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Bifurcation Method for Solving Positive Solutions to a Class of Semilinear Elliptic Equations & Stability Analysis of Solutions

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Abstract—Semilinear elliptic equations are ubiquitous in natural sciences. They give rise to a variety of important phenomena in quantum mechanics, nonlinear optics, astrophysics, etc because they have rich multiple solutions. But the nontrivial solutions of semilinear equations are hard to be solved for the lack of stabilities, such as Lane-Emden equation, Henon equation and Chandrasekhar equation. In this paper, bifurcation method is applied to solving semilinear elliptic equations which are with homogeneous Dirichlet boundary conditions in 2D. Using this method, nontrivial numerical solutions will be computed and visualized in many different domains (such as square, disk, annulus, dumbbell, etc).

Keywords—Semilinear elliptic equations; positive solutions; bifurcation method; isotropy subgroups.

I. Introduction

In this paper, we study semilinear elliptic boundary value problems of the form

$$\begin{cases} \Delta u + f(x, u(x)) = 0, & in \;\; \Omega, \\ u > 0, & in \;\; \Omega, \\ u = 0, & on \;\; \partial\Omega. \end{cases} \quad (1)$$

where Ω is a bounded open domain in $\mathbb{R}^2$, and f is a nonlinear function of x and u . We will deal with $f \equiv u^p, \lambda_1 u + u^p, p > 1$, which are elliptic equations (2),(3) below:

$$\begin{cases} \Delta u + u^p = 0, & in \;\; \Omega, \\ u > 0, & in \;\; \Omega, \\ u = 0, & on \;\; \partial\Omega. \end{cases} \quad (2)$$

$$\begin{cases} \Delta u + \lambda_1 u + u^p = 0, & in \;\; \Omega, \\ u > 0, & in \;\; \Omega, \\ u = 0, & on \;\; \partial\Omega. \end{cases} \quad (3)$$

Since 60's of the 20th century, the existence and multiplicity of solutions to the boundary value problems of the nonlinear elliptic PDEs such as problems (2), (3) have been studied by the monotone iterative method in the ordered Banach space[1,2], the mountain pass lemma and the minmax theorem in the critical point theory[3,4]. It becomes an important field in PDE study. But what distribution and structure the solutions to the BVP of the nonlinear elliptic equations have and how to compute them have attracted the attention of many mathematicians, physicists and engineers.

There are mainly five numerical methods for computing such kinds of problems: the Monotone Iterative Scheme (MIS)[5,6], the Mountain Pass Algorithm (MPA)[7], the High Linking Algorithm (HLA)[8], the Min-Max Algorithm (MMA)[9,10] and the Search Extension Method (SEM)[11]. MIS is based on the monotone iterative methods in the ordered Banach space; MPA, MMA and HLA are based on the numerical implement of the mountain pass lemma and the min-max theorem in the critical point theory. MPA was proposed by Choi and McKenna to compute the solutions with the Morse Index (MI) 0 or 1. Ding, Costa and Chen established HLA for sign-changing solution (MI=2) of semilinear elliptic problems. Li and Zhou designed a new min-max algorithm (MMA) to find multiple saddle points with any Morse index which is more constructive than the traditional min-max theorem. Chen and Xie proposed SEM, which searches the initial guess based on the linear combination of the eigenfunctions of the linearized problem and then gets the better initial guess by the continuation method for the discretized problem by the finite element method.

The advantages of the bifurcation method are computation of the solutions to problem (1) with any Morse index and different symmetries as many as possible and simplification of the computation of problem (1). On the other hand, the difficulty in searching the initial guess in other methods can be solved effectively by the bifurcation method. The bifurcation method is applied successfully to solving the BVP of the Henon equation[12,13].

The organization of our paper is as follows. In Sec.2, we introduce our idea of bifurcation method and present some definitions, theorems which will be used in the following sections. In Sec.3, we use bifurcation method to compute nontrivial positive solution of equation (3) with $\Omega = [0,1] \times [0,1]$ and analyze stability of this solution. In Sec.4, we compute and visualize nontrivial positive solutions of equation (3) on many different complex domains.

II. Theory & Background

Models of (1) arise naturally in physics, engineers, biology and ecology, etc. Although nonlinearities may appear in seemingly endless form, the simplest and most basic form of nonlinearity is the power type. If we set $f \equiv u^p, p > 1$, which

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is (2) we have mentioned, called Lane-Emden (-Fowler) equation and proposed by Chandrasekhar and Fowler[14,15]. If we set $f \equiv |x|^l u^p, p > 1, l > 0$, which is called Henon equation, studied by Henon first[16]. If we set $f \equiv 4\pi(2u+u^2)^{3/2}$, which is called Chandrasekhar equation, proposed by Chandrasekhar, Lieb and Yau[14,17]. The equations above exhibit rich multiplicity of solutions, and draw many researchers interest, and there is a huge body of literature on them[5−11]. Our focus here is positive solutions of equation (1).

We embed parameterλ in (1) and make the following form:

$$\begin{cases} \Delta u + \lambda u + f(x, u(x)) = 0, & in \quad \Omega, \\ u > 0, & in \quad \Omega, \\ u = 0, & on \quad \partial\Omega, \end{cases} \tag{4}$$

where $\lambda \in \mathbb{R}$. According to the bifurcation theory[18,19], Eq.4 has nontrivial solution branches bifurcated from the trivial solution near the bifurcation points. Along the nontrivial solution branches we can get the solutions to problems (1) by the continuation method when the parameterλ goes to 0.

In this paper we will illustrate bifurcation method by embedding (3) to the nonlinear bifurcation problems with parameter of the following form:

$$\begin{cases} \Delta u + \lambda u + u^p = 0, & in \quad \Omega, \\ u > 0, & in \quad \Omega, \\ u = 0, & on \quad \partial\Omega. \end{cases} \tag{5}$$

Now, let's start with some definitions and theorems which we will use to describe the followings.

