

Identifying an unknown source in the Poisson equation by a modified Tikhonov regularization method

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Abstract—In this paper, we consider the problem for identifying the unknown source in the Poisson equation. A modified Tikhonov regularization method is presented to deal with illposedness of the problem and error estimates are obtained with an a priori strategy and an a posteriori choice rule to find the regularization parameter. Numerical examples show that the proposed method is effective and stable.

Keywords—Ill-posed problem; Unknown source; Poisson equation; Tikhonov regularization method; Discrepancy principle

I. INTRODUCTION

In this paper, we consider the following problem [16], [17]:

$$\begin{cases} -u_{xx} - u_{yy} = f(x), & 0 < x < \pi, 0 < y < +\infty, \\ u(0, y) = u(\pi, y) = 0, & 0 \leq y < +\infty, \\ u(x, 0) = 0, u(x, y)|_{y \rightarrow \infty} \text{ bounded}, & 0 \leq x \leq \pi, \\ u(x, 1) = g(x), \end{cases} \quad (1)$$

Our purpose is to identify the source term $f(x)$ from the input data $g(\cdot) := u(\cdot, 1)$. This problem is called the inverse source problem. In practice, the data at $g(x)$ is often obtained on the basis of reading of physical instrument. So only a perturbed data $g^\delta(x)$ can be obtained. We assume that the exact and measured data satisfy

$$\|g - g^\delta\| \leq \delta. \quad (2)$$

where $\delta > 0$ denotes the noisy level, $\|\cdot\|$ denotes the L^2 -norm.

Inverse source problems arise in many branches of science and engineering, e.g. heat conduction, crack identification electromagnetic theory, geophysical prospecting and pollutant detection. The main difficult of these problems is that they are ill-posed (the solution, if it exists, does not depend continuously on the data). Thus, the numerical simulation is very difficult and some special regularization are required. A few papers have present the mathematical analysis and efficient algorithms of these problems. The uniqueness and conditional stability results for these problems can be found in [3], [4], [6], [2], [9], [13]. Some numerical reconstruction schemes can be found in [11], [1], [7], [10], [12], [5], [18], [15], [14], [16], [17].

In [16], [17], a Fourier method and a modified regularization method have been present for the problem (1). In both of methods, the regularization parameters are a priori. It is well

known that the ill posed problem is usually sensitive to the regularization parameter and the a priori bound is usually difficult to be obtained precisely in practice. In this paper, we will use the Tikhonov regularization method to solve the problem (1) in which a modified Tikhonov functional will be used. We will show that the regularization parameter can be chosen by the discrepancy principle. In [19], the method has been used to solve a backward heat equation.

This paper is organized as follows. In section 2, we will give the method to construct approximate solution. The convergence results will be found in section 3. Some numerical results are given in section 4 to show the efficiency of the new method.

II. A MODIFIED TIKHONOV REGULARIZATION METHOD

Let $\phi_l(x) = \sqrt{\frac{2}{\pi}} \sin(lx)$, it is well known that $\{\phi_l(x)\}_{l=1}^\infty$ is an orthonormal basis in $L^2(0, \pi)$, i.e.,

$$\int_0^\pi \phi_l(x) \phi_k(x) dx = \delta_{l,k}, \quad (3)$$

where $\delta_{l,k}$ is the Kronecher symbol. So for any $g \in L^2(0, \pi)$, we can write $g(x) = \sum_{l=1}^\infty \hat{g}_l \phi_l(x)$, where

$$\hat{g}_l = \int_0^\pi g(x) \phi_l(x) dx, \quad l = 0, 1, 2, \dots \quad (4)$$

It can be verified that the solution of problem (1) can be given as

$$f(x) = \sum_{l=1}^\infty \frac{l^2 \hat{g}_l}{1 - e^{-l}} \phi_l(x) =: Tg \quad (5)$$

