Some Remarks About Riemann-Liouville and Caputo Impulsive Fractional Calculus

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Abstract—This paper establishes some closed formulas for Riemann-Liouville impulsive fractional integral calculus and also for Riemann-Liouville and Caputo impulsive fractional derivatives.

Keywords—Riemann-Liouville fractional calculus, Caputo fractional derivative, Dirac delta, Distributional derivatives, Higher-order distributional derivatives.

I. INTRODUCTION

Fractional calculus has been used in a set of applications, mainly, to deal with modelling errors in differential equations and dynamic systems. There are also applications in Signal Processing and sampling and hold algorithms, [1-3]. Fractional integrals and derivatives can be of non-integer orders and even of complex order. This facilitates the description of some problems which are not easily described by ordinary calculus due to modelling errors, [1-5]. There are several approaches for the integral fractional calculus, the most popular ones being the Riemann-Liouville fractional integral. There is also a fractional Riemann-Liouville derivative. However, the well-known Caputo fractional derivative is less involved since the fractional Riemann-Liouville derivative. There is also a fractional calculus, the most popular ones being the fractional Riemann-Liouville impulsive fractional integral calculus and also Caputo impulsive fractional derivatives.

II. GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL

Let us denote the set of positive real numbers by $\mathbb{R}_+ = \{ r \in \mathbb{R} : r > 0 \}$ and left-sided and right-sided Lebesgue integrals, respectively, as:

$$\int_0^x g(\tau) d\tau := \lim_{t \to x^-} \int_0^t g(\tau) d\tau \quad \text{(the identification)}$$

$x = x^-$ is used for all $x$ in order to simplify the notation, and

$$\int_0^x g(\tau) d\tau := \lim_{t \to x^+} \int_0^t g(\tau) d\tau \quad \text{(the identification)}$$

Now, consider real functions $f, \tilde{f} : \mathbb{R}_+ \to \mathbb{R}$, such that

$$\int_0^x (x-t)^{-\mu-1} \tilde{f}(t) dt \text{ exists, } \forall x \in \mathbb{R}_+,$$

fulfilling:

$$f(x) = \tilde{f}(x) + \sum_{i,j \in \mathbb{I}} K_j \delta(x-x_i) \quad \text{(2)}$$

where $\mathbb{I}$ denotes the Dirac delta distribution, $K_j \in \mathbb{R}$ and $x \in \mathbb{I}$ with $x \in \mathbb{I}$ and $x \in \mathbb{I}$.

If we are interested in the whole impulsive set defined via empty or non-empty) partial impulsive strictly ordered denumerable sets:

$$IM(x) := \{ x \in \mathbb{I} : f(x) = K_j \delta(0), x < x_i \} \quad \text{(1)}$$

of indexing set

$$I(x) := \{ x \in \mathbb{I}, x \in \mathbb{I} : x \in \mathbb{I} \}$$

for each $x \in \mathbb{I}$, and

$$IM(x) = \bigcup_{x \in \mathbb{I}} IM(x) \quad \text{(2)}$$

for each $x \in \mathbb{I}$, with the indexing set of $IM$ being

$$I(x) = \bigcup_{x \in \mathbb{I}} IM(x) \quad \text{(2)}$$

studying the fractional derivative of the impulsive function $f : \mathbb{R}_+ \to \mathbb{R}$ then $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}$ is non- uniquely defined as

$$\tilde{f}(x) = f(x) \quad \text{for } x \in \mathbb{I}_+ \\text{and } \tilde{f}(x) = \tilde{f}(x) \quad \text{for } x \in \mathbb{I}_+$$

Note that there exist infinite impulsive cases if the impulsive function $f(x)$ is non- uniquely defined as

$$\tilde{f}(x) = f(x) \quad \text{for } x \in \mathbb{I}_+$$

and $\tilde{f}(x) = \tilde{f}(x)$ if $x \in \mathbb{I}_+$. Note that $IM$ and $I(x)$ have infinite cardinals if there are infinitely many impulsive values of the function $f(t)$.