DEFINITION 1. For $x \in \mathbb{R}^n$, a set $\Sigma_x = \{\gamma \in \Gamma | \gamma x = x\} \subset \Gamma$ is called isotropy subgroup of x.

DEFINITION 2. If $d^i f (i = 1, 2, \cdots, k-1)$ is everywhere differentiable in U, and $d^{k-1} f : U \to \zeta(X, \zeta^{k-2}(X, Y))$ is differentiable at $x_0 \in U$, then

$$d^k f(x_0) \equiv d(d^{k-1} f(x_0)) \in \zeta(X, \zeta(X, \zeta^{k-2}(X, Y))) = \zeta^k(X, Y)$$

is called k-th differential of f at x_0. And

$$d^k f(x_0)(\nu_1, \cdots, \nu_k) = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} f(x_0 + \sum_{i=1}^{k} t_i \nu_i, \lambda)\Big|_{t_1 = \cdots = t_k = 0}. \tag{6}$$

DEFINITION 3. Assume $f(x, \lambda) = 0$ is an equation with symmetry Z_2, $f : X \times \mathbb{R} \to X$, X is a Banach space, X' is a conjugate space of X. Space X and X' can be decomposed into $X = X_s \oplus X_a, X' = X'_s \oplus X'_a$, where $X_s = \{x \in X : Sx = x\}, X_a = \{x \in X : Sx = -x\}, X'_s = \{y \in X' : yS = y\}, X'_a = \{y \in X' : yS = -y\}$. If there exists a singular point (x_0, λ_0) of f_x on the solution branch $C_s = \{(x, \lambda) : x \in X_s\}$ of X_s, and

$$N(f_x^0) = span(\varphi_0), \qquad \varphi_0 \in X_a$$
$$R(f_x^0) = \{y \in X : \psi_0^T y = 0\}. \qquad \psi_0^T \in X'_a$$

together with $\psi_0^T (f_{xx}^0 \nu_\lambda + f_{x\lambda}^0) \varphi_0 \neq 0$, where ν_λ is the unique solution to $f_x^0 \nu_\lambda + f_\lambda^0 = 0$, then (x_0, λ_0) is called a pitchfork bifurcation point of $f(x, \lambda) = 0$ with symmetry Z_2. Furthermore, if we assume:

$$\psi_0^T \varphi_0 \neq 0,$$
$$\psi_0^T (f_{xxx}^0 \varphi_0 \varphi_0 + 3 f_{xx}^0 \varphi_1) \varphi_0 \neq 0,$$

where $\varphi_1 \in X_s$ is the unique solution to $f_{xx}^0 \varphi_0 \varphi_0 + f_x^0 \varphi_1 = 0$, then (x_0, λ_0) is called simple third-order pitchfork bifurcation point of $f(x, \lambda) = 0$ with symmetry Z_2.

THEOREM 1. (Implicit Function Theorem)

(1). Assume $f(x^0, \lambda^0) = 0, x^0 \in X, \lambda^0 \in \mathbb{R}$, where $f : X \times \mathbb{R} \to X$, X is a Banach space,

(2). $f(x, \lambda)$ and $f_x(x, \lambda)$ are continuously differentiable on their open domain,

(3). $f_x^0 \equiv f_x(x^0, \lambda^0)$ is nonsingular, and $\|(f_x^0)^{-1}\| \leq M_0$.

Then there exists $\rho_1 > 0, \rho_2 > 0$, and for all $\lambda \in (\lambda^0 - \rho_2, \lambda^0 + \rho_2)$, there exists $x(\lambda) \in B_{\rho_1}(x^0)$, $B_{\rho_1}(x^0) = \{x \in X \big| \|x - x^0\| < \rho_1\}$, in which $x = x(\lambda)$ with the following properties:

(4). $x(\lambda^0) = x^0$,

(5). $f(x(\lambda), \lambda) = 0$,

(6). for $\lambda \in (\lambda^0 - \rho_2, \lambda^0 + \rho_2)$, $x(\lambda)$ is the unique solution of $f(x(\lambda), \lambda) = 0$,

(7). $x(\lambda)$ is continuous for $\lambda \in (\lambda^0 - \rho_2, \lambda^0 + \rho_2)$.

PROOF. C.f.[18].

THEOREM 2. (Newton-Kantorovich)

Assume $f : X \to X$,X is a Banach space, and

(1). $\exists x^0 \in X$, $f_x(x^0)$ is nonsingular, and $\|(f_x^0)^{-1}\| \leq \beta$,

(2). $\|f_x^{-1}(x^0) f(x^0)\| \leq \alpha$,

(3). $\exists \rho_0^- > 0$, $\forall x, y \in B_{\rho_0^-}(x^0)$, we have $\|f_x(x) - f_x(y)\| \leq \gamma \|x - y\|$,

(4). $\alpha\beta\gamma < \frac{1}{2}$, $\rho_0^- \leq (1 - \sqrt{1 - 2\alpha\beta\gamma})/\beta\gamma$.

Then series $\{x_n\}$ in Newton iteration

$$x_{n+1} = x_n - f_x^{-1}(x_n) f(x_n), \qquad x_0 = x^0,$$

such that

(5). $x_n \in B_{\rho_0^-}(x^0)$,

(6). series $\{x_n\}$ convergence to $x^* \in B_{\rho_0^-}(x^0)$, which is a unique root of $f(x) = 0$ when $x \in B_{\rho_0^+}(x^0)$, $\rho_0^+ = (1 + \sqrt{1 - 2\alpha\beta})/\beta\gamma$.

PROOF. C.f.[20].

THEOREM 3. (Keller Lemma)

Assume $\mathcal{A} : \mathbb{R}^N \times \mathbb{R}^l \to \mathbb{R}^N \times \mathbb{R}^l$, and

$$\mathcal{A} \equiv \begin{pmatrix} A & B \\ C^T & D \end{pmatrix}$$

where $A : \mathbb{R}^N \to \mathbb{R}^N, B : \mathbb{R}^l \to \mathbb{R}^N, C^T : \mathbb{R}^N \to \mathbb{R}^l, D : \mathbb{R}^l \to \mathbb{R}^l$, then

(1). If A is nonsingular, then $\mathcal{A}$ is nonsingular iff $D - C^T A^{-1} B$ is nonsingular.

