

# A Variable Structure MRAC for a Class of MIMO Systems

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**Abstract**—A Variable Structure Model Reference Adaptive Controller using state variables is proposed for a class of multi input–multi output systems. Adaptation law is of variable structure type and switching functions is designed based on stability requirements. Global exponential stability is proved based on Lyapunov criterion. Transient behavior is analyzed using sliding mode control and shows perfect model following at a finite time.

**Keywords**—Adaptive control, Model reference, Variable structure, MIMO system.

## I. INTRODUCTION

MODEL Reference Adaptive Controller with conventional continuous adaptation laws has been investigated extensively in the literatures in two main branches: one assuming full state accessibility [7], and the other assuming accessibility of input and output [7]. Continuous adaptation laws are in the form of pure integral actions and have some problems such as:

- Transient behavior is difficult to analysis.
- Only global (but not asymptotic) stability has been guaranteed.
- Undesirable transient responses and tracking performance
- Lack of robustness

The variable structure systems (VSS) have been studied in great details in the literatures [4, 6, 8]. The basic concept of the variable structure control is that of sliding mode control. Switching control functions are generally designed to generate sliding surfaces, or sliding modes, in the state space [8]. When this is attained, the switching functions keep the trajectory on the sliding surfaces and the closed loop system becomes insensitive, to a certain extent, to parameter variations and disturbances.

The variable structure model reference adaptive controller has been investigated in the literatures in two main branches: one assuming full state accessibility [5], and the other assuming accessibility of input and output [2,3,9,10]. Systems (VSS) have been studied in great details in the literatures.

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## II. PROBLEM DEFINITION

Consider a linear time variant plant with unknown parameters, which their bounds are known. Let the plant be of  $n$ -th order with accessible states and described by the differential equation

$$\dot{X} = AX + BU \quad (1)$$

where  $n \times n$  matrix  $A$  matrix is unknown and of full rank, while the  $n \times m$  matrix  $B$  is assumed to be known and partitioned as

$B = [b_1 \ b_2 \ \dots \ b_m]$ .  $U$  is a  $m$ -dimensional control vector, and  $(A,B)$  is controllable.

The reference model is characterized by the linear time invariant differential equation

$$\dot{X}_m = A_m X_m + B_m V \quad (2)$$

where  $A_m$  is a  $n \times n$  asymptotically stable matrix,  $B_m$  is a known matrix, and  $V$  is a  $m$ -dimensional input vector with bounded elements. The purpose is to find control  $U$  such that the state error

$$e = X - X_m \quad (3)$$

exponentially tends to zero in a finite time.

## III. VARIABLE STRUCTURE ADAPTIVE CONTROLLER

The matrix  $B_m$  of the reference model can be chosen as

$$B_m = B Q^* \quad (4)$$

where  $Q^*$  is a known  $m \times m$  matrix. It is further assumed that an unknown  $m \times n$  matrix  $\theta^*$  exists such that

$$A + B \theta^* = A_m \quad (5)$$

### A. Stability and Switching Function Design

The control  $U$  to the plant, is generated introducing control law

$$U = \Psi X + Q^* V \quad (6)$$

where  $m \times n$  feedback matrix  $\Psi$ , with the elements  $\psi_{ij}$  are adjusted using VS approach by designing switching functions  $\psi_{ij}$  as described in the followings.

Subtracting equation (2) from (1), and using equations (4), (5) and (6), the error equation is obtained as

$$\dot{e} = A_m e + B(\Psi - \Theta^*)X \quad (7)$$

Consider a Lyapunov function of the form

$$\Lambda = e^T P e \quad (8)$$

where P is a positive definite symmetric matrix which satisfies the Lyapunov equation

$$A_m^T P + P A_m = -Q_0 \quad (9)$$

where  $Q_0$  is a positive definite symmetric matrix.

