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Perturbation in the Fractional Fourier Span due to Erroneous Transform Order and Window Function

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Abstract**—**Fractional Fourier Transform is a generalization of the classical Fourier Transform. The Fractional Fourier span in general depends on the amplitude and phase functions of the signal and varies with the transform order. However, with the development of the Fractional Fourier filter banks, it is advantageous in some cases to have different transform orders for different filter banks to achieve better decorrelation of the windowed and overlapped time signal. We present an expression that is useful for finding the perturbation in the Fractional Fourier span due to the erroneous transform order and the possible variation in the window shape and length. The expression is based on the dependency of the time-Fractional Fourier span Uncertainty on the amplitude and phase function of the signal. We also show with the help of the developed expression that the perturbation of span has a varying degree of sensitivity for varying degree of transform order and the window coefficients.

Keywords**—**Fractional Fourier Transform, Perturbation, Fractional Fourier span, amplitude, phase, transform order, filter banks.

I. INTRODUCTION

THE traditional Fourier transform decomposes the signal in terms of sinusoids, which are perfectly localized in frequency, but are not at all localized in time [1]. FrFT expresses the signal in terms of an orthonormal basis formed by linear chirps. Linear chirps are complex signals, whose instantaneous frequency varies linearly with time.

The Kernel for continuous Fractional Fourier Transform is given by [2]:

$$K_\alpha(t,u) = \sqrt{\frac{1-j\cot\alpha}{2\pi}}\, e^{j\frac{t^2+u^2}{2}\cot\alpha - jut\operatorname{cosec}\alpha}$$

Using this kernel of FrFT, the FRFT of signal $x(t)$ with transform order (α) is computed as:

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u)\,dt$$

And x(t) can be recovered from the following equation,

$$x(t) = \int_{-\infty}^{\infty} X_\alpha(u) K_{-\alpha}(u,t)\,du$$

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II. MATHEMATICAL EXPRESSION

We provide here the mathematical development of the expression which relates time span, Fractional Fourier span, transform order, amplitude and phase functions of a signal. It is known that the lower bound for the time frequency product for all classes of signals is given by the following [3]:

If $\sqrt{t}\, x(t) \longrightarrow 0$ for $|t| \to \infty$, then $\Delta_t \Delta_\omega \geq 1/2$ for a signal $x(t)$.

We start with the same assumption for developing the relation. Since, we relate only the spans in time and fractional Fourier domain, which are about the mean, for simplicity, we take the mean value in both the domains as zero, i.e.

$<t> = <\omega> = 0$

Then, we have,

$$\Delta_t^2 \Delta_u^2 = \frac{1}{E^2}\int t^2 |x(t)|^2\,dt \;.\; \int u^2 |X(u)|^2\,du$$

It is easy to establish that,

$$\int u^2 |X(u)|^2\,du = \int uX(u).uX^*(u)du$$
$$= \int h(t)h^*(t)dt$$

With $h(t) = F_\alpha^{-1}\{uX(u)\}$

Using the differentiation property in the Fractional Fourier domain, we can get

$$h(t) = \sin\alpha[-j\frac{d}{dt}x(t) + t\cot\alpha.x(t)]$$

Thus,

$$h^*(t) = \sin\alpha[-j\frac{d}{dt}x^*(t) + t\cot\alpha.x^*(t)]$$

Hence,

$h(t)\,h^*(t)$ can be shown to lead to the following expression,

$$h(t)\,h^*(t) = \sin^2\alpha.\left|\frac{d}{dt}x(t)\right|^2 + t^2\cos^2\alpha|x(t)|^2 +$$
$$t\sin(2\alpha)\phi'(t)|x(t)|^2$$

where *x(t) = A(t) exp (j ϕ (t))* , with the *A(t)* being the real amplitude part and ϕ *(t)* being the real phase of the signal.

We can now arrive at the expression for time-Fractional Fourier span product.

$$\Delta_t^2\Delta_u^2 = \sin^2\alpha\int\left|tx(t)\right|^2 dt \;.\; \int\left|\frac{d}{dt}x(t)\right|^2 dt$$
$$+\cos^2\alpha.\int\left|tx(t)\right|^2 dt.\int\left|tx(t)\right|^2 dt$$
$$+\sin(2\alpha)\int t^2A^2(t).dt.\int t\phi'(t)A^2(t)dt$$

Using Cauchy Schwarz Inequality, the lower bound for the time- Fractional Fourier product can be given dependent on the amplitude and phase functions of a unit energy signal as follows:

For $\sqrt{t}x(t)\rightarrow 0\, for |t|\rightarrow\infty$,

$$\Delta_t^2\Delta_u^2 \geq \frac{\sin^2\alpha}{4} + \cos^2\alpha\left|\int t^2A^2(t)dt\right|^2$$
$$+\sin(2\alpha)\int t^2A^2(t).dt.\int t\phi'(t)A^2(t)dt$$

It is easy to see that the derived expression satisfies the Uncertainty Principle for all the signals in the frequency domain keeping $\alpha = \pi/2$.