(2). If A is singular and $dim N(A) = l$, then $\mathcal{A}$ is nonsingular iff

$$(a) \quad dim R(B) = l, \quad (b) \quad R(B) \cap R(A) = \{0\},$$
$$(c) \quad dim R(C^T) = l, \quad (d) \quad N(A) \cap N(C^T) = \{0\}.$$

(2). If A is singular and $dim N(A) > l$, then $\mathcal{A}$ is singular.

PROOF. C.f.[21,18].

THEOREM 4.

The nontrivial solution branch $(\tilde{x}(\varepsilon), \lambda(\varepsilon)) = (x(\lambda(\varepsilon)) + \varepsilon(\varphi_0 + \omega(\varepsilon)), \lambda(\varepsilon))$ bifurcate from symmetry solution branch $(x(\lambda), \lambda)$ near the symmetry-breaking point (x_0, λ_0) of simple third-order pitchfork bifurcation, then the following assertions are valid.

$$\lambda(0)=\lambda_0, \qquad \omega(0)=0,$$
$$\lambda(-\varepsilon)=\lambda(\varepsilon), \qquad \lambda_\varepsilon(0)=0,$$
$$\lambda_{\varepsilon\varepsilon}(0)=-\frac{\psi_0^T(f_{xxx}^0\varphi_0\varphi_0\varphi_0+3f_{xx}^0\varphi_1\varphi_0)}{3\psi_0^T(f_{xx}^0\varphi_0\nu_\lambda+f_{x\lambda}^0\varphi_0)}, \quad \omega_\varepsilon(0)=\varphi_1/2.$$

PROOF. C.f.[22,18].

III. UNIT SQUARE

(1).Analysis Assume $\Omega=\Omega_0=[0,1]\times[0,1]$, then Eq.5 turns into

$$F(u,\lambda)=\begin{cases}\Delta u+\lambda u+u^p=0, & (x,y)\in\Omega_0,\\ u>0, & (x,y)\in\Omega_0,\\ u=0, & (x,y)\in\partial\Omega_0.\end{cases} \qquad (7)$$

Let $D_4=\{I,R_1,R_2,R_3,S_1,S_2,S_1',S_2'\}$, where

$$\begin{aligned} Iu(x,y)&=u(x,y), & S_1u(x,y)&=u(x,1-y),\\ S_1'u(x,y)&=u(1-x,y), & S_2u(x,y)&=u(y,x),\\ S_2'u(x,y)&=u(1-y,1-x), & R_1u(x,y)&=u(1-y,x),\\ R_2u(x,y)&=u(1-x,1-y), & R_3u(x,y)&=u(y,1-x).\end{aligned}$$

The problem (7) is D_4 equivariant. Especially, if p is odd in (7), Eq.7 is Γ equivariant, where $\Gamma=D_4\times Z_2$, $Z_2=\{I,-I\}$, namely $F(\gamma u,r)=\gamma F(u,r)$. $\forall\gamma\in\Gamma$. The isotropy subgroups of D_4 are $D_4=\{I,R_1,R_2,R_3,S_1,S_2,S_1',S_2'\},\Sigma_R=\{I,R_1,R_2,R_3\},\Sigma_r=\{I,R_2\},\Sigma_1=\{I,S_1\},\Sigma_1'=\{I,S_1'\},\Sigma_2=\{I,S_2\},\Sigma_2'=\{I,S_2'\},\Sigma_d=\{I,R_2,S_2,S_2'\},\Sigma_M=\{I,R_2,S_1,S_1'\}$. Let Σ be one of the above isotropy groups and X^Σ be the invariant subspace of Σ, then the equation (7) yields

$$F_\Sigma(u,\lambda)=0. \quad (u,\lambda)\in X^\Sigma\times\mathbb{R} \qquad (8)$$

Consider the linearized equation of (7) at $u=0$, we get

$$\begin{cases}\Delta\varphi+\lambda\varphi=0, & (x,y)\in\Omega_0,\\ \varphi>0, & (x,y)\in\Omega_0,\\ \varphi=0, & (x,y)\in\partial\Omega_0.\end{cases} \qquad (9)$$

It is well known that Eq.9 always has a trivial solution if we don't consider $\varphi>0$. Further more, Eq.9 has eigenvalues $\lambda_{n,m}=(n^2+m^2)\pi^2$ and corresponding eigenfunctions $\varphi_{n,m}=sin(n\pi x)sin(m\pi y)$. Therefore, $\varphi_{n,m}=sin(n\pi x)sin(m\pi y)$ are roots of Eq.9 when $\lambda=\lambda_{n,m}=(n^2+m^2)\pi^2$. From theory of symmetry-breaking, we know that $\lambda_{n,m}=(n^2+m^2)\pi^2,(n,m=1,2,\cdots)$ are bifurcation points of (9), and there are nontrivial solutions with different symmetries bifurcate from these bifurcate points(see Table 1).

From the analysis above, we know that the solution branch which bifurcates from the first bifurcation point $2\pi^2$ is a positive solution branch. Bifurcation method will be applied to compute the positive solution of (7), and stability analysis of this solution is in subsequent pages.

