mu)(x) \leq \mu(x)$$

We remark that if x is not expressed as $x = as_1s_2 = s_1(b_1 + s_1cs_2)s_2 = s_1s_2d$, then

$$(\mu\mathbf{SS} \cap (\mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS}) \cap \mathbf{SS}\mu)(x) = 0 \leq \mu(x).$$

Thus,

$$(\mu\mathbf{SS} \cap (\mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS}) \cap \mathbf{SS}\mu)(x) \leq \mu(x).$$

Hence μ is a fuzzy quasi-ideal of S. ■

Lemma 3.6. For any non-empty subsets A, B and C of S,

(1) $f_Af_Bf_C = f_{ABC}$
(2) $f_A \cap f_B \cap f_C = f_{A\cap B\cap C}$
(3) $f_A + f_B = f_{A+B}$

Proof: Proof is straight forward. ■

Lemma 3.7. Let Q be an additive subsemigroup of S.

(1) Q is a quasi-ideal of S if and only if f_Q is a fuzzy quasi-ideal of S.
(2) Q is a bi-ideal of S if and only if f_Q is a fuzzy bi-ideal of S.

Proof: Proof of (1) can seen in [8].
Proof of(2) Assume that Q is a bi-ideal of S. Then f_Q is a fuzzy ternary subsemiring of S.

$$f_Qf_Sf_Qf_Sf_Q \leq f_{QSQSQ} \leq f_Q$$

This means that f_Q is a fuzzy bi-ideal of S.
Conversely, let us assume that f_Q is a fuzzy bi-ideal of S. Let x be any element of $QSQSQ$. Then, we have

$$f_Q(x) \geq (f_Qf_Sf_Qf_Sf_Q)(x) = f_{QSQSQ}(x) = 1$$

Thus $x \in Q$ and $QSQSQ \subseteq Q$. Hence Q is a bi-ideal of S. ■

Lemma 3.8. Any fuzzy quasi-ideal of S is a fuzzy bi-ideal of S.

Proof: Let μ be any fuzzy quasi-ideal of S. Then, we have

$$\mu\mathbf{S}\mu\mathbf{S}\mu \subseteq \mu\mathbf{(SSS)S} \subseteq \mu\mathbf{SS},$$
$$\mu\mathbf{S}\mu\mathbf{S}\mu \subseteq \mathbf{S(SSS)}\mu \subseteq \mathbf{SS}\mu,$$
$$\mu\mathbf{S}\mu\mathbf{S}\mu \subseteq \mathbf{SS}\mu\mathbf{SS} \text{ and taking } \{0\} \subseteq \mathbf{S}\mu\mathbf{S}$$

$$\text{so, } \mu\mathbf{S}\mu\mathbf{S}\mu \subseteq \mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS}$$
$$\text{we have, } \mu\mathbf{S}\mu\mathbf{S}\mu \subseteq \mu\mathbf{SS} \cap (\mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS}) \cap \mathbf{SS}\mu \subseteq \mu$$

Hence, μ is a fuzzy bi-ideal of S. ■

Remark 3.9. The converse of Lemma 3.8 does not hold, in general, that is, a fuzzy bi-ideal of a ternary semiring S may not be a fuzzy quasi-ideal of S.

Theorem 3.10. Let μ be a fuzzy subset of S. If μ is a fuzzy left, fuzzy right and lateral ideal of ternary semiring of S, then μ is a fuzzy bi-ideal of S.

Proof: As μ is fuzzy left, right, lateral ideal of S and Lemma 3.5, μ is a fuzzy quasi-ideal of S. Hence by Lemma 3.8, μ is a fuzzy bi-ideal of S. ■

Theorem 3.11.[6] Let μ be a fuzzy subset of S. Then μ is a fuzzy quasi-ideal of S, if and only if μ_t is a quasi-ideal of S, for all $t \in Im(\mu)$.

Theorem 3.12. Let μ be a fuzzy subset of S. Then μ is a fuzzy bi-ideal of S, if and only if μ_t is a bi-ideal of S, for all $t \in Im(\mu)$.