Fig. 1 illustrates the authenticity of the developed expression by plotting the square of lower bound of time-Fractional Fourier span product as a function of time and the transform order for a complex sinusoid with a Gaussian envelope.

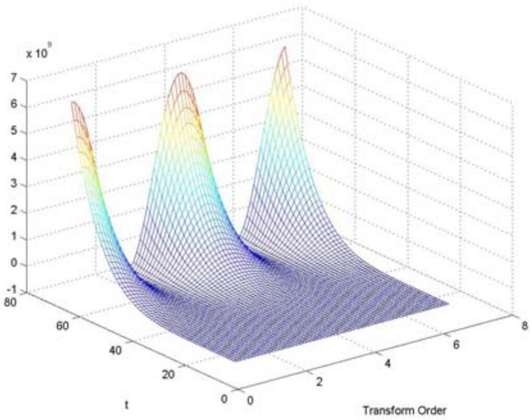

Fig. 1 The minimum span if for the transform orders that are multiples of π/2

III. Perturbation of Fractional Fourier Span

The alteration in the Fractional Fourier span due to error prone transform order and window functions while using the filter banks can lead to varied and undesired levels of compression and noise addition.

Let us consider that a deterministic signal (whose amplitude and phase functions are known) has a time span of Δ_t after getting multiplied by a window function and a Fractional Fourier span of Δ_u corresponding to a transform order of α.

Due to quantization error or erroneous transmission of the different transform orders through a channel, the transform order now becomes α + Δα. In a similar manner, let the time span due to the possible corruption of the window coefficients become Δ. Hence, using the developed mathematical expression, we have,

$$\Delta_t^2\Delta_u^2 \geq \frac{\sin^2\alpha}{4} + \cos^2\alpha\left|\int t^2A^2(t)dt\right|^2$$
$$+\sin(2\alpha)\int t^2A^2(t).dt.\int t\phi'(t)A^2(t)dt$$

And

$$\Delta^2 d^2 \geq \frac{\sin^2(\alpha+\Delta\alpha)}{4}$$
$$+\cos^2(\alpha+\Delta\alpha)\left|\int t^2A^2(t)dt\right|^2$$
$$+\sin(2\alpha+2\Delta\alpha)\int t^2A^2(t).dt.\int t\phi'(t)A^2(t)dt$$

where, d is the perturbed span.

Using the above two equations, it is easy to determine the difference between the perturbed span *d* and the original span.

However, it is important to note the variation of the span with respect to transform order alone and window functions alone. For this we consider a linear chirp with a Gaussian envelope with unit energy of the form,

$$x(t) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}}.e^{-\frac{1}{2}t^2}\;.e^{jt^2}$$

The transform order of π/4 is the optimal for minimum span and hence, we proceed with that. The alteration of square of span with no change in the window function but with varying degree of alteration in the transform order is shown in Table I.

TABLE I
Varying Degree of Alteration in the Span due to Alteration in the Transform Order

Transform Order	Span*Span	Perturbation
0.7854	0.7171	0
0.7864	0.8402	0.1231
0.7904	0.8724	0.1553
0.7954	0.8737	0.1566
0.8854	0.8747	0.1576
1.0854	0.8750	0.1579

We shall now alter the values of the window length (not the window shape) in the same ratio as the transform order and then show the results for the perturbation of the square of Fractional Fourier span. Note that the time span in Table II is changed due to the erroneous window coefficients.

TABLE II
VARYING DEGREE OF ALTERATION IN THE SPAN DUE TO ALTERATION IN THE WINDOW LENGTH

Time Span^2	Span*Span	Perturbation
64.0000	0.8750	0
64.0815	0.8739	0.0011
64.4074	0.8695	0.0055
64.8149	0.8640	0.0110
72.1487	0.7762	0.0988
88.4462	0.6332	0.2418

From the tables shown above, we conclude the following for the Fractional Fourier Span variation due to erroneous transform order and the window coefficients.

The span is more sensitive to the alteration in the transform order for lower degrees of variation, but it is more sensitive to the change in the window functions at higher degrees of variation.

The expression derived in Section II finds its utmost importance in the case where one needs to define a trade-off between the time resolution and the resolution at any transform order domain. In such a case, the medium of analysis presented in this paper helps to evaluate the effects of the both transform order and window coefficients variation on the Fractional Fourier span while making it efficient for adapting the transform order domain resolutions.

IV. CONCLUSION

The development of the lower bound for the time-Fractional Fourier span product with dependence on the amplitude and phase function of the signal, for a generalized class of signals helps us to study the perturbation of the Fractional Fourier span due to error addition in the transform order and the window functions, especially for the case of fractional Fourier filter banks. Also, it is concluded that the sensitivity of the span is variant both of the degree of alteration in the affecting parameters and also o the affecting parameter.

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