Note that the existence of

$$\int_0^x (x-t)^{-\mu-1} \tilde{f}(t) dt \text{ implies that}$$

of

$$\int_0^x (x-t)^{-\mu-1} f(t) dt = \int_0^x (x-t)^{-\mu-1} \tilde{f}(t) dt \text{ if } x \in \mathbb{I}_+,$$

since

$$\int_0^x (x-t)^{-\mu-1} \tilde{f}(t) dt \text{ exists, and that of}$$

$$\int_0^x (x-t)^{-\mu-1} f(t) dt = \int_0^x (x-t)^{-\mu-1} \tilde{f}(t) dt + \sum_{i,j \in \mathbb{I}} K_j \delta(x-x_i) \quad \text{(3)}$$

if $x \in \mathbb{I}_+$.
Theorem 2.1. The extended fractional Riemann-Liouville integrals by considering impulsive functions are defined for any fixed order \( \mu \in \mathbb{R}_+ \) and all \( x \in \mathbb{R}_+ \) by

\[
\left( J^\mu f \right)(x) := \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt
\]

Then, the Caputo fractional derivative of order \( \mu \geq 0 \) with \( m-1 \leq \mu \leq m \), \( m \in \mathbb{Z}_+ \) for any \( x \in \mathbb{R}_+ \):

\[
\left( D^\mu f \right)(x) := \frac{1}{\Gamma(m-\mu)} \left( \frac{d}{dx} \right)^m \left( \int_0^x (x-t)^{m-\mu-1} f(t) dt \right)
\]

The following particular cases follow from this formula for \( \mu = m-1 \):

(a) \( \mu = -1; m = 0 \) yields \( \left( D^{-1} f \right)(x) = \int_0^x f(t) dt \) which is the standard integral of the function \( f \). This case does not verify the “derivative constraint” \( 0 \leq m-1 \leq \mu \in \mathbb{R}_+ \) leading to an integral result.

(b) \( \mu = 0; m = 1 \) yields \( \left( D^0 f \right)(x) = f(x) \) which so that \( D^0 f \) is the identity operator

(c) \( \mu = 1; m = 2 \) yields \( \left( D^1 f \right)(x) = f'(x) \)

(d) \( \mu = 2; m = 3 \) yields \( \left( D^2 f \right)(x) = f''(x) \) which is the standard first-derivative of the function \( f \).

Compared to the parallel cases with the Caputo fractional derivative, note that the Riemann-Liouville fractional derivative, compared to the Caputo corresponding one, does not depend on the conditions at zero of the function and its derivatives. Define the Kronecker delta \( \delta(a,b) \) of any pair of real numbers \( (a,b) \) as \( \delta(a,b) = 1 \) if \( a = b \) and \( \delta(a,b) = 0 \) if \( a \neq b \) and then evaluate recursively the Riemann-Liouville fractional derivative of order \( \mu \geq 0 \) from the above formula by using Leibniz's differentiation rule by noting that, since \( \mu \neq m-j-1 \), \( j \in \mathbb{Z}_+ \), only the differential part corresponding to the differentiation of the integrand is non zero for \( j > m-\mu \). This yields the following result:

Theorem 3.1. Assume that \( f \in C^{m-1}(\mathbb{R}_+ \setminus \{0\}) \) and its \( m-1 \)th derivative exists everywhere in \( \mathbb{R}_+ \). Then, the Caputo fractional derivative of order \( \mu \geq 0 \) with \( m-1 \leq \mu \in \mathbb{R}_+ \leq m \), \( m \in \mathbb{Z}_+ \) for any \( x \in \mathbb{R}_+ \):

\[
\left( D^\mu f \right)(x) := \frac{1}{\Gamma(m-\mu)} \left( \frac{d}{dx} \right)^m \left( \int_0^x (x-t)^{m-\mu-1} f(t) dt \right)
\]