TABLE I
THE SOLUTION WITN DIFFERENT SYMMETRIES OF EQ.9

Bifurcation point λ	Number of nontrivial solution	Symmetry
$2\pi^2$	1	D_4
$5\pi^2$	2	Σ_1,Σ_2
$8\pi^2$	1	Σ_d
$10\pi^2$	2	D_4,Σ_M
$13\pi^2$	2	Σ_1,Σ_2
$17\pi^2$	2	Σ_1,Σ_2
$18\pi^2$	1	D_4
$20\pi^2$	2	Σ_d,Σ_r

(2).Algorithm For $\lambda_0=\lambda_{1,1}=2\pi^2,\varphi_{1,1}=sin(\pi x)sin(\pi y)$, let

$$\begin{aligned} L&=\Delta+2\pi^2,\\ X&=\{u\big|u\in C^2(\Omega_0),u|_{\partial\Omega_0}=0\},\\ Y&=\{u\big|u\in C^0(\Omega_0)\}.\end{aligned}$$

we define inner product by $\langle u,v\rangle=4\int_0^1\int_0^1 uvdxdy$, L is a Fredholm self-adjoint operator with index zero, and

$$N(L^*)=N(L)=span\{\varphi_{1,1}\}:=span\{\varphi_0\}, \qquad (10)$$

where $N(L)$ and $N(L^*)$ are the null space of L and L^* respectively. Space X and Y have the decomposition

$$X=N(L)\oplus M, \qquad Y=N(L)\oplus R(L),$$

where $M=N(L)^\perp\cap X$, $R(L)$ is the range of L.

Let P be the orthogonal projector from Y to $R(L)$

$$Pz=z-(z,\varphi_0)\varphi_0. \quad z\in Y$$

Eq.7 is equivalent to

$$PF(\tau\varphi_0+\omega,\mu+\lambda_0)=0, \qquad \tau\in\mathbb{R},\omega\in M \qquad (11)$$
$$\langle\varphi_0,F(\tau\varphi_0+\omega,\mu+\lambda_0)\rangle=0. \qquad (12)$$

where $\mu=\lambda-\lambda_0,u=\tau\varphi_0+\omega$. Since $PF_\omega(0,\lambda_0)=PF_u(0,\lambda_0)=PL=L$, and L restricted in M is regular, Eq.11 has a unique solution $\omega=\omega(\tau,\mu)$ which satisfies $\omega(0,0)=0$ by Theorem 1.

Substituting $\omega(\tau,\mu)$ into (12) yields

$$g(\tau,\mu)=\langle\varphi_0,F(\tau\varphi_0+\omega(\tau,\mu),\mu+\lambda_0)\rangle=0. \qquad (13)$$

Then we get

$$F(u,\lambda)=F(\tau\varphi_0+\omega,\mu+\lambda_0)=\Delta\omega+\lambda_0\omega+h(\tau,\mu), \quad (14)$$

where $h(\tau,\mu)=\mu(\tau\varphi_0+\omega)+(\tau\varphi_0+\omega)^p,\omega=\omega(\tau,\mu)$. From Definition 2 above, we get

$$\begin{aligned}(d^kF)_{(0,\lambda_0)}(\nu_1,\cdots,\nu_k)&=\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_k}F(\sum_{i=1}^k t_i\nu_i,\lambda)\Big|_{t_1=\cdots=t_k=0}\\ &=\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_k}\left(\Delta(\sum_{i=1}^k t_i\nu_i)+\lambda_0(\sum_{i=1}^k t_i\nu_i)+(\sum_{i=1}^k t_i\nu_i)^p\right)\Bigg|_{t_1=\cdots=t_k=0}\\ &=\begin{cases}0, & k\neq p \quad \& \quad k\geq 2,\\ p!\sum_{i=1}^p\nu_i, & k=p.\end{cases}\end{aligned} \qquad (15)$$

Especially

$$(d^kF)_{(0,\lambda_0)}(\underbrace{\varphi_0,\cdots,\varphi_0}_{k}) = \begin{cases} 0, & k\neq p \quad \& \quad k\geq 2, \\ p!\varphi_0^k, & k=p. \end{cases} \tag{16}$$

Differentiating Eq.11 with respect to τ, we get

$$PdF(\varphi_0+\omega_\tau)=0, \tag{17}$$

which is evaluated at $(0,0)$ leads to $L\omega_\tau(0,0)=0$ due to $dF(0,\lambda_0)=L,\ \varphi_0\in N(L),\ PL=L$. Since $\omega_\tau(0,0)\in M$ and L restricted in M is regular, it follows that

$$\omega_\tau(0,0)=0. \tag{18}$$

Similarly, differentiating Eq.12 with respect to τ, we get

$$g_\tau(\tau,\mu)=\langle\varphi_0, dF(\varphi_0+\omega_\tau)\rangle, \tag{19}$$

therefore

$$g_\tau(0,0)=\langle\varphi_0, dF_{(0,\lambda_0)}\varphi_0\rangle=0. \tag{20}$$

Similarly, from (15) we get

$$\omega_{\tau^2}(0,0)=\begin{cases} -2L^{-1}P\varphi_0^2, & p=2, \\ 0, & p\geq 3. \end{cases} \tag{21}$$

$$g_\tau(0,0)=\begin{cases} \frac{128}{9\pi^2}, & p=2, \\ 0, & p\geq 3. \end{cases} \tag{22}$$

$$\omega_{\tau^3}(0,0)=\begin{cases} 12L^{-1}(P(\varphi_0L^{-1}P\varphi_0^2)), & p=2, \\ -6L^{-1}(P\varphi_0^3), & p=3, \\ 0, & p\geq 4. \end{cases} \tag{23}$$

$$g_{\tau^3}(0,0)=\begin{cases} -12\langle\varphi_0L^{-1}P\varphi_0^2,\varphi_0\rangle, & p=2, \\ \frac{27}{8}, & p=3, \\ 0, & p\geq 4. \end{cases} \tag{24}$$