It is apparent that the exact data \hat{g}_l must decay faster than the rate l^{-2} . However, in general, the measured function g^δ does not possess such a decay property. So

$$Tg^\delta = \sum_{l=1}^\infty \frac{l^2 \hat{g}_l^\delta}{1 - e^{-l}} \phi_l(x) \quad (6)$$

can not give a reliable approximation for f . In the following, we apply the Tikhonov regularization method to reconstruct a new function h^δ from the perturbed data g^δ and Th^δ will give a reliable approximation of f . Before doing that we impose an a priori bound on the unknown source

$$\|f\|_p \leq E, \quad p > 0, \quad (7)$$

where $E > 0$ is a constant and $\|\cdot\|_p$ denotes the norm in Sobolev space which is defined by [8] as follows:

$$\|f\|_p = \left(\sum_{l=1}^{\infty} (1+l^2)^p |\hat{f}_l|^2 \right)^{\frac{1}{2}}. \quad (8)$$

We let $h^\delta = h^{\alpha,\delta}$ be the minimizer of the Tikhonov functional

$$\Phi(h) = \|h - g^\delta\|^2 + \alpha \|Th\|_p^2, \quad (9)$$

where $\alpha > 0$ is a regularization parameter. If we let $h^\delta(x) = \sum_{l=1}^{\infty} \hat{h}_l^\delta \phi_l(x)$, it can be verified that \hat{h}_l^δ satisfy [?]

$$\left(1 + \alpha(1+l^2)^p \left(\frac{l^2}{1-e^{-l}} \right)^2 \right) \hat{h}_l^\delta = \hat{g}_l^\delta. \quad (10)$$

So we can get

$$\hat{h}_l^\delta = \frac{1}{1 + \alpha(1+l^2)^p \left(\frac{l^2}{1-e^{-l}} \right)^2} \hat{g}_l^\delta \quad (11)$$

That is to say

$$h^\delta(x) = \sum_{l=1}^{\infty} \frac{1}{1 + \alpha(1+l^2)^p \left(\frac{l^2}{1-e^{-l}} \right)^2} \hat{g}_l^\delta \phi_l(x). \quad (12)$$

Then the approximation solution can be given as

$$f^{\alpha,\delta} = Th^\delta = \sum_{l=1}^{\infty} \frac{l^2}{1 + \alpha(1+l^2)^p \left(\frac{l^2}{1-e^{-l}} \right)^2} \hat{g}_l^\delta \phi_l(x). \quad (13)$$

III. THE CHOICES OF REGULARIZATION PARAMETER α AND CONVERGENCE RESULTS

In this section, we consider an a priori strategy and an a posteriori choice rule to find the regularization parameter. Under each choice of the regularization parameter, the convergence estimate can be obtained. The follows lemmas are needed

Lemma 1: For any $l \in \mathbb{N}$, we have

$$l^2 \leq \frac{l^2}{1-e^{-l}} \leq 1+l^2, \quad (14)$$

Lemma 2: If $\|Th\|_p \leq E$, then we have

$$\|Th\| \leq \|h\|_{\frac{p}{p+2}} \|Th\|_{\frac{2p}{p+2}}. \quad (15)$$

Proof: By the Hölder inequality and lemma 1,

$$\begin{aligned} \|Th\|^2 &= \sum_{l=1}^{\infty} \left(\frac{l^2}{1-e^{-l}} \right)^2 |\hat{h}_l|^2 \\ &= \sum_{l=1}^{\infty} \left(\frac{l^2}{1-e^{-l}} \right)^2 |\hat{h}_l|^{\frac{4}{p+2}} |\hat{h}_l|^{\frac{2p}{p+2}} d\xi \\ &\leq \left[\sum_{l=1}^{\infty} \left[\left(\frac{l^2}{1-e^{-l}} \right)^2 |\hat{h}_l|^{\frac{4}{p+2}} \right]^{\frac{p+2}{2}} \right]^{\frac{2}{p+2}} \\ &\quad \left[\sum_{l=1}^{\infty} \left(|\hat{h}_l|^{\frac{2p}{p+2}} \right)^{\frac{p+2}{p}} \right]^{\frac{2}{p+2}} \\ &\leq \sum_{l=1}^{\infty} \left[\left(\frac{l^2}{1-e^{-l}} \right)^{p+2} |\hat{h}_l|^2 \right]^{\frac{2}{p+2}} \|h\|_{\frac{2p}{p+2}} \\ &\leq \left[\sum_{l=1}^{\infty} (1+l^2)^p \left(\frac{l^2}{1-e^{-l}} \right)^2 |\hat{h}_l|^2 \right]^{\frac{2}{p+2}} \|h\|_{\frac{2p}{p+2}} \\ &= \|h\|_{\frac{2p}{p+2}}^2 \|Th\|_{\frac{4}{p+2}}^2. \end{aligned} \quad (16)$$

A. The a priori choice rule

We assume that we have obtained an E in (7), then we take

$$\alpha_1 = \frac{\delta^2}{E^2} \quad (17)$$

as the a priori rule. We can obtain the following theorem.