Differentiating (8) with respect to time along the trajectory (7) yields

$$\begin{aligned} \dot{\Lambda} &= -e^T Q_0 e + 2B^T P e (\Psi - \Theta^*)X \\ &= -e^T Q_0 e + 2 \sum_{i=1}^m \left[ b_i^T P e \sum_{j=1}^n (\psi_{ij} - \theta_{ij}^*) x_j \right] \end{aligned} \quad (10)$$

Now, introducing the switching functions  $\psi_{ij}$  as

$$\psi_{ij} = -\overline{\theta}_{ij} \operatorname{sgn}(b_i^T P e x_j), \quad \overline{\theta}_{ij} > |\theta_{ij}^*| \quad (11)$$

and substitute into (10), yields

$$\begin{aligned} \dot{\Lambda} &= -e^T Q_0 e \\ &+ 2 \sum_{i=1}^m \left\{ b_i^T P e \sum_{j=1}^n \left[ -\overline{\theta}_{ij} \operatorname{sgn}(b_i^T P e x_j) - \theta_{ij}^* \right] x_j \right\} \\ &= -e^T Q_0 e \\ &- 2 \sum_{i=1}^m \left\{ \sum_{j=1}^n \left[ \overline{\theta}_{ij} |b_i^T P e x_j| + \theta_{ij}^* (b_i^T P e x_j) \right] \right\} \end{aligned} \quad (12)$$

The terms in the summation are always positive, therefore  $\dot{\Lambda} < 0$  and regarding (8) it can be concluded that  $\|e\|$  decreases at least exponentially.

### B. Existence of Sliding Mode

Here it is shown that the hyper surfaces

$$S_k = b_k^T P e = 0 \quad (13)$$

are always sliding hyper surfaces for the system. Let examine the following reaching conditions [15, 9].

$$S_k \dot{S}_k < 0, \quad \left| \dot{S}_k \right| > 0 \quad (14)$$

Regarding equations (7) and (11), we have

$$\begin{aligned} S_k \dot{S}_k &= S_k b_k^T P A_m e + S_k b_k^T P B (\Psi - \Theta^*) X \\ &= S_k b_k^T P A_m e \\ &\quad + S_k \sum_{i=1}^m \left[ b_k^T P b_i \sum_{j=1}^n (\psi_{ij} - \theta_{ij}^*) x_j \right] \\ &= S_k (b_k^T P A_m) e \\ &+ S_k \sum_{i=1}^m \left\{ b_k^T P b_i \sum_{j=1}^n \left[ -\overline{\theta}_{ij} \operatorname{sgn}(b_i^T P e x_j) - \theta_{ij}^* \right] x_j \right\} \\ &= S_k (b_k^T P A_m) e \\ &\quad - |S_k| \sum_{i=1}^m \left\{ b_k^T P b_i \operatorname{sgn}(S_k S_i) \sum_{j=1}^n \left[ \overline{\theta}_{ij} + \theta_{ij}^* \operatorname{sgn}(b_i^T P e x_j) \right] |x_j| \right\} \\ &\quad \left\langle |S_k| \left[ \varepsilon_1 \|e\| - \sum_{i=1}^m \left\{ b_k^T P b_i \operatorname{sgn}(S_k S_i) \sum_{j=1}^n \left[ \overline{\theta}_{ij} + \theta_{ij}^* \operatorname{sgn}(b_i^T P e x_j) \right] |x_j| \right\} \right] \right\rangle \end{aligned} \quad (15)$$

where  $\varepsilon_1$  is a positive constant. Now we note that

$$S_k S_i = b_k^T P e b_i^T P e = b_k^T P e e^T P b_i.$$

The matrix  $e e^T$  is a positive semi-definite matrix, hence  $P e e^T P$  is also positive semi-definite, and therefore  $S_k S_i = b_k^T P e e^T P b_i$  should has the same sign as  $b_k^T P b_i$  or should be zero. Hence the terms in the summation in (15) are always positive or partly zero, since  $\|e\|$  exponentially tends to zero, it can be concluded that if we have

$$\|X(t)\| > \lambda > 0, \quad \forall t > t_0 \quad (16)$$

then there exists  $T > t_0$  such that

$$S_k \dot{S}_k < 0, \quad \forall t \geq T \quad (17)$$

Condition  $\left| \dot{S}_k \right| > 0$  in the relation (14) is considered here. One can writes

