About Analysis and Modelling of the Open Message Switching System

Saulius. Minkevičius, *Institute of Mathematics and Informatics and Vilnius Gediminas Technical University,*
Genadijus Kulvietis, *Vilnius Gediminas Technical University* this

Abstract—The modern queueing theory is one of the powerful tools for a quantitative and qualitative analysis of communication systems, computer networks, transportation systems, and many other technical systems. The paper is designated to the analysis of queueing systems, arising in the networks theory and communications theory (called open queueing network). The authors of this research in the sphere of queueing theory present the theorem about the law of the iterated logarithm (LIL) for the queue length of a customers in open queueing network and its application to the mathematical model of the open message switching system.

Keywords—models of information systems, open message switching system, open queueing network, queue length of a customers, heavy traffic, a law of the iterated logarithm.

I. Problem Formulation

At first the authors continues the research in queueing theory about LIL in queueing systems and presents theorem for the queue length of a customers in open queueing network. We note that papers on the queue length in heavy traffic for a multiphase queue and open queueing network are sparse. Kogan et al [6], [7] investigated tandem queues with blocking in heavy traffic. In Minkevičius [8], functional limit theorems for the queue length of customers in multiphase queues in various conditions of heavy traffic are proved. [3] surveyed some results on the queue length in multiphase queues. [9] presented laws of the iterated logarithm for a total queue length of customers and a queue length of customers in multiphase queues. In [10], functional limit theorems on complex transient phenomena for a queue length in multiphase queues are proved. Knessl et al [4], [5] presented diffusion approximations for a two-node tandem queue with general renewal process input and independent exponential service times at each node.

Note that the research of the LIL in more general systems than the queueing system $GI/G/1$ or multiphase queueing systems and open queueing network has just started (see [1]). In [8], [9], the LIL is proved in heavy traffic for the queue length of customers, waiting time of a customer, a virtual waiting time of a customer in a multiphase queueing system. In [8], [9], [12], also gives the proof of the theorem on the

S. Minkevičius is with the Department of System Analysis, Institute of Mathematics and Informatics, Akademijos 4, 08663 Vilnius, Lithuania and Vilnius Gediminas Technical University, Sauletekio 11, 10223 Vilnius, Lithuania e-mail: (stst@ktl.mii.lt).

Genadijus Kulvietis is with Department of Information Technology, Vilnius Gediminas Technical University, Sauletekio 11, 10223 Vilnius, Lithuania e-mail: (gk@fm.vtu.lt).

Manuscript received September 20, 2010.

LIL under the conditions of heavy traffic for a virtual waiting time of a customer in the open Jackson network.

In this paper, we investigated an open queueing network model in heavy traffic. The LIL for the queue length of customers in an open queueing network has been presented. The main tool for the analysis of these queueing systems in heavy traffic is a functional LIL for renewal process (the proof can be found in [13] and [2]).

The service discipline is "first come, first served" (FCFS). We consider open queueing networks with the FCFS service discipline at each station and general distributions of inter-arrival and service times. The queueing network we studied has k single server stations, each of which has an associated infinite capacity waiting room. Every station has an arrival stream from outside the network, and the arrival streams are assumed to be mutually independent renewal processes. Customers are served in the order of arrival and after service they are randomly routed to either another station in the network, or out of the network entirely. Service times and routing decisions form mutually independent sequences of independent identically distributed random variables.