Theorem 3: Suppose that the conditions (2) and (7) hold, $f^{\alpha_1,\delta}$ is defined by (13) and (17), then

$$\|f^{\alpha_1,\delta} - f\| = (\sqrt{2} + 1)E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}} \quad (18)$$

Proof: Owing to $h^{\alpha_1,\delta}$ is the minimizer of (9), we can obtain

$$\|h^{\alpha_1,\delta} - g^\delta\|^2 \leq \Phi(h^{\alpha_1,\delta}) \leq \Phi(g) = \|g - g^\delta\|^2 + \alpha_1 \|Tg\|_p^2 \leq 2\delta^2 \quad (19)$$

and

$$\|Th^{\alpha_1,\delta}\|_p^2 \leq \frac{1}{\alpha_1} \Phi(h^{\alpha_1,\delta}) \leq \frac{1}{\alpha_1} \Phi(g) \leq 2E^2 \quad (20)$$

So we have

$$\|h^{\alpha_1,\delta} - g\| \leq \|h^{\alpha_1,\delta} - g^\delta\| + \|g - g^\delta\| \leq (\sqrt{2} + 1)\delta \quad (21)$$

$$\|T(h^{\alpha_1,\delta} - g)\|_p \leq \|Th^{\alpha_1,\delta}\|_p + \|Tg\|_p \leq (\sqrt{2} + 1)E \quad (22)$$

The assertion of theorem will be obtained by lemma 2. ■

B. The a posteriori choice rule

In the case of

$$\|g^\delta\| \geq \delta, \quad (23)$$

the Morozov's discrepancy principle is used as an a posteriori rule in this paper, i.e., choosing α from the following equation

$$\|h^{\alpha_2,\delta} - g^\delta\| = \delta. \quad (24)$$

Theorem 4: Suppose that the condition (2),(7) and (23) hold, $f^{\alpha_2,\delta}$ is defined by (13) and (24), then

$$\|f^{\alpha_2,\delta} - f\| = 2E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (25)$$

Proof: Owing to $h^{\alpha_2,\delta}$ is the minimizer of (9),

$$\begin{aligned} \|h^{\alpha_2,\delta} - g^\delta\|^2 + \alpha_2 \|Th^{\alpha_2,\delta}\|_p^2 &= \Phi(h^{\alpha_2,\delta}) \\ &\leq \Phi(g) = \|g - g^\delta\|^2 + \alpha_2 \|Tg\|_p^2 \end{aligned} \quad (26)$$

So by (2), (7) and (24),

$$\|Th^{\alpha_2,\delta}\|_p^2 \leq \|f\|_p^2 + \frac{1}{\alpha_2} (\|g - g^\delta\|^2 - \delta^2) \leq \|f\|_p^2 \leq E^2 \quad (27)$$

Hence

$$\|T(h^{\alpha_2,\delta} - g)\|_p \leq \|f\|_p + \|Th^{\alpha_2}\|_p \leq 2E. \quad (28)$$

Moreover,

$$\|h^{\alpha_2,\delta} - g\| \leq \|h^{\alpha_2,\delta} - g^\delta\| + \|g - g^\delta\| \leq 2\delta \quad (29)$$

■ The assertion of theorem will be obtained by lemma 2. ■

IV. NUMERICAL TESTS

In practical problem, the perturbed data of functions is usually given at nodes. In this case, our approach can be realized by a discrete sine transform(DST).

