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Some Remarks About Riemann-Liouville and Caputo Impulsive Fractional Calculus

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Abstract—This paper establishes some closed formulas for Riemann- Liouville impulsive fractional integral calculus and also for Riemann- Liouville and Caputo impulsive fractional derivatives.

***Keywords*—**Rimann- Liouville fractional calculus, Caputo fractional derivative, Dirac delta, Distributional derivatives, High-order distributional derivatives.

I. Introduction

FRACTIONAL calculus has been used in a set of applications, mainly, to deal with modelling errors in differential equations and dynamic systems. There are also applications in Signal Processing and sampling and hold algorithms, [1-3]. Fractional integrals and derivatives can be of non-integer orders and even of complex order. This facilitates the description of some problems which are not easily descxribed by ordinary calculus due to modelling errors, [1-5]. There are several approaches for the integral fractional calculus, the most popular ones being the Riemann-Liouville fractional integral. There is also a fractional Riemann- Liouville derivative. However, the well-known Caputo fractional derivative are less involved since the associated integral operator manipulates the derivatives of the primitive function under the integral symbol. This paper extends the basic fractional differ-integral calculus to impulsive functions described through the use of Dirac distributions and Dirac distributional derivatives, [5], of real fractional orders. In the general case, it is admitted a presence of infinitely many impulsive terms at certain isolated point of the relevant function domains. Control Theory topics in [6-9] could be reformulated under thefractional formalism considered in this paper.

II. Generalized Riemann-Liouville Fractional Integral

Let us denote the set of positive real numbers by $\boldsymbol{R}_+ = \{ r \in \boldsymbol{R} : r > 0 \}$ and left-sided and right-sided Lebesgue integrals, respectively, as:

$\int_0^x g(\tau) d\tau := \lim_{t \to x \equiv x^-} \int_0^t g(\tau) d\tau$ (the identification $x \equiv x^-$ is used for all x in order to simplify the notation), and

$$\int_0^{x^+} g(\tau) d\tau := \lim_{t \to x^+} \int_0^t g(\tau) d\tau$$

Now, consider real functions $f, \bar{f} : \boldsymbol{R}_+ \to \boldsymbol{R}$,such that $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ exists, $\forall x \in \boldsymbol{R}_+$, fulfilling:

$$f(x) = \bar{f}(x) + \sum_{x_i \in IMP} K_i \delta(x - x_i) = \bar{f}(x) + \sum_{i \in I(\infty)} K_i \delta(x - x_i)$$

$\delta(x)$ denotes the Dirac delta distribution, $K_i \delta(0) = f(x_i^+) - f(x_i)$ with $K_i \in \mathbf{R}$; $\forall i \in I(\infty) \subset \boldsymbol{Z}_+$, [5], and $IMP := \bigcup_{x \in \boldsymbol{R}_+} IMP(x) = \bigcup_{x \in \boldsymbol{R}_+} IMP(x^+)$ of indexing set $I(\infty)$ is the whole impulsive set defined via empty or non-empty) partial impulsive strictly ordered denumerable sets:

$$IMP(x) := \{ x_i \in \mathbf{R}_+ : f(x_i^+) - f(x_i) = K_i \delta(0), x_i < x \} \quad (1)$$

of indexing set $I(x) := \{ i \in \boldsymbol{Z}_{0+} : x_i \in IMP(x) \} \subset I(x^+) \subset \boldsymbol{Z}_+$, for each $x \in \boldsymbol{R}_+$; and

$$IMP(x) \subset IMP(x^+)_+ := \{ x_i \in \boldsymbol{R}_+ : f(x_i^+) - f(x_i) = K_i \delta(0), x_i \le x^+ \} \subset \boldsymbol{R} \quad (2)$$