$$\begin{aligned} \dot{S}_k &= b_k^T P \dot{e} = b_k^T P A_m e + b_k^T P B (\Psi - \Theta^*) X \\ &= b_k^T P A_m e - \sum_{i=1}^m \left[ b_k^T P b_i \sum_{j=1}^n (\psi_{ij} - \theta_{ij}^*) x_j \right] \\ &= b_k^T P A_m e \\ &+ \sum_{i=1}^m \left\{ b_k^T P b_i \sum_{j=1}^n \left[ -\overline{\theta}_{ij} \operatorname{sgn}(b_i^T P e x_j) - \theta_{ij}^* \right] x_j \right\} \end{aligned} \quad (18)$$

since  $\|e\|$  exponentially tends to zero, and consider to equation (16), it can be seen at least one of the terms in the summation is nonzero and we have

$$\left| \dot{S} \right| > 0, \quad \text{for } t > T \quad (19)$$

Relations (17) and (19) mean that for all  $t > T$ , the surface  $S_k = b_k^T P e = 0$  is a sliding surface.

### C. Stability in the Presence of Bounded Disturbance

If a disturbance vector  $W(t)$  with bounded components, acts on the plant input, the error equation (7) becomes

$$\dot{e} = A_m e + B(\Psi - \Theta^*)X + B W \quad (20)$$

Consider the Lyapunov function (8), we have

$$\begin{aligned} \dot{\Lambda} &= -e^T Q_0 e \\ &= +2 \sum_{i=1}^m \left[ b_i^T P e \sum_{j=1}^n (\psi_{ij} - \theta_{ij}^*) x_j \right] \\ &\quad + 2e^T P B W \end{aligned} \quad (21)$$

and regard to switching functions  $\psi_{ij}$  as defined by equation (11), one can write

$$\begin{aligned} \dot{\Lambda} &= -e^T Q_0 e \\ &\quad - 2 \sum_{i=1}^m \left\{ \sum_{j=1}^n \left[ \overline{\theta_{ij}} \left| b_i^T P e x_j \right| + \theta_{ij}^* \left( b_i^T P e x_j \right) \right] \right\} \\ &\quad + 2e^T P B W \end{aligned} \quad (22)$$

Now, defining  $\Delta\theta_m$  and  $\overline{w}$  as

$$\begin{aligned} \Delta\theta_m &= \min_{i,j} (\overline{\theta_{ij}} - \theta_{ij}^*), \quad \overline{w} = \sup_{i}^{t>0} |w_i(t)|, \\ \sum x &= \sum_{j=1}^n |x_j| \end{aligned} \quad (23)$$

one can concluded that

$$\dot{V} < -\rho_1 \|e\|^2 - \rho_2 (\Delta\theta_m) (\sum x) \|e\| + \rho_3 \overline{w} \|e\| \quad (24)$$

where  $\rho_1$ ,  $\rho_2$ , and  $\rho_3$  are positive constants.

From relation (24), it is understandable that  $e$  should have a residual set as

$$\|e\| < \rho_4 \overline{w} \quad (25)$$

where  $\rho_4$  is a positive constant.

### D. Existence of Sliding Mode in the Presence of Bounded Disturbance

As mentioned in section (I.3), the error equation in the presence of bounded disturbance  $W(t)$ , will be in the form of equation (20).

In the same manner as followed in section (I.2) one can writes:

$$\begin{aligned} S_k \dot{S}_k &= S_k b_k^T P A_m e + S_k b_k^T P B (\Psi - \Theta^*) X \\ &\quad + S_k b_k^T P B W \\ \langle S_k | & \left[ \varepsilon_1 \|e\| - \sum_{i=1}^m \left\{ b_k^T P b_i \operatorname{sgn}(S_k S_i) \sum_{j=1}^n \left[ \overline{\theta_{ij}} + \theta_{ij}^* \operatorname{sgn}(b_i^T P e x_j) \right] |x_j| \right\} + \operatorname{sgn}(S_k) b_k^T P B W \right] \end{aligned} \quad (26)$$

since  $\|e\|$  exponentially tends to a residual domain specified by relation (25), it can be concluded that if we have

$$\|X(t)\| > \lambda_1 \overline{w}, \quad \forall t > t_0 \quad (27)$$

where  $\lambda_1$  is a positive constant, then there exists  $T > t_0$  such that

$$S_k \dot{S}_k < 0, \quad \forall t \geq T \quad (28)$$

Condition  $\left| \dot{S}_k \right| > 0$  can be proved in the same manner as applied for equations (18) and (19).