and if \( x \notin \text{IMP} \), since \( I(x^+) = I(x) \), then

\[
\left( J^\mu f \right)(x^+) = \left( J^\mu f \right)(x).
\]
If \( f \in PC^k(R_+, R) \) with \( f^{(k)}(x) \) being discontinuous of first class then \( f^{(m-1)}(x) = \delta (j(x)) \) with \( j(x) = m-1-k(x) \), one uses to obtain the right value of (8) the perhaps high-order distributional derivatives formula:

\[
\int (m-1)(x) - f^{(m-1)}(x) = \frac{(-1)^k k!}{x^k} f^{(m-1-k)}(x) |\delta(0)| = \infty
\]

to yield

\[
(D^\mu f)(x) = \frac{1}{(m-\mu)} \left[ \frac{(-1)^{(m-\mu)} x^{(m-\mu)}}{(m-\mu)!} \right] f^{(m-\mu)}(x) - \int \delta(0) \delta(\mu, m-1) + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-i) + \sum_{j=0}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-j)
\]

If \( \mu = m-1 \) then

\[
(D^{m-1} f)(x) = f^{(m-1)}(x) + \sum_{j=0}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right)
\]

provided that \( \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \) exists for \( x \in R_+ \). This is guaranteed if \( f(t) \) is Lebesgue-integrable on \( R_+ \), \( f \in C^{m-1} (R_+, R) \) and \( f^{m-1} \) exists everywhere in \( R_+ \). The correction (10) applies when the derivative does not exist. \( \Box \)

If \( \mu = m-1 \) with \( m-1 \leq \mu (\in R_+) \leq m \) then after defining the impulsive sets, its associated indexing sets and the function \( \tilde{f} : R_+ \rightarrow R \) as for the extended Riemann-Liouville fractional integral, one gets:

\[
\int \frac{1}{(m-\mu)} \left[ \frac{(-1)^{(m-\mu)} x^{(m-\mu)}}{(m-\mu)!} \right] f^{(m-\mu)}(x) - \int \delta(0) \delta(\mu, m-1) + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-i) + \sum_{j=0}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-j)
\]

\[
= \frac{1}{(m-\mu)} \left[ \frac{(-1)^{(m-\mu)} x^{(m-\mu)}}{(m-\mu)!} \right] f^{(m-\mu)}(x) - \int \delta(0) \delta(\mu, m-1) + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-i) + \sum_{j=0}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-j)
\]

\[
= \frac{1}{(m-\mu)} \left[ \frac{(-1)^{(m-\mu)} x^{(m-\mu)}}{(m-\mu)!} \right] f^{(m-\mu)}(x) - \int \delta(0) \delta(\mu, m-1) + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-i) + \sum_{j=0}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-j)
\]

IV. GENERALIZED CAPUTO FRACTIONAL DERIVATIVE

Assume that \( f \in C^{m-1} (R_+, R) \) and its \( m-th \) derivative exists everywhere in \( R_+ \). Then, the Caputo fractional derivative of order \( \mu \geq 0 \) with \( m-1 \leq \mu (\in R_+) < m \), \( m \in Z_+ \) is for any \( x \in R_+ \):

\[
(D^\mu f)(x) = \frac{1}{(m-\mu)} \left[ \frac{(-1)^{(m-\mu)} x^{(m-\mu)}}{(m-\mu)!} \right] f^{(m-\mu)}(x) - \int \delta(0) \delta(\mu, m-1) + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-i) + \sum_{j=0}^{m-1} [j - \mu] \left( \int_0^x (x-t)^{-(\mu+i)} f(t) dt \right) \delta(\mu, m-j)
\]

The following particular cases occur with \( \mu = m-1 \) leading to

\[
(D^{m-1} f)(x) = \int_0^x f(t) dt = f^{(m-1)}(x) - f^{(m-1)}(0^+)
\]

(a) \( \mu = -1; m = 0 \) yields \( (D^{-1} f)(x) = f^{(-1)}(x) - f^{(-1)}(0^+) \) which is an integral result \( f \). Note that this case does not verify the “derivative constraint” \( 0 \leq \mu (\in R_+) < m \) leading to an integral result.