of indexing set $I(x) \subset I(x^+) := \{ i \in \boldsymbol{Z}_{0+} : x_i \in IMP(x^+) \} \subset \boldsymbol{Z}_+$, for each $x \in \boldsymbol{R}_+$ with the indexing set of IMP being $I(\infty) = \bigcup_{x \in IMP(x)} I(x) = \bigcup_{x \in IMP(x^+)} I(x^+)$. If we are interested in studying the fractional derivative of the impulsive function $f : \boldsymbol{R}_+ \to \boldsymbol{R}$ then $\bar{f} : \boldsymbol{R}_+ \to \boldsymbol{R}$ is non- uniquely defined as $\bar{f}(x) = f(x)$ for $x \in \boldsymbol{R}_+ \setminus IMP$, and $f(x_i) = \bar{f}(x_i)$, $f(x_i^+) = f(x_i) + K_i \delta(0) = \bar{f}(x_i) + K_i \delta(0)$, for $x_i \in IMP$ with $\bar{f}(x^+) \in \boldsymbol{R}$ (non-uniquely) defined being bounded arbitrary (for instance, being zero or $\bar{f}(x^+) = f(x)$) if $x \in IMP$. Note that IMP and $I(\infty)$ have infinite cardinals if there are infinitely many impulsive values of the function f(t).

Note that the existence of $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ implies that of $\int_0^x (x-t)^{\mu-1} f(t) dt = \int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ if $x \notin IMP(x)$, since $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ exists, and that of

$$\int_0^{x_i^+} (x-t)^{\mu-1} f(t) dt = \int_0^{x_i} (x-t)^{\mu-1} \bar{f}(t) dt + (x - x_i)^{\mu-1} \left(f(x_i^+) - f(x_i) \right)$$
if $x_i \in IMP(x^+)$ (3)

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Theorem 2.1. The extended fractional Riemann- Liouville integrals by considering impulsive functions are defined for any fixed order $\mu\in\boldsymbol{R}_+$ and all $x\in\boldsymbol{R}_+$ by

$$\left(J^{\mu} f\right)(x):=\frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1} f(t)dt$$

$$=\frac{1}{\Gamma(\mu)}\left(\int_0^x (x-t)^{\mu-1}\bar{f}(t)dt+\sum_{i\in I(x)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right)\right)$$

$$=\frac{1}{\Gamma(\mu)}\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{\mu-1}f(t)dt$$

$$+\frac{1}{\Gamma(\mu)}\int_{x^+_{n(x)}}^{x}(x-t)^{\mu-1}f(t)dt+\sum_{i\in I(x)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right) \tag{4}$$

$$\left(J^{\mu} f\right)\left(x^+\right):=\frac{1}{\Gamma(\mu)}\int_0^{x^+}(x-t)^{\mu-1}f(t)dt$$

$$=\frac{1}{\Gamma(\mu)}\left(\int_0^x (x-t)^{\mu-1}\bar{f}(t)dt+\sum_{i\in I(x^+)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right)\right)$$

$$=\frac{1}{\Gamma(\mu)}\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{\mu-1}f(t)dt$$

$$+\frac{1}{\Gamma(\mu)}\sum_{i\in I(x^+)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right) \tag{5}$$

$$\left(J^{0} f\right)\left(x^+\right)\equiv\left(J^{0} f\right)(x):=f(x)$$

where $\Gamma:\boldsymbol{R}_{0+}\to\boldsymbol{R}_+$ is the Γ - function , [1-5] and $n:IMP\to\boldsymbol{Z}_+$ is defined by $n(x)=card\, I(x)=card\, IMP(x)$. □

Note that if $x\in IMP$ then

$$\left(J^{\mu} f\right)\left(x^+\right)=\frac{1}{\Gamma(\mu)}\sum_{i\in I(x^+)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{\mu-1}f(t)dt$$

$$+\frac{1}{\Gamma(\mu)}\sum_{i\in I(x^+)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right)$$

$$=\left(J^{\mu} f\right)(x)+\left(x-x_{n(x)}\right)^{\mu-1}\left(f\left(x^+_{n(x)}\right)-f\left(x_{n(x)}\right)\right)$$

$$\neq\left(J^{\mu} f\right)(x)=\frac{1}{\Gamma(\mu)}\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}}(x-t)^{\mu-1}f(t)dt$$

$$+\frac{1}{\Gamma(\mu)}\sum_{i\in I(x)}(x-x_i)^{\mu-1}\left(f\left(x_i^+\right)-f(x_i)\right) \tag{6}$$

and if $x\notin IMP$, since $I(x^+)=I(x)$, then $\left(J^{\mu} f\right)\left(x^+\right)=\left(J^{\mu} f\right)(x)$.