### E. Average Control

If control signal in equation (6) is written as

$$U = U^o + Q^* V, \quad U^o = \Psi X \quad (29)$$

where  $U^o$  and  $Q^* V$  are variable structure and continuous parts of controller, respectively. When sliding mode is occurred, actually the average control  $\overline{U}^o$  can be used [ ], which elements are the outputs of  $m$  first order filters

$$\tau_i \overline{u}^o_i + \overline{u}^o_i = u^o_i = \sum_{j=1}^n (\psi_{ij} x_j), \quad i = 1, 2, \dots, m \quad (30)$$

where time constants  $\tau_i$  are sufficiently small. Then the average control  $\overline{U}$  can be obtained as

$$\overline{U} = \overline{U}^o + Q^* V \quad (31)$$

The block diagram is presented in Fig. 1.

#### IV. SIMULATION RESULTS

In this section, simulation results are presented to show the performance of the proposed schemes and comparing these with the conventional schemes. The example consists of a time variant system with parameter variation. Effect of input disturbance on the controller is verified.

##### Example

Consider a time-variant system described by

$$\dot{X} = AX + BU \quad (32)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -9.81(1 + .1 \sin t) \\ .1 \sin t & .0000157(1 + .1 \sin t) & .1 \sin t \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \quad (33)$$

Choose a reference model described by equation

$$\dot{X}_m = A_m X_m + B_m V \quad (34)$$

where

$$A_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -9.8 \\ .85 & 1 & -5 \end{bmatrix}, \quad B_m = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \quad (35)$$

hence, from (4) and (5) it can be concluded that

$$Q^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \theta_{11}^* = \theta_{12}^* = 0,$$

$$\theta_{13}^* = -0.981 \sin(t), \quad \theta_{21}^* = 0.85 - 0.1 \sin(t),$$

$$\theta_{22}^* = 1 - 0.0000157(1 - 0.1 \sin(t)),$$

$$\theta_{23}^* = -5 - 0.1 \sin(t) \quad (36)$$

therefore, with regard to equation (11) we can choose

$$\bar{\Theta} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1.1 & 5.2 \end{bmatrix} \quad (37)$$

Thus the VS-MRA Controller can be designed using equations (11) and (6) for unfiltered, or equations (11) and (30-31) for filtered control. System was simulated, responses was compared with the responses of system with conventional

adaptation law as  $\dot{\theta} = -B^T p e X^T$ , and control law

$U = \theta X + Q^* V$ . The input signal was

$$V = \begin{bmatrix} 1 + \sin(2\pi t) \\ 1 + \sin(2\pi t) \end{bmatrix},$$

and initial conditions was  $X(0) = [0 \ 0 \ 0]^T$ , and

$X_m(0) = [0 \ 0 \ 0]^T$ . Responses are presented in Fig. 2, for

filtered control without disturbance, and in Fig. 3, for filtered control in the presence of a disturbance as

$$W = \begin{bmatrix} 1 + \sin(5\pi t) \\ 1 + \sin(3\pi t) \end{bmatrix}.$$

#### V. CONCLUSION

A variable structure model reference adaptive controller has been proposed for MIMO systems. Structure of the switching functions was designed based on exponential stability requirements. Magnitude of the switching functions then can be determined using reaching conditions of sliding mode. Relation (5) is used to determine the bounds on elements  $\theta_{ij}^*$ , and these elements must be limited.

This controller has some significant advantages compared to the conventional model reference adaptive controller. Global exponential stability is proved without requirements on persistence of excitation. Then it was shown that, it is always possible to introduce sliding mode into the system. Transient behavior was analyzed and showed perfect model following at a finite time. Insensitivity with respect to input disturbances was investigated and showed preference to the conventional schemes. Simulation was presented to clear the theoretical results.