While $p\geq 4$, we can get

$$\omega_{\tau^k}(0,0)=0, \qquad k=2,\cdots,p-1 \tag{25}$$

$$\omega_{\tau^p}(0,0)=-p!L^{-1}P\varphi_0^p, \tag{26}$$

$$g_{\tau^k}(0,0)=0, \qquad k=2,\cdots,p-1 \tag{27}$$

Similarly, we can get

$$\omega_{\mu^k}(0,0)=0, \qquad k\in Z^+ \tag{28}$$

$$g_{\mu^k}(0,0)=0, \qquad k\in Z^+ \tag{29}$$

$$g_{\tau^p}(0,0)=\begin{cases} 4p!(\frac{p!!}{(p+1)!!})^2, & p \text{ is odd} \\ \frac{16p!}{\pi^2}(\frac{p!!}{(p+1)!!})^2, & p \text{ is even} \end{cases} \tag{30}$$

$$g_{\tau^{p+1}}(0,0)=0, \quad p\geq 4 \tag{31}$$

$$\omega_{\tau\mu}(0,0)=0, \tag{32}$$

$$g_{\tau\mu}(0,0)=1, \tag{33}$$

$$\omega_{\tau^{k_1}\mu^{k_2}}(0,0)=0, \quad k_1\in Z^+, k_2\in Z^+ \ \&\ k_2\geq 2 \tag{34}$$

$$g_{\tau^{k_1}\mu^{k_2}}(0,0)=0. \quad k_1\in Z^+, k_2\in Z^+ \ \&\ k_2\geq 2 \tag{35}$$

$$\omega_{\tau^2\mu}(0,0)=\begin{cases} 2L^{-1}(PL^{-1}P\varphi_0^2)), & p=2, \\ 0. & p\geq 3 \end{cases} \tag{36}$$

$$g_{\tau^2\mu}(0,0)=\begin{cases} -2\langle L^{-1}P\varphi_0^2,\varphi_0\rangle, & p=2, \\ 0. & p\geq 3 \end{cases} \tag{37}$$

$$g_{\tau^{k-1}\mu}(0,0)=0, \quad k=3,\cdots,p \tag{38}$$

$$g_{\tau^p\mu}(0,0)=\langle\varphi_0,-p!L^{-1}P\varphi_0^p\rangle. \tag{39}$$

Therefore we have approximately

$$\omega(\tau,\mu)=\begin{cases} \frac{1}{2}\omega_{\tau^2}(0,0)\tau^2+\frac{1}{6}\omega_{\tau^3}(0,0)\tau^3+\frac{1}{2}\omega_{\tau^2\mu}(0,0)\tau^2\mu+O(\tau^4), & p=2 \\ \frac{1}{6}\omega_{\tau^3}(0,0)\tau^3+O(\tau^4), & p=3 \\ \frac{1}{p!}\omega_{\tau^p}(0,0)\tau^p+O(\tau^{p+1}). & p\geq 4 \end{cases} \tag{40}$$

$$g(\tau,\mu)=\begin{cases} \tau\mu+\frac{64}{9\pi^2}\tau^2-2\langle\varphi_0L^{-1}P\varphi_0^2,\varphi_0\rangle\tau^3-\langle L^{-1}P\varphi_0^2,\varphi_0\rangle\tau^2\mu+\cdots, & p=2 \\ \tau\mu+\frac{9}{16}\tau^3+\cdots, & p=3 \\ \tau\mu+\frac{16}{\pi^2}(\frac{p!!}{(p+1)!!})^2\tau^p-\langle\varphi_0,L^{-1}P\varphi_0^p\rangle\tau^p\mu+\cdots, & p\geq 4 \ \&\ p \text{ is even} \\ \tau\mu+4(\frac{p!!}{(p+1)!!})^2\tau^p+\cdots. & p\geq 4 \ \&\ p \text{ is odd} \end{cases} \tag{41}$$

Next we want to get the approximative analytic solution of (7). Here we deal with Eq.7 while $p=3, \lambda_1=1$. Substituting $\mu=\lambda_1-\lambda_0$ into (41), we can get

$$\tau=\frac{4}{3}\sqrt{-\mu}=\frac{4}{3}\sqrt{-(\lambda_1-\lambda_0)}=\frac{4}{3}\sqrt{2\pi^2-1}.$$

Then we have

$$u=\frac{4}{3}\sqrt{2\pi^2-1}\times\varphi_0+\omega(\frac{4}{3}\sqrt{2\pi^2-1},1-2\pi^2). \tag{42}$$

In order to know $\omega(\frac{4}{3}\sqrt{2\pi^2-1},1-2\pi^2)$ in (42), we get $\omega(\tau,\mu)=\frac{1}{6}\omega_{\tau^3}(0,0)\tau^3+O(\tau^4)$ from (40). When $p=3$, differentiating Eq.11 with respect to τ three times, then we get

$$6P\varphi_0^3+L\omega_{\tau^3}(0,0)=0. \tag{43}$$

Due to

$$\varphi_0^3=sin^3(\pi x)sin^3(\pi y)=\frac{9}{16}sin(\pi x)sin(\pi y)-\frac{3}{16}sin(\pi x)sin(3\pi y) -\frac{3}{16}sin(3\pi x)sin(\pi y)+\frac{1}{16}sin(3\pi x)sin(3\pi y), \tag{44}$$

we get $P\varphi_0^3=Psin^3(\pi x)sin^3(\pi y)=\frac{1}{16}sin(3\pi x)sin(3\pi y)$, together with ω_{τ^3} is restricted in X_{D^4}, then we can let $\omega_{\tau^3}(0,0)=Csin(3\pi x)sin(3\pi y)$, where C is a undetermined constant. Substituting $\omega_{\tau^3}(0,0)$ into (43) yields

$$\omega_{\tau^3}(0,0)=\frac{3}{128\pi^2}sin(3\pi x)sin(3\pi y). \tag{45}$$

From (42) we can get the approximative positive analytic solution of Eq.7

$$u=\frac{4}{3}\sqrt{2\pi^2-1}sin(\pi x)sin(\pi y)+\frac{(2\pi^2-1)^{\frac{3}{2}}}{128\pi^2}sin(3\pi x)sin(3\pi y). \tag{46}$$

(3).Stability analysis

We always have trivial solution branch $(u,\lambda)=(0,\lambda)$ for equation $F(u,\lambda)=u+\lambda u+u^p=0$. When the trivial solution branch cross the bifurcation point $(0,\lambda_0)$, one eigenvalue of F_u^0 equals zero, and all the others are less than 0. So we can know that the sign of the "special" eigenvalue determines stability of the trivial solution when eigenvalue λ cross λ_0.