III. Generalized Riemann-Liouville Fractional Derivative

Assume that $f\in C^{m-1}\left(\boldsymbol{R}_+,\boldsymbol{R}\right)$ and its $m-th$ derivative exists everywhere in $\boldsymbol{R}_+$. Then, the Caputo fractional derivative of order $\mu\ge 0$ with $m-1\le\mu(\in\mathbf{R}_+)\le m$, $m\in\boldsymbol{Z}_+$ is for any $x\in\boldsymbol{R}_+$:

$$\left(D^{\mu} f\right)(x):=\left(\frac{d}{dx}\right)^m\left(J^{m-\mu} f\right)(x)$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^m\left(\int_0^x (x-t)^{m-\mu-1}f(t)dt\right) \tag{7}$$

The following particular cases follow from this formula for $\mu=m-1$:

(a) $\mu=-1;\, m=0$ yields $\left(D^{-1} f\right)(x)=\int_0^x f(t)dt$ which is the standard integral of the function f . This case does not verifies the "derivative constraint " $0\le m-1\le\mu(\in\boldsymbol{R}_+)<m$ leading to an integral result.

(b)) $\mu=0;\, m=1$ yields $\left(D^{0} f\right)(x)=f(x)$ which so that $D^0 f$ is the identity operator

(c) $\mu=1;\, m=2$ yields $\left(D^{1} f\right)(x)=f^{(1)}(x)$

(d) $\mu=2;\, m=3$ yields $\left(D^{2} f\right)(x)=f^{(2)}(x)$ which is the standard first- derivative of the function f.

Compared to the parallel cases with the Caputo fractional derivative, note that the Riemann- Liouville fractional derivative, compared to the Caputo corresponding one, does not depend on the conditions at zero of the function and its derivatives. Define the Kronecker delta $\delta(a,b)$ of any pair of real numbers (a,b) as $\delta(a,b)=1$ if $a=b$ and $\delta(a,b)=0$ if $a\neq b$ and then evaluate recursively the Riemann – Liouville fractional derivative of order $\mu\ge 0$ from the above formula by using Leibniz´s differentiation rule by noting that , since $\mu\neq m-j;\forall\, j(\in\boldsymbol{Z}_+)>1$, only the differential part corresponding to the differentiation of the integrand is non zero for $j>m-\mu$. This yields the following result:

Theorem 3.1. Assume that $f\in C^{m-2}\left(\boldsymbol{R}_+,\boldsymbol{R}\right)$ and $f^{(m-1)}$ exists everywhere in $\boldsymbol{R}_+$ and that $f(t)$ is integrable on $\mathbf{R}_+$, then:

$$\left(D^{\mu} f\right)(x)=\frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^m\left(\int_0^x (x-t)^{m-\mu-1}f(t)dt\right)$$

$$=\frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^{m-1}$$

$$\left[\int_0^x (m-\mu-1)(x-t)^{m-\mu-2}f(t)dt+f(x)\delta(\mu,m-1)\right]$$

$$=\frac{1}{\Gamma(m-\mu)}f^{(m-1)}(x)\delta(\mu,m-1)$$

$$+\frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^{m-1}\left(\int_0^x (m-\mu-1)(x-t)^{m-\mu-2} f(t)dt\right)$$

$$=\frac{1}{\Gamma(m-\mu)} f^{(m-1)}(x)\delta(\mu,m-1)$$

$$+\frac{1}{\Gamma(m-\mu)}\left(\sum_{i=1}^{m-2}\prod_{j=i+1}^{m-1}[j-\mu]\right) f^{(i)}(x)\delta(\mu,m-i)$$

$$+\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)} f(t)dt\right) \quad (8)$$