(b) \( \mu = 0; m = 1 \) yields

\[
(D^0 f)(x) = f(0^+) = f(0^+) - f(0^+)
\]

(c) \( \mu = 1; m = 2 \) yields

\[
(D^1 f)(x) = f^{(0)}(x) - f^{(0)}(0^+)
\]

(d) \( \mu = 2; m = 3 \) yields

\[
(D^2 f)(x) = f^{(2)}(x) - f^{(2)}(0^+)
\]

We can extend the above formula to real functions with impulsive \( m-th \) derivative as follows. Assume that \( f \in C^{m-2} (R_+, R) \) with bounded piecewise \( (m-1)-th \)
derivative existing everywhere in $\mathbb{R}_+$ for some $0 < t < T$. Then

\[
\frac{d}{dx^n} f(x) = \frac{d}{dx^n} \left( f(x) - f(x^-) \right)
\]

(16)

Now, consider $f \in C^{m-1}(0, \infty)$ with $0 < m < \infty$. Then

\[
\frac{d}{dx^n} f(x) = \frac{d}{dx^n} \left( f(x) - f(x^-) \right)
\]

(17)

Continuous in $\mathbb{R}_+$ except possibly on a non-empty discrete set $\text{IMP}$. Define a non-impulsive real function $\hat{f}: \mathbb{R}_+ \rightarrow \mathbb{R}$ defined as

\[
\hat{f}(x) = f(x) - f(x^-)
\]

(18)

and if $f \in C^{m-1}(0, \infty)$ then

\[
\frac{d}{dx^n} f(x) = \frac{d}{dx^n} \left( f(x) - f(x^-) \right)
\]

(19)

where $n: \text{IMP} \rightarrow \mathbb{Z}_+$ is a discrete function defined by $n(x) = \text{card } I(x) - \text{card } \text{IMP}(x)$.

Assume that $f \in PC^{m-1}(\mathbb{R}_+, \mathbb{R})$ and $x$ is a discontinuity point of first class of $f(x)$ for some $j \in m-1 \cup \{0\}$. Then, $f(x)$ is impulsive for $\ell \in \ell$ and $j \in j - 1$ th impulsive sets of the function $f$ on $(0, \infty)$ for $x \in \mathbb{R}_+$ as:

\[
\text{IMP}_{j+1}(x) := \left\{ x \in \mathbb{R}_+ : 0 < f(x^+) - f(x^-) < \infty \right\}
\]

(20)

This leads directly the definition of the following impulsive sets:

\[
\text{IMP}_{j+1} := \left\{ x \in \mathbb{R}_+ : 0 < f(x^+) - f(x^-) < \infty \right\}
\]

(21)

which can be empty. Thus, if $z \in \text{IMP}_{j+1}$ then $f(x^+) = f(x^-)$ exists with identical left and right
limits, 
\[ f^{(j)}(x^+) - f^{(j)}(x^-) = K = \mathcal{K}(x) \neq 0 \] and
\[ f^{(j)}(x^-) = \mathcal{K} \delta(0) \]
with successive higher-order derivatives represented by higher-order Dirac distributional derivatives.

The above definitions yield directly the following simple results:

**Assertion 5.2.** \( x \in \text{IMP} \Rightarrow x \in \text{IMP}_j \) for a unique \( j = j(x) \in \mathbb{N} \).

**Proof.** Proceed by contradiction. Assume that \( x \in \left( \text{IMP}_{j+1} \cap \text{IMP}_{j+1} \right) \) for \( i, j \neq i \in \mathbb{N} - \{0\} \). Then:
\[ 0 < |f^{(j)}(x^+) - f^{(j)}(x^-)| < \infty \]
\[ 0 < |f^{(j)}(x^+) - f^{(j)}(x^-)| < \infty \]
Assume with no loss of generality that \( j = i + k > i \) for some \( k \leq m - i - 1 \in \mathbb{N} \). Then,
\[ f^{(j)}(x^+) = f^{(j)}(x^-) = f^{(j)}(x^+) - f^{(j)}(x^-) \delta(0) = \infty \]
with \( x \in \mathbb{R} \). If \( f^{(j)}(x^+) = f^{(j)}(x^-) \neq 0 \) which contradicts
\[ 0 < |f^{(j)}(x^+) - f^{(j)}(x^-)| < \infty \]
so that \( i = j \). \( \square \)