As mentioned in Sec.2, the meaning of $\varphi_0,\psi_0,\nu_\lambda,\varphi_1$ are given by Definition 3. In addition, we construct l_0 which satisfies $l_0^T\varphi_0=1$.

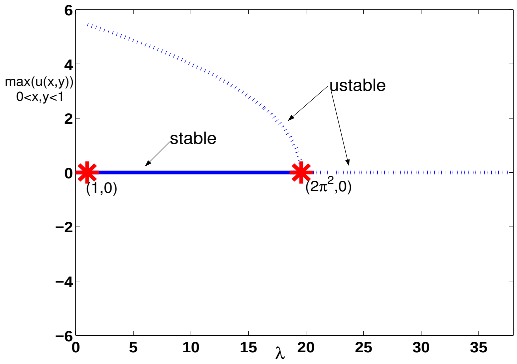

Fig.1. Trivial and nontrivial positive solution branches of (7).

Now we consider the stability of the trivial solution branch$(u,\lambda)=(0,\lambda)$, let

$$G(z,\lambda)=\begin{pmatrix}F_u(u,\lambda)\varphi-\sigma\varphi\\ l_0^T\varphi-1\end{pmatrix}=0, \quad (47)$$

where $z=(\varphi,\sigma)$, $z_0=(\varphi_0,0)$, differentiating $G(z,\lambda)=0$ with respect to z at (u_0,λ_0) we get $G_z^0=\begin{pmatrix}F_u^0 & -\varphi_0\\ l_0^T & 0\end{pmatrix}$. From $\psi_0^T\varphi_0\neq 0$, we know G_z^0 is nonsingular by Keller lemma. So we get that (47) has a unique solution branch $(z(\lambda),\lambda)=(\varphi(\lambda),\sigma(\lambda),\lambda)$ which satisfies $\sigma(\lambda_0)=0,\varphi(\lambda_0)=\varphi_0$ by the Implicit function theorem. Differentiating $F_u(u,\lambda)\varphi-\sigma\varphi=0$ with respect to λ at $\lambda=\lambda_0$, then we have

$$(F_{uu}^0u_\lambda^0+F_{u\lambda}^0)\varphi_0+F_u^0\varphi'(\lambda_0)-\sigma'(\lambda_0)\varphi_0=0, \quad (48)$$

so we get $\sigma'(\lambda_0)=\frac{\psi_0^T(F_{uu}^0\nu_\lambda^0+F_{u\lambda}^0)\varphi_0}{\psi_0^T\varphi_0}>0$. From analysis above, we know the trivial solution branch $(u,\lambda)=(0,\lambda)$ is stable for $\lambda<\lambda_0$, is unstable when $\lambda>\lambda_0$(see Fig.1).

REMARK 1. In fact, during actual computation, we always

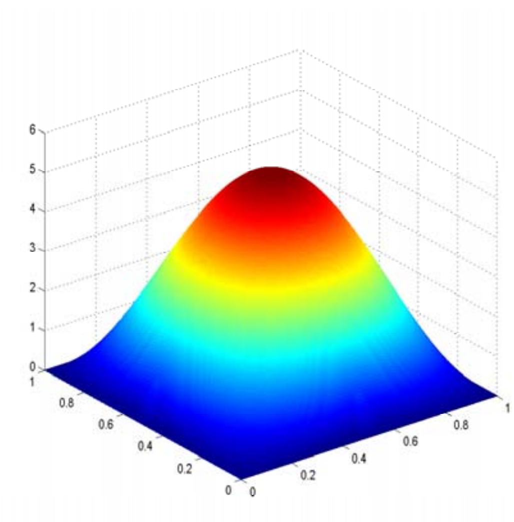

Fig.2. Approximative positive analytic solution of (7) while $p=3,\lambda_1=1$.

use $\frac{1}{\varepsilon}[F_u(u_0+\varepsilon\nu_\lambda,\lambda_0)-F_u(u_0,\lambda_0)]$ to substitute $F_{uu}^0\nu_\lambda$ in $\sigma'(\lambda_0)$. Similarly we can solve others such as $F_{u\lambda}^0$, etc.

Then we discuss stability of the nontrivial solution branch$(u,\lambda)=(\widetilde{u}(\varepsilon),\lambda(\varepsilon))$, let

$$H(z,\varepsilon)=\begin{pmatrix}F_u(\widetilde{u}(\varepsilon),\lambda(\varepsilon))\varphi(\varepsilon)-\sigma_1\varphi(\varepsilon)\\ l_0^T\varphi-1\end{pmatrix}=0, \quad (49)$$

where $z=(\varphi,\sigma_1)$, $z_0=(\varphi_0,0)$, $H_z^0=\begin{pmatrix}F_u^0 & -\varphi_0\\ l_0^T & 0\end{pmatrix}$ is nonsingular at $\varepsilon=0$. So we get that Eq.49 has a unique solution branch $(z(\varepsilon),\varepsilon)=(\varphi(\varepsilon),\sigma_1(\varepsilon),\varepsilon)$ which satisfies $\sigma_1(\lambda_0)=0$, $\varphi(\lambda_0)=\varphi_0$ by the Implicit function theorem. Differentiating $F_u(\widetilde{u}(\varepsilon),\lambda(\varepsilon))\varphi(\varepsilon)-\sigma_1\varphi(\varepsilon)=0$ with respect to ε at $\varepsilon=0$, we get

$$[F_{uu}^0(\nu_\lambda\lambda_\varepsilon(0)+\varphi_0)+F_{u\lambda}^0\lambda_\varepsilon(0)]\varphi_0+F_u^0\varphi'(0)-\sigma_1'(0)\varphi_0=0. \quad (50)$$