If $f\in PC^k(\boldsymbol{R}_+,\boldsymbol{R})$ with $f^{(k)}(x)$ being discontinuous of first class then $f^{m-1}(x)=\delta^{(j(x))}(x)$ with $j(x)=m-1-k(x)$, one uses to obtain the right value of (8) the perhaps high-order distributional derivatives formula:

$$\left|f^{(m-1)}(x^+)-f^{(m-1)}(x)\right|=$$
$$\frac{(-1)^k k!}{x^k}\left|f^{(m-1-k)}(x^+)-f^{(m-1-k)}(x)\right|\delta(0)=\infty \quad (9)$$

to yield

$$(D^\mu f)(x^+)=\frac{1}{\Gamma(m-\mu)}$$
$$\left[\frac{(-1)^{k(x)}k(x)!}{x^{k(x)}}\left|f^{(m-1-k(x))}(x^+)-f^{(m-1-k(x))}(x)\right|\delta(0)\delta(\mu,m-1)\right.$$
$$+\left(\sum_{i=1}^{m-2}\prod_{j=i+1}^{m-1}[j-\mu]\right)f^{(i)}(x)\delta(\mu,m-i)$$
$$\left.+\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)} f(t)dt\right)\right] \quad (10)$$

If $\mu=m-1$ then

$$(D^{m-1} f)(x)$$
$$=f^{(m-1)}(x)+\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-m} f(t)dt\right) \quad (11)$$

provided that $\left(\int_0^x (x-t)^{-(\mu+1)} f(t)dt\right)$ exists for $x\in\mathbf{R}_+$ (which is guaranteed if $f(t)$ is Lebesgue-integrable on $\mathbf{R}_+$), $f\in C^{m-2}(\boldsymbol{R}_+,\boldsymbol{R})$ and f^{m-1} exists everywhere in $\mathbf{R}_+$. The correction (10) applies when the derivative does not exist. □

If $\mu\neq m-1$ with $m-1\le\mu(\in\mathbf{R}_+)\le m$ then after defining the impulsive sets, its associated indexing sets and the function $\bar{f}:\mathbf{R}_+\to\mathbf{R}$ as for the extended Riemann- Liouville fractional integral, one gets:

$$(D^\mu f)(x)$$
$$=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\left(\int_0^x (x-t)^{-(\mu+1)} f(t)dt\right)$$
$$=$$
$$=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\sum_{i\in I(x)\cup\{0\}}\int_{x_i^+}^{x_{i+1}} (x-t)^{-(\mu+1)} f(t)dt$$
$$+\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\int_{x_{n(x)}^+}^{x} (x-t)^{-(\mu+1)} f(t)dt$$

$$+\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]\sum_{i\in I(x)}(x-x_i)^{-(\mu+1)}\left(f(x_i^+)-f(x_i)\right) \quad (12)$$

$$(D^\mu f)(x^+)=\frac{1}{\Gamma(m-\mu)}\left[\prod_{j=0}^{m-1}[j-\mu]\right]$$
$$\times\left(\int_0^x (x-t)^{-(\mu+1)}\bar{f}(t)dt+\sum_{i\in I(x)}(x-x_i)^{-(\mu+1)}\left(f(x_i^+)-f(x_i)\right)\right) \quad (13)$$

IV. Generalized Caputo Fractional Derivative

Assume that $f\in C^{m-1}(\boldsymbol{R}_+,\boldsymbol{R})$ and its $m-th$ derivative exists everywhere in $\boldsymbol{R}_+$. Then, the Caputo fractional derivative of order $\mu\ge 0$ with $m-1\le\mu(\in\boldsymbol{R}_+)<m$, $m\in\boldsymbol{Z}_+$ is for any $x\in\boldsymbol{R}_+$:

$$(D_*^\mu f)(x):=\left(J^{m-\mu} f^{(m)}\right)(x)$$
$$=\frac{1}{\Gamma(m-\mu)}\int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt \quad (14)$$

; $m-1\le\mu<m$, $m\in\boldsymbol{Z}_+$, $x\in\boldsymbol{R}_+$

The following particular cases occur with $\mu=m-1$ leading to

$$(D_*^{m-1} f)(x)=\int_0^x f^{(m)}(t)dt=f^{(m-1)}(x)-f^{(m-1)}(0^+) \quad (15)$$

(a) $\mu=-1; m=0$ yields $(D_*^{-1} f)(x)=f^{(-1)}(x)-f^{(-1)}(0^+)$ which is an integral result f . Note that this case does not verifies the "derivative constraint" $0\le\mu(\in\boldsymbol{R}_+)<m$ leading to an integral result.