The basic components of the queueing network are arrival processes, service processes, and routing processes. In particular, there are mutually independent sequences of independent identically distributed random variables $\left\{z_n^{(j)}, n \geq 1\right\}$, $\left\{S_n^{(j)}, n \geq 1\right\}$ and $\left\{\Phi_n^{(j)}, n \geq 1\right\}$ for $j = 1, 2, \ldots, k$; defined on the probability space. Random variables $z_n^{(j)}$ and $S_n^{(j)}$ are strictly positive, and $\Phi_n^{(j)}$ have support in $\{0, 1, 2, \ldots, k\}$. We define $\mu_j = \left(M\left[S_n^{(j)}\right]\right)^{-1} > 0$, $\sigma_j = D\left(S_n^{(j)}\right) > 0$ and $\lambda_j = \left(M\left[z_n^{(j)}\right]\right)^{-1} > 0$, $a_j = D\left(z_n^{(j)}\right) > 0$, $j = 1, 2, \ldots, k$; with all of these terms assumed finite. Denote $p_{ij} = P\left(\Phi_n^{(i)} = j\right) > 0$, $j = 1, 2, \ldots, k$. In the context of the queueing network, the random variables $z_n^{(j)}$ function as interarrival times (from outside the network) at the station j, while $S_n^{(j)}$ is the nth service time at the station j, and $\Phi_n^{(j)}$ is a routing indicator for the nth customer served at the station j. If $\Phi_n^{(i)} = j$ (which occurs with probability p_{ij}), then the nth customer served at the station i is routed to the station j. When $\Phi_n^{(i)} = 0$, the associated customer leaves the network. The matrix P is called a routing matrix.

First let us define $Q_j(t)$ as the queue length of customers at the jth station of the queueing network in time t; $\hat{\sigma}_j^2 =$

$$(\lambda_j)^3 \cdot Dz_n^{(j)} + \sum_{i=1}^{k} (\mu_i)^3 \cdot DS_n^{(i)} \cdot (p_{ij})^2 + (\mu_j)^3 \cdot DS_n^{(j)} > 0,$$

World Academy of Science, Engineering and Technology
International Journal of Computer and Information Engineering
Vol:4, No:11, 2010

$$\hat{\beta}_j = \lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} - \mu_j > 0, \, j = 1, 2, \ldots, k.$$

We assume that the following condition is fulfilled:

$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} > \mu_j, \; j = 1, 2, \ldots, k. \tag{1}$$

Note that this condition quarantees that, with probability one there exists a queue length of customers and this queue length of customers is constantly growing.

II. MAIN RESULT

One of the results of the paper is a theorem on the LIL for the queue length of customers in an open queueing network.

Theorem 2.1. *If conditions (1) are fulfilled, then*

$$P \left(\overline{\lim_{t \to \infty}} \frac{Q_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = 1 = \right)$$

$$= P \left(\underline{\lim_{t \to \infty}} \frac{Q_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = -1 \right) = 1,$$

$j = 1, 2, \ldots, k$ *and* $a(t) = \sqrt{2t \ln \ln t}$.

Proof: The proof of theorem is connected with the proof of Theorem 3.1 from paper Minkevičius et al [11], and we omit this. The proof of Theorem 2.1 is complete. ∎

III. ON THE MODEL OF THE OPEN MESSAGE SWITCHING FACILITY

In this part of the paper, we will present an application of the proved theorem - a mathematical model of open message switching system.

As noted in the introduction, open network queueing systems are of special interest both in theory and in practical applications. Such systems consist of several service nodes, and each arriving customer is served in the order of arrival and after service they are randomly routed to either another station in the network, or out of the network entirely. A typical example is provided by queueing systems with identical service. Such systems are very important in applications, especially to open message switching systems. In fact, in many comunication systems the transmission times of the customers do not vary in the delivery process.

So, we investigate a message switching system which consists of k service nodes and in which $S_n^{(j)} = S_n$, $j = 1, 2, \ldots, k$ (the service process is identical in nodes of the system).

Also, let us note $\mu = (M[S_n])^{-1} > 0$, $\tilde{\sigma}_j^2 = (\lambda_j)^3 \cdot Dz_n^{(j)} + (\mu)^3 \cdot DS_n \cdot \left(1 + \sum_{i=1}^{k} (p_{ij})^2\right)$, $\beta_j = \lambda_j - \mu \cdot (1 - \sum_{i=1}^{k} p_{ij}) > 0$, $j = 1, 2, \ldots, k$.

We assume that the following conditions are fulfilled:

$$\beta_j > 0, j = 1, 2, \ldots, k. \tag{2}$$

Similarly as in the proof of Theorem 2.1, we present the following theorem and corollary on the LIL for the queue length of messages in open message switching systems.