From Theorem 4, we know $\lambda_\varepsilon(0)=0$, $\psi_0^TF_{uu}^0\varphi_0\varphi_0=0$ at simple third-order pitchfork bifurcation point. By (50), we get $\sigma_1'(0)$, $\varphi'(0)=\varphi_1$, so we know stability of the nontrivial solution branch is determined by $\sigma_1''(0)$. In order to know $\sigma_1''(0)$, we differentiate $F_u(\widetilde{u}(\varepsilon),\lambda(\varepsilon))\varphi(\varepsilon)-\sigma_1\varphi(\varepsilon)=0$ with respect to ε two times at $\varepsilon=0$, and then we have

$$(F_{uuu}^0\widetilde{u}_\varepsilon^0\widetilde{u}_\varepsilon^0+F_{uu}^0\widetilde{u}_{\varepsilon\varepsilon}^0+F_{u\lambda}^0\lambda_{\varepsilon\varepsilon}^0)\varphi_0+2F_{uu}^0\widetilde{u}_\varepsilon^0\varphi'(0)+F_u^0\varphi''(0)-\sigma_1''(0)\varphi_0=0. \quad (51)$$

Substituting $\varphi'(0)=\varphi_1,\widetilde{u}_\varepsilon^0=\varphi_0,\widetilde{u}_{\varepsilon\varepsilon}^0=\nu_\lambda\lambda_{\varepsilon\varepsilon}^0+\varphi_0$ into (51), we can compute $\sigma''(0)$. For example, let $p=3$, and then we have numerical result $\sigma''(0)>0$, so we know that the bifurcation is subcritical, and the nontrivial solution branch is unstable(see Fig.1).

Simultaneously, we can use (41) to illustrate stability of Eq.7. By theorem of singularity[23,24], we know that Eq.7 is strong equivalence with

$$\begin{cases}\tau\mu+\frac{16}{\pi^2}(\frac{p!!}{(p+1)!!})^2\tau^p=0, & p \text{ is even}\\ \tau\mu+4(\frac{p!!}{(p+1)!!})^2\tau^p=0. & p \text{ is odd}\end{cases} \quad (52)$$

That is, Eq.7 has the same qualitative property as Eq.52 near the bifurcation point $\lambda_0=2\pi^2$. Assume $p=3,\lambda_1=1$, Eq.7 has the same stability as $\tau\mu+\frac{9}{16}\tau^3=0$ near the bifurcation point $\lambda_0=2\pi^2$, so we also know that the bifurcation point $\lambda_0=2\pi^2$ is of pitchfork type and the nontrivial solution branch is unstable. We can find that two methods above are in perfect accord with each other, that is, system (7) has subcritical bifurcation at $\lambda=2\pi^2$. We use bifurcation numerical method which will be showed in Sec.4 to get a bifurcation graph from $\lambda=2\pi^2$ to$\lambda=1$(see Fig.1).

In addition, it's easy to know that equation $F(u,\lambda)=\Delta u+\lambda u+u^3=0$ is Z_2 equivariant, that is, this equation also has a nontrivial unstable negative solution branch, which in not shown in Fig.1.

IV. Other complex domains

(1). **Numerical method**

The algorithms here are the same as in Sec.3. The only difference is that the bifurcation point λ_0 and the corresponding eigenfunctions φ_0 must be computed numerically.

We use finite difference method to discrete Eq.5, then we get numerical solutions of (2) and (3). A more detailed algorithm is following:

Step 1: General domain Ω is divided homogeneously, the five-point difference scheme is used to discrete Laplace operator Δ , and it keeps the same symmetry as the original problem for the discretized problem to simplify the computation, where we choose $N = 200, h = 1/200$.

Step 2: We store the data of the five-point difference operator Δ_h in A , and compute the eigenvalue problem $A\varphi_h = \lambda_h\varphi_h$, where φ_h and λ_h are approximations to φ_0 and λ_0.

Step 3: Let

$$u = \tau\varphi_h + \omega, \qquad \eta = \lambda - \lambda_h, \tag{53}$$

where τ is a small parameter, and ω satisfies $(\varphi_h, \omega) = 0$. Substituting (53) into (5), we have

$$\begin{cases} \Delta\omega + (\eta + \lambda_h)\omega + \eta\tau\omega + (\tau\varphi_h + \omega)^p = 0, & (x, y) \in \Omega_0, \\ \omega > 0, & (x, y) \in \Omega_0, \\ \omega = 0, & (x, y) \in \partial\Omega_0, \\ (\varphi_h, \omega) = 0. \end{cases} \tag{54}$$

The Gauss-Newton method is used to solve this nonlinear equation for different τ from $\tau = 0$ to end $\tau = \tau_{end}$. τ_{end} must be chosen big enough in order that the nontrivial solutions of (54) are faraway from the trivial solution.

Step 4: Continue λ until $\lambda = \lambda_1$, and then we get the nontrivial solution $u(x, y)$ of (3) and plot it.

(2).**Visualization of positive solutions of (3) in many complex domains**($p = 3, \lambda_1 = 1$)

TABLE II
THE SOLUTIONS WITN DIFFERENT SYMMETRIES TO EQ.3
($p = 3, \lambda_1 = 1$)

Shape of the domain	Bifurcation point λ	Symmetry
Unit square(Fig.3)	19.739	D_4
Unit disk(Fig.4)	22.976	$O(2)$
L-shaped domain(Fig.5)	38.576	Σ_2'
Unsymmetrical annulus(Fig.6)	49.102	Σ_1
Annulus(Fig.7)	212.166	$O(2)$
The exterior of a "Butterfly"(Fig.8)	64.805	Σ_d
Heart(Fig.9)	60.555	Σ_1'
Crisscross(Fig.10)	57.610	D_4
Ellipse(Fig.11)	56.382	Σ_M
Dumbbell shaped domain(Fig.12)	189.157	Σ_M

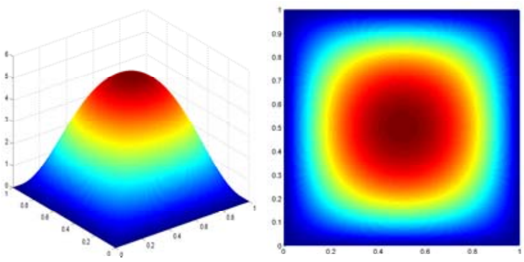

Fig.3. Positive solution of (3) on square with $p = 3, \lambda_1 = 1$.