(b)) $\mu=0; m=1$ yields $(D_*^0 f)(x)=f^{(0)}(x)-f^{(0)}(0^+)=f(x)-f(0^+)$

(c) $\mu=1; m=2$ yields $(D_*^1 f)(x)=f^{(1)}(x)-f^{(1)}(0^+)$

(d) $\mu=2; m=3$ yields $(D_*^2 f)(x)=f^{(2)}(x)-f^{(2)}(0^+)$

We can extend the above formula to real functions with impulsive $m-th$ derivative as follows. Assume that $f\in C^{m-2}(\mathbf{R}_+,\mathbf{R})$ with bounded piecewise $(m-1)-th$

derivative existing everywhere in $\boldsymbol{R}_+$ and $f^{(m)}(x) \equiv \dfrac{d^m f(x)}{dx^m}$ being impulsive with $f^{(m)}(x_i) = K_i\,\delta(0) = \left(f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)\right)\delta(0)$

; $\forall x_i \in IMP$, equivalently , $\forall i \in I(\infty)$, at the eventual discontinuity points $x_i > 0$ at the impulsive set $IMP := \bigcup_{x \in \boldsymbol{R}_+} IMP(x)$, where the partial impulsive sets are re-defined as follows:

$$IMP(x) := \left\{x_i \in \mathbf{R}_+ : f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i) = K_i ,\, x_i < x\right\} \subset IMP(x^+) \tag{16}$$

$$IMP(x^+) := \left\{x_i \in \mathbf{R}_+ : f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i) = K_i ,\, x_i \le x^+\right\} \subset IMP(x^+) \tag{17}$$

Now, consider $f \in C^{m-1}(0,\infty)$ with $f^{(m)}(x) \equiv \dfrac{d^m f(x)}{dx^m}$ being almost everywhere piecewise continuous in $\boldsymbol{R}_+$ except possibly on a non-empty discrete impulsive set IMP .Define a non-impulsive real function $\bar{f} : \boldsymbol{R}_+ \to \boldsymbol{R}$ defined as $\bar{f}^{(m)}(x) = f^{(m)}(x)$ for $x \in \boldsymbol{R}_+ \setminus IMP$, and $f^{(m)}(x_i) = \bar{f}^{(m)}(x_i)$, $f^{(m)}(x_i) = \bar{f}^{(m)}(x_i) + K_i\,\delta(0)$ for $x_i \in IMP$ with $\bar{f}^{(m)}(x^+) = f^{(m)}(x)$; $x \in IMP$ (defined being bounded arbitrary (for instance, zero) if $x \in IMP$. Through a similar reasoning as that used for Riemann- Liouville fractional integral by replacing the function $f : \boldsymbol{R}_+ \to \boldsymbol{R}$ by its m-th derivative, one obtains the following result:

Theorem 4.1. The Caputo fractional derivative of order $\mu \in \boldsymbol{R}_+$ satisfying $m-1 < \mu \le m$; $m \in \boldsymbol{Z}_+$ and all $x \in \boldsymbol{R}_+$ is given below:

$$\left(D_*^{\mu} f\right)(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)\,dt$$

$$= \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t)\,dt + \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1}\left(f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)\right)\delta(x-x_i)$$

$$= \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t)\,dt + \frac{1}{\Gamma(m-\mu)} \int_{x_{n(x)}^+}^{x} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t)\,dt + \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1}\left(f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)\right)\delta(x-x_i) \tag{18}$$

$$\left(D_*^{\mu} f\right)(x^+) := \frac{1}{\Gamma(m-\mu)} \int_0^{x^+} (x-t)^{m-\mu-1} f^{(m)}(t)\,dt = \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t)\,dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} (x-x_i)^{m-\mu-1}\left(f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)\right)\delta(x-x_i) \tag{19}$$

where $n : IMP \to \boldsymbol{Z}_+$ is a discrete function defined by $n(x) = card\, I(x) = card\, IMP(x)$. □