Theorem 3.1. *If conditions (2) are fulfilled, then*

$$P \left(\overline{\lim_{t \to \infty}} \frac{Q_j(t) - \beta_j \cdot t}{\tilde{\sigma}_j \cdot a(t)} = 1 \right) =$$

$$= P \left(\underline{\lim_{t \to \infty}} \frac{Q_j(t) - \beta_j \cdot t}{\tilde{\sigma}_j \cdot a(t)} = -1 \right) = 1,$$

$j = 1, 2, \ldots, k$ *and* $a(t) = \sqrt{2t \ln \ln t}$.

Corollary 3.1. *If conditions (2) are fulfilled, then for fixed $\varepsilon > 0$ there exists $t(\varepsilon)$ such that for every $t \geq t(\varepsilon)$,*

$$(1 - \varepsilon) \cdot \tilde{\sigma}_j \cdot a(t) + \beta_j \cdot t \leq Q_j(t) \leq (1 + \varepsilon) \cdot \tilde{\sigma}_j \cdot a(t) + \beta_j \cdot t,$$

$j = 1, 2, \ldots, k$, *with probability one.*

IV. ON THE MODEL OF THE MULTISTAGE MESSAGE SWITCHING FACILITY

Finally we present investigation of seperate case of Theorem 3.1 (case of multistage message switchning system). We investigate here a k-phase queue (i.e., after a customer has been served in the j-th phase of the queue, he is routed to the $j + 1$-th phase of the queue, and, after the service in the k-th phase of the queue, he leaves the queue). Let us denote by t_n the time of arrival of the n-th customer; by $S_n^{(j)}$– the service time of the n-th customer in the j-th phase; $z_n = t_{n+1} - t_n$; $j = 1, 2, \cdots, k$.

So, we investigate a multistage message switching system which consists of k service nodes and in which $S_n^{(j)} = S_n$, $j = 1, 2, \ldots, k$ (the service process is identical in phases of the system).

Next, denote $Q_j(t)$ as the queue length of messages in j-th phase of multistage message switching system at time moment t; $v_j(t) = \sum_{i=1}^{j} Q_i(t)$ stands the total queue length of messages until j-phase of the multistage message switching system at time moment t, $j = 1, 2, \ldots, k$ and $t > 0$.

Let us define $\beta_1 = (MS_n)^{-1}, \beta_0 = (Mz_n)^{-1}, \alpha = \beta_0 - \beta_1$, $\hat{\sigma}_1^2 = DS_n \cdot (MS_n)^{-3} > 0$, $\hat{\sigma}_0^2 = Dz_n \cdot (Mz_n)^{-3} > 0$, $\tilde{\sigma}^2 = \hat{\sigma}_1^2 + \hat{\sigma}_0^2$.

We assume that the following condition is fulfilled:

$$\alpha > 0. \tag{3}$$

In [9] is presented theorem and corollary about total queue length of messages in multistage message switching system.

Theorem 4.1. *If condition (3) is fulfilled, then*

$$P \left(\overline{\lim_{t \to \infty}} \frac{v_j(t) - \alpha \cdot t}{\tilde{\sigma} \cdot a(t)} = 1 \right) =$$

$$= P \left(\underline{\lim_{t \to \infty}} \frac{v_j(t) - \alpha \cdot t}{\tilde{\sigma} \cdot a(t)} = -1 \right) = 1,$$

$j = 1, 2, \ldots, k$ *and* $a(t) = \sqrt{2t \ln \ln t}$.

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International Journal of Computer and Information Engineering
Vol:4, No:11, 2010

V. COMPUTING EXAMPLE

We see that Theorem 4.1 implies that for fixed $\varepsilon > 0$ there exists $t(\varepsilon)$ such that for every $t \geq t(\varepsilon)$,

$$(1-\varepsilon) \cdot \tilde{\sigma} \cdot a(t) + \alpha \cdot t \leq v_j(t) \leq (1+\varepsilon) \cdot \tilde{\sigma} \cdot a(t) + \alpha \cdot t, \quad (4)$$

where $a(t) = \sqrt{2t \ln \ln t}$, $\beta_1 = (MS_n)^{-1}, \beta_0 = (Mz_n)^{-1}, \alpha = \beta_0 - \beta_1 > 0$, $\hat{\sigma}_1^2 = DS_n \cdot (MS_n)^{-3} > 0$, $\hat{\sigma}_0^2 = Dz_n \cdot (Mz_n)^{-3} > 0$, $\tilde{\sigma}^2 = \hat{\sigma}_1^2 + \hat{\sigma}_0^2$, $j = 1, 2, \ldots, k$.