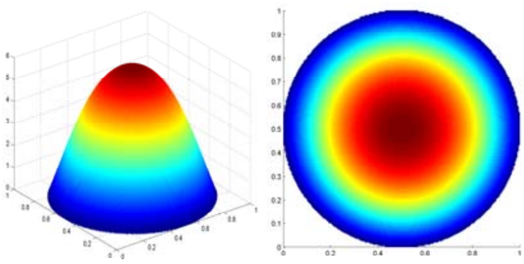

Fig.4. Positive solution of (3) on disk with $p = 3, \lambda_1 = 1$.

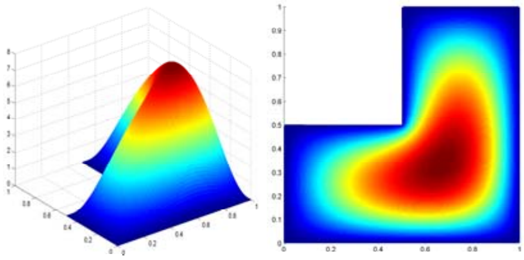

Fig.5. Positive solution of (3) on a L-shaped domain with $p = 3, \lambda_1 = 1$.

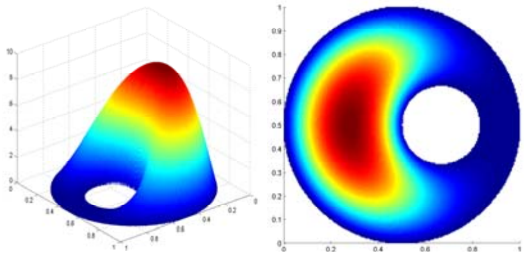

Fig.6. Positive solution of (3) on unsymmetrical annulus with $p = 3$, $\lambda_1 = 1$.

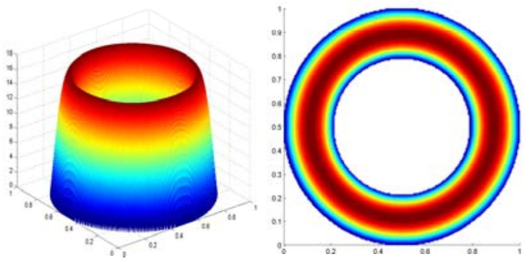

Fig.7. Positive solution of (3) on annulus with $p = 3, \lambda_1 = 1$.

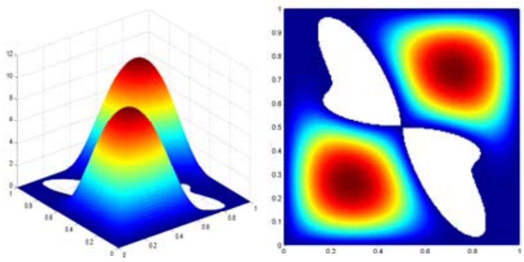

Fig.8. Positive solution of (3) on the exterior of a "Butterfly" with $p = 3$, $\lambda_1 = 1$.

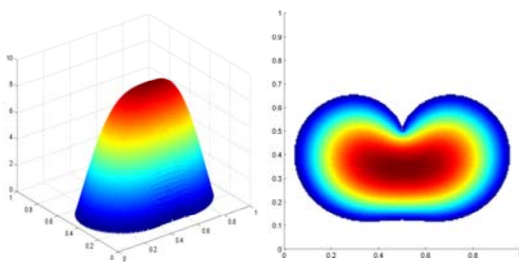

Fig.9. Positive solution of (3) on a heart domain with $p = 3, \lambda_1 = 1$.

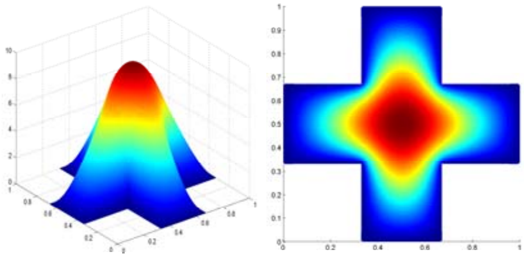

Fig.10. Positive solution of (3) on crisscross with $p = 3, \lambda_1 = 1$.

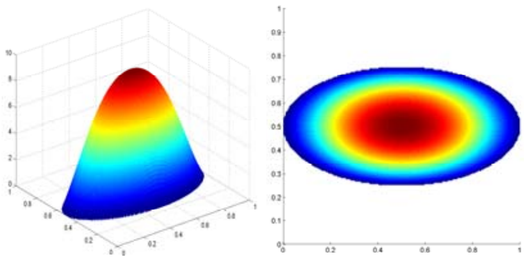

Fig.11. Positive solution of (3) on ellipse with $p = 3, \lambda_1 = 1$.

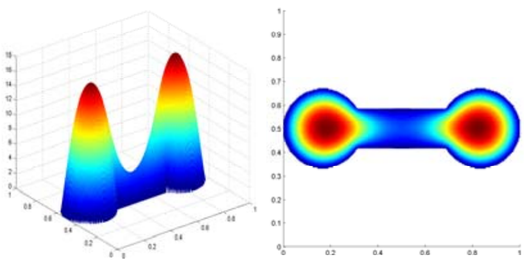

Fig.12. Positive solution of (3) on a dumbbell shaped domain with $p = 3$, $\lambda_1 = 1$.

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