Give $N - 1$ knots

$$x_i = i\pi/N, i = 1, \dots, N - 1.$$

The noisy vector $\mathbf{g}^\delta = (\mathbf{g}_1^\delta, \dots, \mathbf{g}_{N-1}^\delta)$ of the vector $\mathbf{g} = (\mathbf{g}_1, \dots, \mathbf{g}_{N-1}) = (g(x_1), \dots, g(x_{N-1}))$ are given by

$$\mathbf{g}_i^\delta = \mathbf{g}_i + \epsilon_i \quad (30)$$

where $\{\epsilon_j\}_{j=0}^N$ are generated by Function $\text{randn}(N, 1) \times \delta_1$ in Matlab, and the condition

$$\|\mathbf{g}^\delta - \mathbf{g}\|_{l^2} \leq \delta$$

is assumed.

Then the expansion

$$\mathbf{g}_i^\delta = \sum_{l=1}^{N-1} \hat{\mathbf{g}}_l^\delta \sin(lx_i), \quad i = 1, \dots, N - 1$$

can be obtained, where the coefficients

$$\hat{\mathbf{g}}_l^\delta = \frac{2}{N} \sum_{i=1}^{N-1} \mathbf{g}_i^\delta \sin(lx_i), \quad l = 1, 2, \dots, N - 1.$$

Then we can give the approximate function as follows

$$f^{\alpha, \delta}(x) = \sum_{l=1}^{N-1} \frac{l^2}{1 + \alpha(1 + l^2)^p} \frac{1}{1 - e^{-l}} \hat{\mathbf{g}}_l^\delta \sin(lx). \quad (31)$$

Example 1 In this example, we let

$$f(x) = \sum_{l=1}^{\infty} e^{-l} \sin lx, \quad (32)$$

accordingly,

$$g(x) = \sum_{l=1}^{\infty} \frac{(1 - e^{-l})e^{-l}}{l^2} \sin(lx) \quad (33)$$

It is obviously that the condition (7) holds for any $p \in \mathbb{R}^+$. We can not give the analytic expressions of f and g , so we take

$$f(x_i) = \sum_{l=1}^{M_1} e^{-l} \sin lx_i, \quad (34)$$

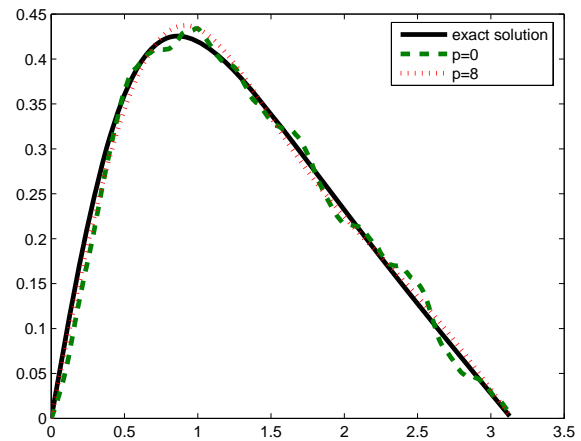
and

$$g(x_i) = \sum_{l=1}^{M_1} \frac{(1 - e^{-l})e^{-l}}{l^2} \sin(lx_i) \quad (35)$$

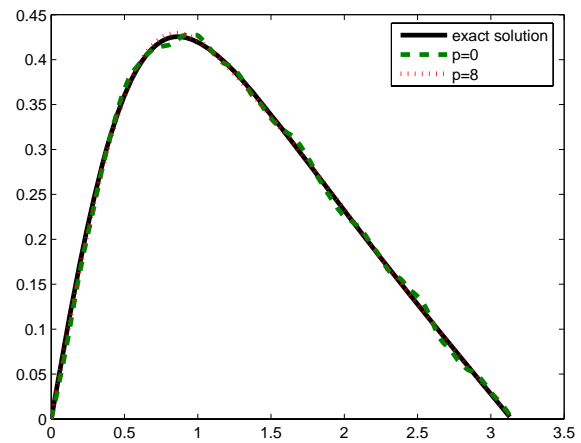
in practical computing, where M_1 is an integer such that $\sum_{l=M_1}^{\infty} e^{-2l} \leq 10^{-20}$. The numerical results for taking $p = 0, 2, 4, 8$ in (9) are displayed in Table 1. We can see that when δ decreases from 0.1 to 0.0001, the relative errors become smaller and when p increases, the rates of convergence become larger.