Note that if $x \in IMP$ then

$$\left(D_*^{\mu} f\right)(x^+) = \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t)\,dt + \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} (x-x_i)^{m-\mu-1}\left(f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)\right)\delta(x-x_i)$$

$$= \left(D_*^{\mu} f\right)(x) + \left(x - x_{n(x)}\right)^{m-\mu-1}\left(f^{(m-1)}(x_{n(x)}^+) - f^{(m-1)}(x_{n(x)})\right)\delta(0)$$

$$\ne \left(D_*^{\mu} f\right)(x) = \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t)\,dt + \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1}\left(f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)\right)\delta(x-x_i)$$

and if $x \notin IMP$, since $I(x^+) = I(x)$, then $\left(D_*^{\mu} f\right)(x^+) = \left(D_*^{\mu} f\right)(x)$. The above formalism applies when $f^{(m-1)} : \boldsymbol{R}_+ \to \boldsymbol{R}$ is piecewise continuous with isolated first- class discontinuity points, that is $f \in PC^{m-1}(\boldsymbol{R}_+, \boldsymbol{R})$ implying that $f \in C^{m-2}(\boldsymbol{R}_+, \boldsymbol{R})$. A more general situation arises when the discontinuities can point-wise arise for points of the function itself of for any successive derivative up- till order m. This would lead to a more general description than that given as follows. Define partial sets of positive integers as $\bar{k} := \{1, 2, \ldots, k\}$

Assume that $f \in PC^{j}(\boldsymbol{R}_+, \boldsymbol{R})$ and x is a discontinuity point of first class of $f^{(j)}(x)$ for some $j \in \overline{m-1} \cup \{0\}$. Then , $f^{(j+\ell)}(x)$ are impulsive for $\ell \in \overline{m-j}$ of high order being increasing with ℓ . Define the (j+1) – th impulsive sets of the function *f* on $(0,x) \subset \boldsymbol{R}$ as:

$$IMP_{j+1}(x) := \left\{z \in \boldsymbol{R}_+ : z < x,\ 0 < \left| f^{(j)}(z^+) - f^{(j)}(z) \right| < \infty\right\};\ j \in \overline{m-1} \cup \{0\},\ x \in \boldsymbol{R}_+ \tag{20}$$

This leads directly the definition of the following impulsive sets:

$$IMP_{j+1} := \left\{x \in \boldsymbol{R}_+ :\ 0 < \left| f^{(j)}(x^+) - f^{(j)}(x) \right| < \infty\right\} \equiv \bigcup_{x \in \boldsymbol{R}_+} IMP_{j+1}(x) \tag{21}$$

$$IMP := \left\{x \in \boldsymbol{R}_+ :\ 0 < \left| f^{(j)}(x^+) - f^{(j)}(x) \right| < \infty,\ some\ j \in \overline{m-1} \cup \{0\}\right\} \equiv \bigcup_{x \in \boldsymbol{R}_+} \left(\bigcup_{j \in \overline{m-1} \cup \{0\}} IMP_{j+1}(x) \right) \tag{22}$$

which can be empty . Thus , if $z \in IMP_{j+1}$ then $f^{(j-1)}(x^+) = f^{(j-1)}(x)$ exists with identical left and right limits, $f^{(j)}\left(x^{+}\right)-f^{(j)}(x)=K=K(x)\neq 0$ and $f^{(j)}(x)=K\delta(0)$ with successive higher-order derivatives represented by higher- order Dirac distributional derivatives

The above definitions yield directly the following simple results:

Assertion 5.2. $x\in IMP \Rightarrow x\in IMP_j$ for a unique $j=j(x)\in \overline{m}$.

Proof: Proceed by contradiction. Assume that $x\in\left(IMP_{i+1}\cap IMP_{j+1}\right)$ for $i,j(\neq i)\in \overline{m-1}\cup\{0\}$.Then:

$0<\left|f^{(i)}\left(x^{+}\right)-f^{(i)}(x)\right|<\infty$;

$0<\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|<\infty$

Assume with no loss of generality that $j=i+k>i$ for some $k(\leq m-i-1)\in \boldsymbol{Z}_{+}$. Then,

$$\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|=\left|f^{(i+k)}\left(x^{+}\right)-f^{(i+k)}(x)\right|$$
$$=\frac{(-1)^{k}\,k\,!}{x^{k}}\left|f^{(i)}\left(x^{+}\right)-f^{(i)}(x)\right|\delta(0)=\infty$$

with $x\in \boldsymbol{R}_{+}$. If $\left|f^{(i)}\left(x^{+}\right)-f^{(i)}(x)\right|\neq 0$ which contradicts $0<\left|f^{(i)}\left(x^{+}\right)-f^{(i)}(x)\right|<\infty$ so that $i=j$. □

Assertion 5.3. $x\in IMP \Rightarrow$
$\left(x\in IMP_j \Leftrightarrow \exists\ a\ unique\ j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty\right)$

Furthermore, such a unique *j=j(x)* satisfies $\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|>0$.