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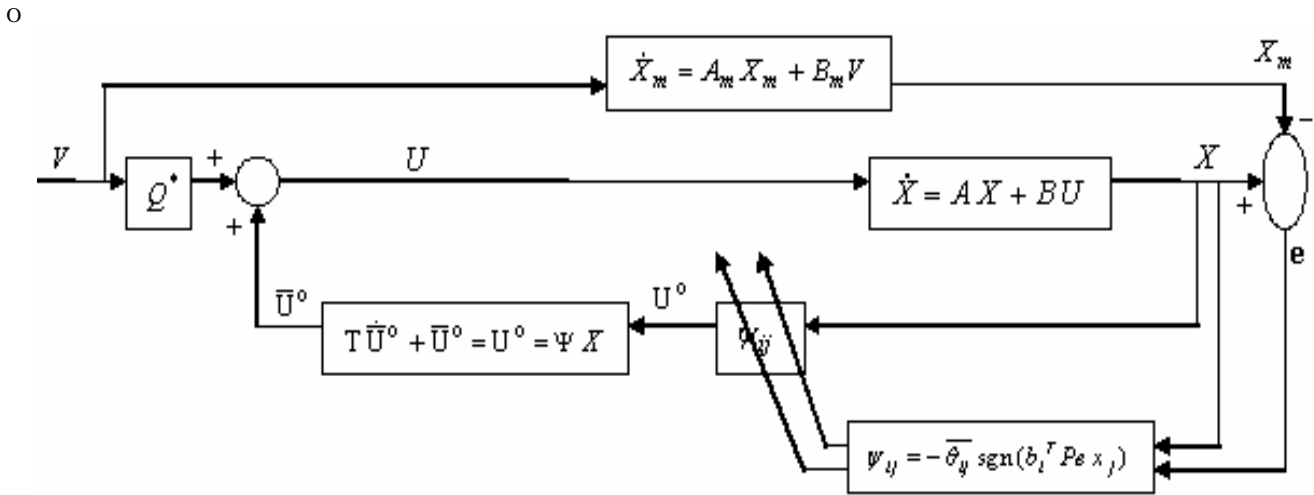