Proof: Let μ be a fuzzy bi-ideal of S. Let $t \in Im(\mu)$. Suppose $x, y, z \in S$ such that $x, y, z \in \mu_t$.Then

$$\mu(x) \geq t, \mu(y) \geq t, \mu(z) \geq t$$

, and

$$min\{\mu(x), \mu(y), \mu(z)\} \geq t.$$

As μ is a fuzzy bi-ideal, $\mu(x-y) \geq t$ and thus $x-y \in \mu_t$. Let $u \in S$. Suppose $u \in \mu_t\mathbf{S}\mu_t\mathbf{S}\mu_t$. Then there exist $x, y, z \in \mu_t$ and $s_1, s_2, \in S$ such that $u = xs_1ys_2z$. Then,

$$(\mu\mathbf{S}\mu\mathbf{S}\mu)(u) = \mu(xs_1ys_2z)$$

$$\geq min\{\mu(x), \mu(y), \mu(z)\} \geq min\{t, t, t\} = t.$$

Therefore, $(\mu\mathbf{S}\mu\mathbf{S}\mu)(u) \geq t$. As μ is a bi-ideal of S, $\mu(u) \geq t$ implies $u \in \mu_t$. Hence μ_t is a bi-ideal of S.
Conversely, let us assume that μ_A is a bi-ideal of S, $t \in Im(\mu)$.Let $p \in S$. Consider

$$(\mu\mathbf{S}\mu\mathbf{S}\mu)(p) = \sup_{p=xs_1ys_2z}\Big\{min\{\mu(x), \mu(y), \mu(z)\}\Big\}$$

Let $\mu(x) = t_1 < \mu(y) = t_2 < \mu(z) = t_3$. Then, $\mu_{t_1} \supseteq \mu_{t_2} \supseteq \mu_{t_3}$. Thus $x, y, z \in \mu_{t_1}$ and $p = xs_1ys_2z \in \mu_{t_1}\mathbf{S}\mu_{t_1}\mathbf{S}\mu_{t_1} \subseteq \mu_{t_1}$. This implies $\mu(p) \geq t_1$ and hence $\mu\mathbf{S}\mu\mathbf{S}\mu \leq \mu$. Therefore, μ is a fuzzy bi-ideal of S. ■

Definition 3.13 Let S and T be two ternary semirings. Let f be a mapping which maps from S into T. Then f is called a homomorphism of S into T if

(i) $f(a+b) = f(a) + f(b)$ and
(ii) $f(abc) = f(a)f(b)f(c)$ for all $a, b, c \in S$

Theorem 3.14. If λ is a fuzzy bi-ideal of a ternary semiring S and μ is a fuzzy ternary subsemiring of S, then $(\lambda \cap \mu)$ is a fuzzy bi-ideal of S.

Proof: Let λ be a fuzzy bi-ideal and μ be a fuzzy ternary subsemiring of S. Clearly $(\lambda \cap \mu)$ is a fuzzy ternary subsemiring of S. Next we prove that $(\lambda \cap \mu)$ is a fuzzy bi-ideal of ternary semiring S. Let $t \in S$ and $s_1, s_2, x, y, z \in S$ such that $t = xs_1ys_2z$.
Consider
$((\lambda \cap \mu)\mathbf{S}(\lambda \cap \mu)\mathbf{S}(\lambda \cap \mu))(t)$

$$= \sup_{t=xs_1ys_2z}\Big\{min\{(\lambda \cap \mu)(x), \mathbf{S}(s_1), (\lambda \cap \mu)(y), \mathbf{S}(s_2), (\lambda \cap \mu)(z)\}\Big\}$$

$$= \sup_{t=xs_1ys_2z}\Big\{min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\}\Big\}$$