Proof: The existence is direct by contradiction. If $\neg\exists\, j=j(x)\in \overline{m-1}\cup\{0\}$ such that $\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|<\infty$ then $x\notin IMP$. Now, assume there exist two nonnegative integers $i=i(x)=\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty$ and $j=j(x)=i+k=\left|f^{(i+k-1)}\left(x^{+}\right)-f^{(i+k-1)}(x)\right|<\infty$; for some $k\in\overline{m-i}$. But for $x>0$,

$$\infty=\frac{(-1)^{k}\,k\,!}{x^{k}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|\delta(0)$$
$$=\left|f^{(i+k-1)}\left(x^{+}\right)-f^{(i+k-1)}(x)\right|<\infty$$

which is a contradiction. Then,

$$x\in IMP_j \Rightarrow \exists\ j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty$$

which is unique. Also, from the definition of the impulsive sets $IMP_i(x)$, $\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|<\infty \Rightarrow x\in\bigcup_{i\in\overline{j}\cup\{0\}} IMP_i(x)$

Now , assume that $x\in\bigcup_{i\in\overline{j-1}\cup\{0\}} IMP_i(x)$. Thus, $0<\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|<\infty \Rightarrow \left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|=\infty$ from the definition of the impulsive sets. Then, $x\in IMP_j(x)$.The opposite logic implication

$$j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty \Rightarrow x\in IMP_j$$

is proved. Then, it has been fully proved that $x\in IMP \Rightarrow$
$\left(x\in IMP_j \Leftrightarrow \exists\ a\ unique\ j=j(x)=\max_{i\in\overline{m}}\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|<\infty\right)$

Now, establish again a contradiction by assuming that

$$j=j(x)=\left|f^{(k-1)}\left(x^{+}\right)-f^{(k-1)}(x)\right|=\max$$
$$\left|f^{(i-1)}\left(x^{+}\right)-f^{(i-1)}(x)\right|=0<\infty\,;\ \forall k\in\overline{m}$$

what contradicts $x\in IMP$. This proves that the unique *j=j(x)* implying and being implied by $x\in IMP_j$ satisfies $\left|f^{(j-1)}\left(x^{+}\right)-f^{(j-1)}(x)\right|>0$. □

Using the necessary – high order distributional derivatives, one gets that

$$x\in IMP \Rightarrow f^{(m)}(x)=\frac{(-1)^{m-j}(m-j)!}{x^{m-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0)$$

; with $j\in\overline{m-1}\cup\{0\}$ being uniquely defined so that $0<\left|f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right|<\infty$. Thus, the m-th distributional derivative of $f:\boldsymbol{R}_{+}\to\boldsymbol{R}$ can be represented as:

$$f^{(m)}(x)=\bar{f}^{(m)}(x)$$
$$+\sum_{x_i\in IMP_{j_i+1}}\frac{(-1)^{j_i}(m-j_i)!}{x_i^{m-j_i}}\left(f^{(j_i)}\left(x_i^{+}\right)-f^{(j_i)}\left(x_i\right)\right)\delta\left(x-x_i\right)$$

, $x\in\boldsymbol{R}_{+}$

with $j_i=j_i\left(x_i\right)$ being uniquely defied for each $x_i\in IMP$ so that $x_i\in IMP_{j_i}$, where $\bar{f}\in C^{m-1}\left(\boldsymbol{R}_{+},\boldsymbol{R}\right)$ with everywhere continuous first-derivative defined as $\bar{f}^{(j)}(x)=f^{(j)}(x)$; $x\in\boldsymbol{R}_{+}$, $\bar{f}(0)=f(0)$. The above formula is applicable if $f\notin PC^{m}\left(\boldsymbol{R}_{+},\boldsymbol{R}\right)$ but it is also applicable if $f\in PC^{m}\left(\boldsymbol{R}_{+},\boldsymbol{R}\right)$ yielding:

$$f^{(m)}\left(x^{+}\right)=f^{(m)}(x)=\bar{f}^{(m)}(x)\ \text{ if }\ x\notin IMP$$

$$f^{(m)}(x)=\bar{f}^{(m)}(x)$$
$$f^{(m)}\left(x^{+}\right)=f^{(m)}(x)+\frac{(-1)^{m-j}(m-j)!}{x^{m-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0)$$

if $x\in IMP$

$$f^{(m-1)}(x)=\bar{f}^{(m-1)}(x)$$
$$f^{(m-1)}\left(x^{+}\right)=f^{(m-1)}(x)+\frac{(-1)^{m-j}(m-1-j)!}{x^{m-1-j}}\left(f^{(j)}\left(x^{+}\right)-f^{(j)}(x)\right)\delta(0)$$

if $x\in IMP$ and $j<m-1$

$$f^{(m-1)}(x)=\bar{f}^{(m-1)}(x)$$

$$f^{(m-1)}(x^{+})=f^{(m-1)}(x)+\left(f^{(m-1)}(x^{+})-f^{(m-1)}(x)\right)$$ if $x\in IMP$ and $j=m-1$

for a unique $j=j(x)\in\overline{m-1}\cup\{0\}$ from Assertion 1. Denote further sets related to impulses as follows:

$IMP(x):=\{z\in IMP : z<x\}$; $IMP(x^{+}):=\{z\in IMP : z\le x\}$; $\forall x\in \boldsymbol{R}_{+}$

being indexed by two subsets of integers of the same corresponding cardinals defined by:

$I(x)=\bar{j}=\overline{j(x)}$ indexing the members z_i of $IMP(x)$ in increasing order

$I(x^{+})$, being either $I(x)$ or $I(x)+1$, indexing the members z_i of $IMP(x^{+})$ in increasing order

The following result holds:

Theorem 5.4. The Caputo fractional derivative of $f:\boldsymbol{R}_{+}\to\boldsymbol{R}$ of order $\mu\in\boldsymbol{R}_{+}$ satisfying $m-1<\mu\le m$; $m\in\boldsymbol{Z}_{+}$ and all $x\in\boldsymbol{R}_{+}$ is after using distributional derivatives becomes in the most general case:

$$\left(D_{*}^{\mu}f\right)(x):=\frac{1}{\Gamma(m-\mu)}\int_{0}^{x}(x-t)^{m-\mu-1}f^{(m)}(t)\,dt$$

$$=\frac{1}{\Gamma(m-\mu)}\Bigg(\int_{0}^{x}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)\,dt+\sum_{i\in I(x)}(\text{-}1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^{+})-f^{(j(x_i))}(x_i)\right)\hat{\delta}(x-x_i)\Bigg)$$

$$=\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)\cup\{0\}}\int_{x_i^{+}}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)\,dt+\frac{1}{\Gamma(m-\mu)}\int_{x_{n(x)}^{+}}^{x}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)\,dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x)}(\text{-}1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}\times\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^{+})-f^{(j(x_i))}(x_i)\right)\quad(23)$$

$$\left(D_{*}^{\mu}f\right)(x^{+}):=\frac{1}{\Gamma(m-\mu)}\int_{0}^{x^{+}}(x-t)^{m-\mu-1}f^{(m)}(t)\,dt$$

$$=\frac{1}{\Gamma(m-\mu)}\int_{0}^{x}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)\,dt+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^{+})}(\text{-}1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}\times\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^{+})-f^{(j(x_i))}(x_i)\right)$$

$$=\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^{+})\cup\{0\}}\int_{x_i^{+}}^{x_{i+1}}(x-t)^{m-\mu-1}\bar{f}^{(m)}(t)\,dt$$
$$+\frac{1}{\Gamma(m-\mu)}\sum_{i\in I(x^{+})}(\text{-}1)^{m-j(x_i)-1}(x-x_i)^{m-\mu-1}\times\frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}}\left(f^{(j(x_i))}(x_i^{+})-f^{(j(x_i))}(x_i)\right)\quad(24)$$

□

Note that $\left|\left(D_{*}^{\mu}f\right)(x^{+})\right|=\infty$ if $x=x_i\in IMP$, as expected.

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