Fig. 1 The block diagram

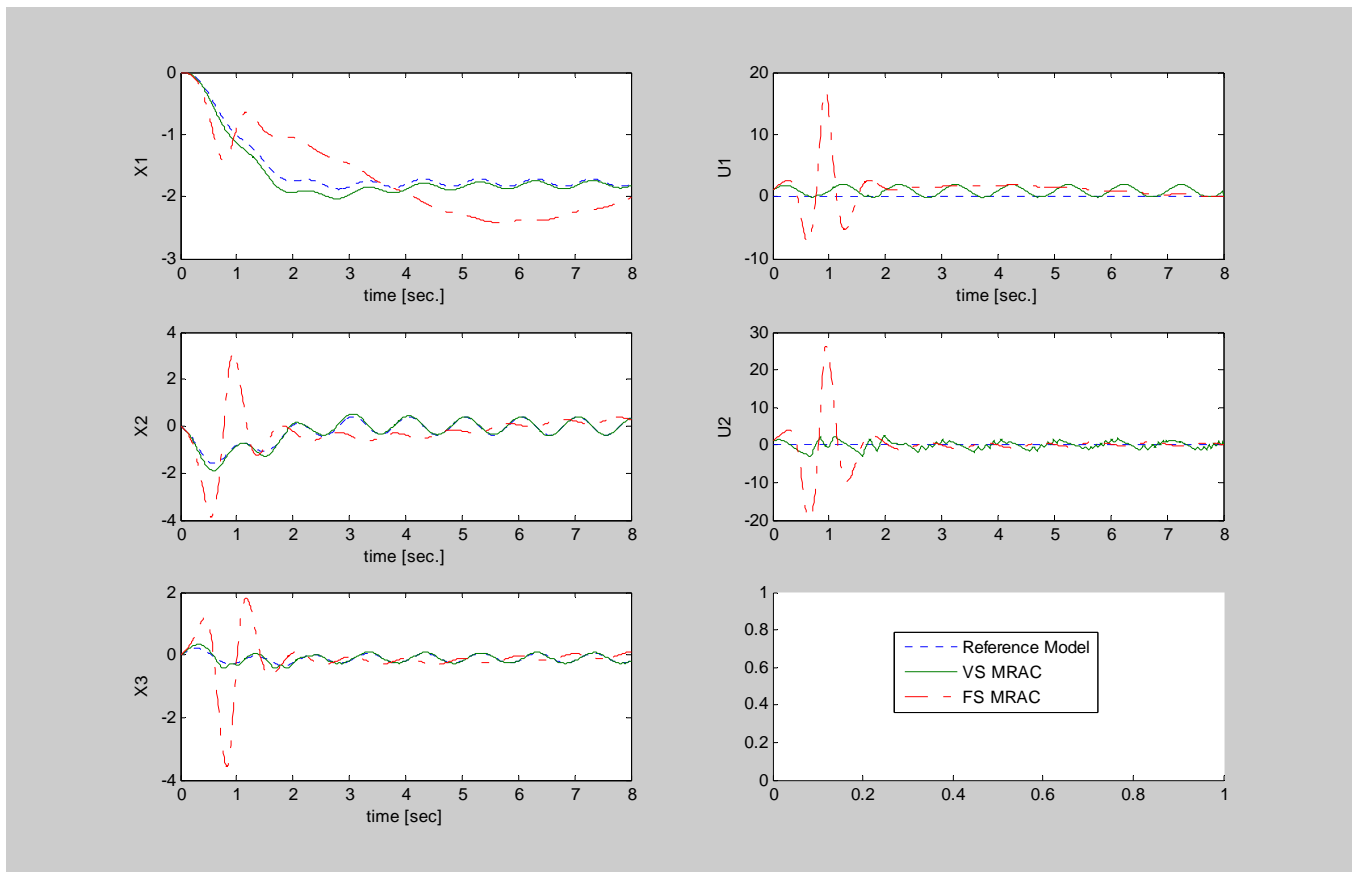


Fig. 2 Responses and controls for systems without disturbance

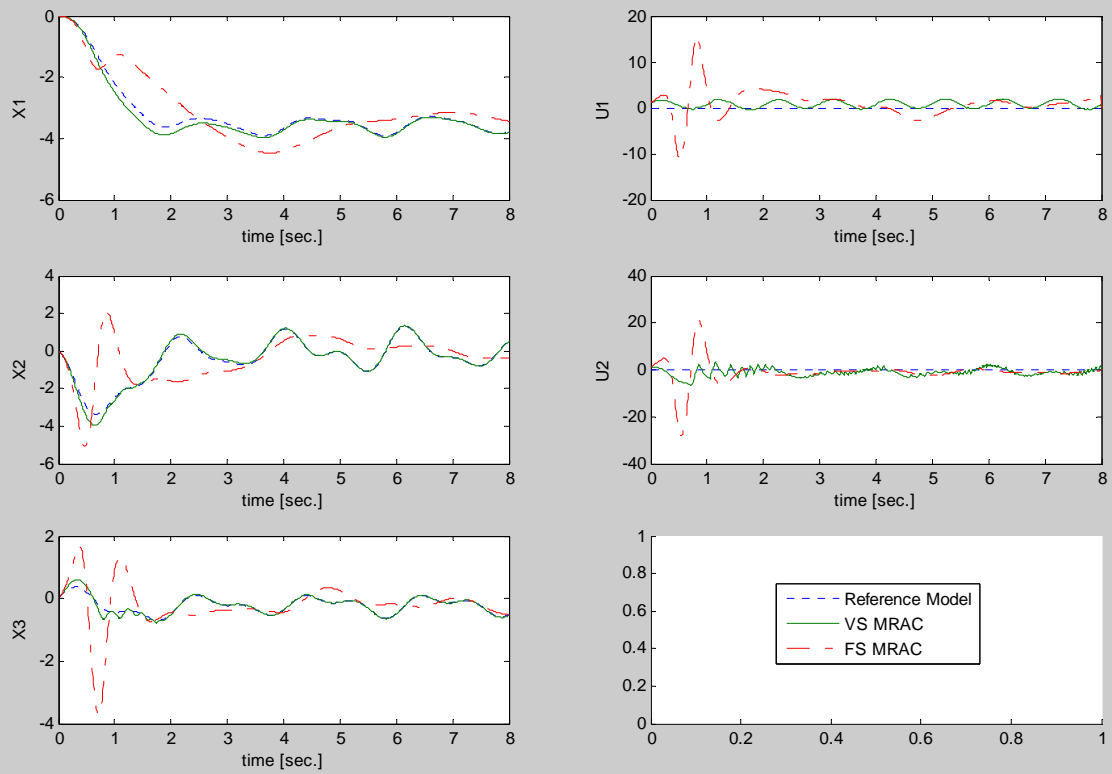


Fig. 3 Responses and controls for systems with disturbance