From this we can obtain

$$1 - \varepsilon) \cdot \tilde{\sigma} \cdot a(t) + \alpha \cdot t \leq v_j(t) \leq (1+\varepsilon) \cdot \tilde{\sigma} \cdot a(t) + \alpha \cdot t,$$

$$|M(v_j(t) - \alpha \cdot t) - \{(1-\varepsilon) \cdot \tilde{\sigma} \cdot a(t)\}| \leq 2 \cdot \varepsilon \cdot \tilde{\sigma} \cdot a(t), \quad (5)$$

$$M\left(\frac{v_j(t) - \alpha \cdot t}{\tilde{\sigma} \cdot a(t)}\right) - (1+\varepsilon)| \leq 2 \cdot \varepsilon, \; j = 1, 2, \ldots, k.$$

So from (5) we can get

$$Mv_j(t) \sim \alpha \cdot t + (1+\varepsilon) \cdot \tilde{\sigma} \cdot a(t), \; j = 1, 2, \ldots, k. \quad (6)$$

$Mv_j(t)$ is average total queue length of messages until j-th phase of the multistage message switching system at time moment t, $j = 1, 2, \ldots, k$ and $t > 0$.

We see from (6) that $Mv_j(t)$ consists of linear function and nonlinear slowly increasing function $(1+\varepsilon) \cdot \tilde{\sigma} \cdot a(t)$, $j = 1, 2, \ldots, k$.

Now we present a technical example from the computer network practice. Assume that messages arrive at the computer v_1 at the rate λ of 21 per hour during business hours. These messages are served at a rate μ of 20 per hour in the computer v_1. After service in the computer v_1 messages arrive at the second computer v_2. Also we note that messages are served at a rate μ of 20 per hour in the computer v_2. So, messages is served in computers v_1, v_2,...,v_k, and after messages are served in computer v_k, they leave computer network.

So,
$\beta_1 = (MS_n)^{-1} = \mu = 20$, $\beta_0 = (Mz_n)^{-1} = \lambda = 21$, $\alpha = 21 - 20 = 1$, $DS_n = 1/\mu = 1/20$, $Dz_n = 1/\mu = 1/21$, $\hat{\sigma}_1^2 = DS_n \cdot (MS_n)^{-3} = (1/\mu)^2 = 0.0025$, $\hat{\sigma}_0^2 = DS_n \cdot (MS_n)^{-3} = (1/\mu)^2 = 0.0023$, $\tilde{\sigma}^2 \sim 0.0047$, $\tilde{\sigma} \sim 0.0685$, $\varepsilon = 0.001$, $t \geq 100$.

Thus,

$$Mv_j(t) \sim \alpha \cdot t + (1+\varepsilon) \cdot \tilde{\sigma} \cdot a(t) = (1.00) \cdot t + (0.0685) \cdot a(t), \quad (7)$$

$j = 1, 2, \ldots, k$. From (7) we get

$$\frac{Mv_j(t)}{t} = (1.00) + (0.0685) \cdot \sqrt{\frac{2 \ln \ln t}{t}}, \; j = 1, 2, \ldots, k. \quad (8)$$

Now we present figure for $\frac{Mv_j(t)}{t}$, $j = 1, 2, \ldots, k$, when $100 \leq t \leq 1000$, $\varepsilon = 0.001$ (see (8) and Table 1).

Table 1 Summary of computing results

Time t	$\frac{Mv_j(t)}{t}$, $j = 1, 2, \ldots, k$
100	1.0120
200	1.0088
300	1.0074
400	1.0065
500	1.0059
600	1.0054
700	1.0050
800	1.0047
900	1.0045
1000	1.0043

We see that when $\alpha = 0.01 > 0$, average total queue length of messages is about 1 in all message system.

Corollary 5.1. *Average total queue length of messages system direcly depends of traffic coefficient α and time t and is the same in all phases of multistage message system.*

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