**Assertion 5.3.** \( x \in \text{IMP} \Rightarrow \exists \) a unique \( j = j(x) = \max_{i \in \mathbb{N}} \left| f^{(i)}(x^+) - f^{(i)}(x^-) \right| < \infty \)

Furthermore, such a unique \( j = j(x) \) satisfies
\[ f^{(j)}(x^+) - f^{(j)}(x^-) > 0 \]

**Proof.** The existence is direct by contradiction. If \( \exists \) \( j(x) \) such that
\[ \left| f^{(j)}(x^+) - f^{(j)}(x^-) \right| < \infty \] then \( x \notin \text{IMP} \). Now, assume there exist two nonnegative integers \( i = i(x) = \max_{i \in \mathbb{N}} \left| f^{(i)}(x^+) - f^{(i)}(x^-) \right| < \infty \) and
\( j = j(x) = i + k = \max_{i \in \mathbb{N}} \left| f^{(i+k)}(x^+) - f^{(i+k)}(x^-) \right| < \infty \); for some \( k \leq m - i \). But for \( x > 0 \),
\[ x \in \text{IMP} \Rightarrow \exists \ j = j(x) = \max_{i \in \mathbb{N}} \left| f^{(i)}(x^+) - f^{(i)}(x^-) \right| < \infty \]
which is unique. Also, from the definition of the impulsive sets \( \text{IMP}_j(x) \).

Now, assume that \( x \in \mathbb{R} \). Then:
\[ j = j(x) = \max_{i \in \mathbb{N}} \left| f^{(i)}(x^+) - f^{(i)}(x^-) \right| < \infty \]

from the definition of the impulsive sets. Then, \( x \in \text{IMP} \).

The opposite logic implication
\[ j = j(x) = \max_{i \in \mathbb{N}} \left| f^{(i)}(x^+) - f^{(i)}(x^-) \right| \]

is proved. Then, it has been fully proved that \( x \in \text{IMP} \Rightarrow \)
\[ x \in \text{IMP}_j \Rightarrow \exists \text{ a unique } j = j(x) = \max_{i \in \mathbb{N}} \left| f^{(i)}(x^+) - f^{(i)}(x^-) \right| < \infty \]

Now, establish again a contradiction by assuming that
\[ j = j(x) = \max_{i \in \mathbb{N}} \left| f^{(i)}(x^+) - f^{(i)}(x^-) \right| \]

what contradicts \( x \in \text{IMP} \). This proves that the unique \( j = j(x) \) implying and being implied by \( x \in \text{IMP}_j \) satisfies
\[ f^{(j)}(x^+) - f^{(j)}(x^-) > 0 \]

Using the necessary high order distributional derivatives, one gets that
\[ x \in \text{IMP} \Rightarrow f^{(m)}(x) = \left( \frac{(-1)^{m-j}(m-j)!}{x^{m-j}} \right) f^{(j)}(x^+) \delta(0) \]

with \( j = j(x) \) being uniquely defined so that
\[ 0 < f^{(j)}(x^+) - f^{(j)}(x^-) < \infty \]. Thus, the m-th distributional derivative of \( f : \mathbb{R} \rightarrow \mathbb{R} \) can be represented as:
\[ f^{(m)}(x) = \tilde{f}^{(m)}(x) - \sum_{j \in \text{IMP}_j, i} \left( \frac{(-1)^{m-j}(m-j)!}{x^{m-j}} \right) f^{(j)}(x^+) \delta(0) \]
\( x \in \mathbb{R} \), \( j \) being uniquely defined for each \( x \) so that
\( x \in \text{IMP}_j \), where \( \tilde{f} \in C \mathbb{R} \) with everywhere continuous first-derivative defined as \( \tilde{f}^{(j)}(x) = f^{(j)}(x) \); \( x \in \mathbb{R} \), \( \tilde{f}(0) = 0 \). The above formula is applicable if \( f \notin PC \mathbb{R} \) but it is also applicable if \( f \notin PC \mathbb{R} \) yielding:
\[ f^{(m)}(x) = \tilde{f}^{(m)}(x) \]
\( x \in \text{IMP} \)
\[ f^{(m)}(x) = \tilde{f}^{(m)}(x) \]
"
\begin{align*}
  f^{(m-1)}(x) &= \tilde{f}^{(m-1)}(x) \\
  f^{(m-1)}(x^+) &= f^{(m-1)}(x) + \left( f^{(m-1)}(x^+) - f^{(m-1)}(x) \right) \\
  &\text{if } x \in \text{IMP} \text{ and } j = m-1
\end{align*}