Let $min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\} = t$. This implies that $(\lambda \cap \mu)(x) \geq t$, $(\lambda \cap \mu)(y) \geq t$ and $(\lambda \cap \mu)(z) \geq t$. Then $x, y, z \in (\lambda_t \cap \mu_t)$. As λ is the fuzzy bi-ideal and μ is the fuzzy ternary subsemiring, $(\lambda_t \cap \mu_t)$ is a bi-ideal of S. Hence, $xs_1ys_2z \in (\lambda_t \cap \mu_t)$. This implies

$$(\lambda \cap \mu)(xs_1ys_2z) \geq t$$

$$= min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\}.$$

Thus,

$$min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\}$$

$$\leq (\lambda \cap \mu)(xs_1ys_2z)$$

This shows that

$$\sup_{t=xs_1ys_2z} min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\}$$

$$\leq (\lambda \cap \mu)(xs_1ys_2z)$$

Thus, we have

$$((\lambda \cap \mu)\mathbf{S}(\lambda \cap \mu)\mathbf{S}(\lambda \cap \mu))(t) \leq (\lambda \cap \mu)(t)$$

Hence,

$$((\lambda \cap \mu)\mathbf{S}(\lambda \cap \mu)\mathbf{S}(\lambda \cap \mu)) \leq (\lambda \cap \mu)$$

and $(\lambda \cap \mu)$ is a fuzzy ideal of S. ■

IV. Regular Ternary Semiring

A ternary semiring S is called regular if for every $a \in S$, there exists an x in S such that $axa = a$. **Lemma 4.1.** A ternary semiring S is regular if and only if

$$\mu * \gamma * \lambda = \mu \cap \gamma \cap \lambda$$

for every fuzzy right ideal μ, fuzzy left ideal λ and fuzzy lateral ideal γ of S.

Proof: Straight forward from Theorem 5.1 in [5] ■

Theorem 4.2. For a ternary semiring S, the following conditions are equivalent:

(1) S is regular
(2) $\mu = \mu * S * \mu * S * \mu$, for every fuzzy bi-ideal μ of S.
(3) $\mu = \mu * S * \mu * S * \mu$, for every fuzzy quasi-ideal μ of S

Proof: (1)⇒(2) First assume that (1) holds. Let μ be any fuzzy bi-ideal of S, and a any element of S. Then since S is regular, there exists an element x in S such that $a = axa(= axaxa)$. Then we have

$$(\mu * S * \mu * S * \mu)(a)$$

$$= \sup_{a=\sum_{finite} x_iy_iz_i} min \{\mu(x_i), (S * \mu * S)(y_i), (\mu)(z_i)\}$$

$$\geq min\{\mu(a), (S * \mu * S)(xax), (\mu)(a)\}$$

$$= min\Big\{\mu(a), \sup_{xax=\sum_{finite} p_iq_ir_i} [min\{S(p_i), \mu(q_i), S(r_i)\}], \mu(a)\Big\}$$

$$\geq min\Big\{\mu(a), min\{S(x), \mu(a), S(x)\}, \mu(a)\Big\}$$

$$= min\Big\{\mu(a), min\{1, \mu(a), 1\}, \mu(a)\Big\} = \mu(a),$$

and so $\mu * S * \mu * S * \mu \subseteq \mu$. Since μ is a fuzzy bi-ideal of S, the converse inclusion holds. Thus we have $\mu * S * \mu * S * \mu = \mu$

(2)⇒(3) Since any fuzzy quasi-ideal of S is a fuzzy bi-ideal of S by Lemma 3.8.
(3)⇒(1) Assume (3) holds. Let Q be any quasi-ideal of S, and a any element of Q. Then it follows from Lemma 3.7 (1)

that the characteristic function f_Q is a quasi-ideal of S.Then we have

$$f_{QSQSQ}(a) = (f_Q * f_S * f_Q * f_S * f_Q)(a) = f_Q(a) = 1$$

and so, $a \in QSQSQ$. Thus $Q \subseteq QSQSQ$. On the other hand, Q is a quasi-ideal of S

$$QSQSQ \subseteq (QSS \cap SQS \cap SSQ)$$

$$QSQSQ \subseteq (QSS \cap SSQSS \cap SSQ)$$

then,

$$QSQSQ \subseteq (QSS \cap (SQS + SSQSS) \cap SSQ) \subseteq Q$$

and so we have $QSQSQ = Q$ and hence, by [5, Theorem 3.4], S is a regular ternary semiring.

■

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