Example 2 In this example, we let

$$f(x) = \sum_{l=1}^{\infty} \frac{(1 + l^2)^{-2}}{l} \sin lx, \quad (36)$$



(a) The exact solution and its approximation with $\delta_1 = 0.01$



(b) The exact solution and its approximation with $\delta_1 = 0.001$

Fig. 1. Results of Example 1

accordingly,

$$g(x) = \sum_{l=1}^{\infty} \frac{(1 + l^2)^{-2}}{l} \sin(lx) \quad (37)$$

We also can not give the analytic expressions of f and g , so we take

$$f(x_i) = \sum_{l=1}^{M_2} \frac{(1 + l^2)^{-2}}{l} \sin lx_i, \quad (38)$$

and

$$g(x_i) = \sum_{l=1}^{M_1} \frac{(1 + l^2)^{-2}}{l} \sin(lx_i) \quad (39)$$

in practical computing, where M_2 is an integer such that $\sum_{l=M_2}^{\infty} \frac{(1 + l^2)^{-2}}{l} \leq 10^{-20}$.

In this example, the condition (7) holds only for $p < 2$, while we also compute with $p = 0, 2, 4, 8$. Results are given in Table 2, we can see that the results are close for $p \geq 2$,

TABLE I
 NUMERICAL RESULTS OF EXAMPLE 1

δ_1	$p = 0$	$p = 2$	$p = 4$	$p = 8$
1e-1	0.2266	0.2578	0.2620	0.2628
1e-2	0.0631	0.0450	0.0453	0.0462
1e-3	0.0212	0.0109	0.0103	0.0104
1e-4	0.0069	0.0021	0.0019	0.0018

TABLE II
 NUMERICAL RESULTS OF EXAMPLE 1

δ_1	$p = 0$	$p = 2$	$p = 4$	$p = 8$
1e-1	0.0692	0.0655	0.0785	0.0810
1e-2	0.0375	0.0140	0.0167	0.0175
1e-3	0.0162	0.0023	0.0029	0.0034
1e-4	0.0054	0.0006	0.0007	0.0007

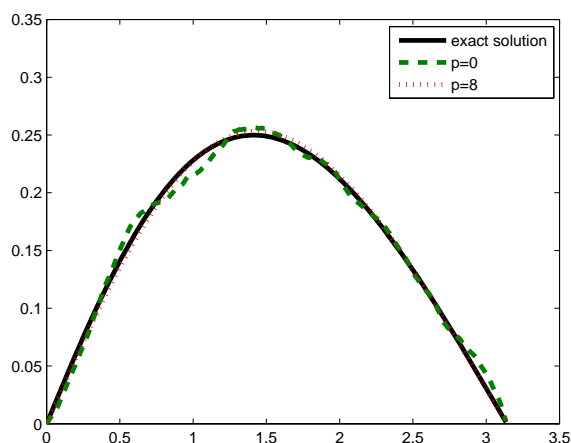
which indicate that the method can work well even if p is greater than the smooth scale of f .

V. CONCLUSION

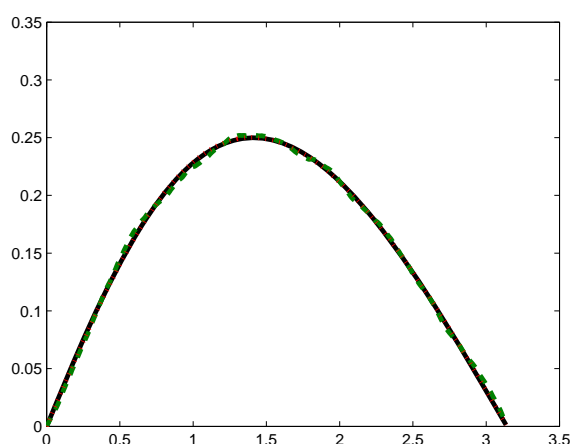
We have proposed a new method to Identify the unknown source in the Poisson equation. Theoretical analysis as well as experience from computations indicate that the proposed method works well. Moreover, the proposed method is readily extendable to solve some other ill posed problems, but these investigations are deferred to future work.

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(a) The exact solution and its approximation with $\delta_1 = 0.01$



(b) The exact solution and its approximation with $\delta_1 = 0.001$

Fig. 2. Results of Example2

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