for a unique \( j = j(x) \in \mathbb{m-1} \cup \{0 \} \) from Assertion 1. Denote further sets related to impulses as follows:

\begin{align*}
  \text{IMP}(x) &= \{ z \in \text{IMP} : z < x \} \\
  \text{IMP}(x^+) &= \{ z \in \text{IMP} : z \leq x \}
\end{align*}

being indexed by two subsets of integers of the same corresponding cardinals defined by:

\[ I(x) = \tilde{j} = j(x) \]

indexing the members \( z_i \) of \( \text{IMP}(x) \) in increasing order \( I(x^+) \), being either \( I(x) \) or \( I(x) + 1 \), indexing the members \( z_i \) of \( \text{IMP}(x^+) \) in increasing order \( I(x^+) \).

The following result holds:

**Theorem 5.4.** The Caputo fractional derivative of \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) of order \( \mu \in \mathbb{R}_+ \) satisfying \( m-1 < \mu \leq m \), where \( m \in \mathbb{Z}_+ \) and all \( x \in \mathbb{R}_+ \) is after using distributional derivatives becomes in the most general case:

\begin{align*}
  \left( D^\mu f \right)(x) &= \frac{1}{\Gamma(m-\mu)} \int_0^t (t-x)^{m-\mu-1} \tilde{f}^{(m)}(t) dt \\
  &= \frac{1}{\Gamma(m-\mu)} \left( \int_0^t (t-x)^{m-\mu-1} \tilde{f}^{(m)}(t) dt + \sum_{i \in I(x)} (m-j(x)_i-1) \right) \\
  &= \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)_+, j \in \{0\}} \int_{x_i}^{x_i+} (t-x)_i^{m-\mu-1} \tilde{f}^{(m)}(t) dt \\
  &\quad + \frac{1}{\Gamma(m-\mu)} \int_{x_i}^{x_i+} (t-x)_i^{m-\mu-1} \tilde{f}^{(m)}(t) dt \\
  &\quad + \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (m-j(x)_i-1) \int_{x_i}^{x_i+} (t-x)_i^{m-\mu-1} \tilde{f}^{(m)}(t) dt \\
  &\quad \times \left( f(j(x)_i) \left( x_i^+ \right) - f(j(x)_i) \left( x_i \right) \right) + \left( m-j(x)_i-1 \right) \tilde{f}^{(m)}(x_i) (x_i)_i^{m-\mu-1} (23)
\end{align*}

\begin{align*}
  \left( D^\mu f \right)(x^+) &= \frac{1}{\Gamma(m-\mu)} \int_0^t (t-x)^{m-\mu-1} \tilde{f}^{(m)}(t) dt \\
  &= \frac{1}{\Gamma(m-\mu)} \int_0^t (t-x)^{m-\mu-1} \tilde{f}^{(m)}(t) dt \\
  &\quad + \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (m-j(x)_i-1) \tilde{f}^{(m)}(x_i) (x_i)x^{m-\mu-1} \\
  &\quad \times \left( f(j(x)_i) \left( x_i^+ \right) - f(j(x)_i) \left( x_i \right) \right)
\end{align*}

Note that \( \left( D^\mu f \right)(x^+) = \infty \) if \( x = x_i \in \text{IMP} \), as expected.

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