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Statistical Analysis of First Order Plus Dead-time System using Operational Matrix

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Abstract**—**To increase precision and reliability of automatic control systems, we have to take into account of random factors affecting the control system. Thus, operational matrix technique is used for statistical analysis of first order plus time delay system with uniform random parameter. Examples with deterministic and stochastic disturbance are considered to demonstrate the validity of the method. Comparison with Monte Carlo method is made to show the computational effectiveness of the method.

Keywords**—**First order plus dead-time, Operational matrix, Statistical analysis, Walsh function.

I. INTRODUCTION

ONE of the modern approaches to build mathematical models and solve problems in control theory is to use the generalized spectral theory and matrix operator theory. These techniques, known as the spectral or operational matrix, are based on finite-dimensional approximation of the mathematical model of a system using the orthogonal expansions.

In this work, the operational matrix is used to take into account the influence of random changes in the parameters of the first order plus dead-time control system on the statistical characteristics of its output when the disturbance is deterministic or stochastic.

II. MOMENT OF ARBITRARY ORDER FOR UNIFORM RANDOM VARIABLE

The analysis method of stochastic systems described in [1] can be applied not only to systems with normally distributed random parameters, but also to systems with random parameters whose laws of the distribution are different from normal. For normal random variable, the moments of arbitrary order can be expressed through the cumulants [1]. For certain types of distributions, the moment expansions of the arbitrary order are much easier. In particular, for the uniform distributed random variable x on the interval $[a,b]$ with the probability density

$$f_x(x) = \frac{1}{b-a} \tag{1}$$

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The moment of arbitrary order random variable x is

$$\alpha^x_k = \frac{1}{b-a}\int_a^b x^k dx = \frac{b^{k+1}-a^{k+1}}{(k+1)(b-a)} \tag{2}$$

In the spectral models of stochastic systems, we transform random factors a_i in the form of

$$a_i = \overline{a}_i + a_i^r \tag{3}$$

where $\overline{a}_i = M[a_i]$ is the mean of a_i; a_i^r is the random central component.

Thus, if a random variable a_i is in the interval $[g_i^L, g_i^R]$, the central random variable a_i^r will be in the range

$$V_i = [g_i^L - M[a_i]; g_i^R - M[a_i]] \tag{4}$$

V_i can also be defined as follows

$$V_i = [-r_i \quad r_i] \tag{5}$$

where r_i is half the length of the interval; g_i^L, g_i^R are the left and the right borders, respectively.

Then, the k[th] order moment of a_i^r will be determined on the basis of (2) as following

$$\alpha_1^{a_i^r} = \frac{1}{2r_i}\int_{-r_i}^{r_i} x dx = \frac{r_i^2 - r_i^2}{2r_i} = 0$$

$$\alpha_2^{a_i^r} = \frac{1}{2r_i}\int_{-r_i}^{r_i} x^2 dx = \frac{r_i^3 + r_i^3}{3*2r_i} = \frac{r_i^2}{3}$$

$$\vdots$$

Note that all odd moments are zero, since the segment is symmetrically about 0. Thus, the general formula, which determines the k[th] moments of central uniformly distributed random variables, has the form

$$\alpha_k^{a_i^r} = \begin{cases} 0 & k-odd \\ \dfrac{r_i^k}{(k+1)} & k-even \end{cases} \tag{6}$$

III. Statistical Analysis for First Order Plus Dead-Time (FOPDT) Using Operational Matrix

1. Stochastic operational matrix of FOPDT

Consider a first order plus dead-time system with uniform random gain in Fig.1

$$\frac{K}{Ts+1}e^{(-Ls)} \tag{7}$$

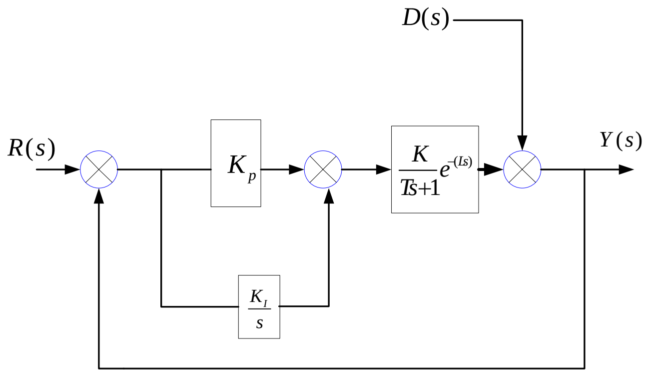

Fig. 1. Control system block diagram of FOPDT system

Let introduce the output and disturbance signals in the form of Fourier series

$$\begin{cases} D(t) \simeq D_l(t) = \sum_{k=1}^{l} c^D{}_k \varphi_k(t) = \Phi^T(t)C^d \\ Y(t) \simeq Y_l(t) = \sum_{k=1}^{l} c^Y{}_k \varphi_k(t) = \Phi^T(t)C^Y \end{cases} \tag{8}$$

where

$\Phi^T(t) = [\varphi_1, ..., \varphi_l]$ is a set of orthonormal basis(superscript T denotes operation transpose).

C^d is the vector coefficient of Fourier expansion or spectral characteristic of disturbance.

C^y is the vector coefficient or spectral characteristic of output signal.

Using the Pade approximation for a delay term, one can get operational matrix for the delay term as A_{pa}. Using the block matrix algebra for operational matrix [3], operational matrix for the open loop system is

$$A_l = A_c A_{pa}(TI + A_i)^{-1} A_i K \tag{9}$$

where A_c is operational matrix of proportional integral (PI) controller :

$$A_c = K_p I + K_i A_i \tag{10}$$

I is unity matrix and A_i is operational matrix of integrator

Let denote

$$A_1 = A_c A_{pa}(TI + A_i)^{-1} A_i \tag{11}$$

Relations between spectral characteristics of disturbance and output are given by

$$C^y = (I + A_1 K)^{-1} C^d \tag{12}$$

Denoting $A_0 = (I + A_1 \overline{K})^{-1}$ and using geometric series gives

$$(I + A_1 K)^{-1} = (I + A_1 \overline{K} + K_r A_1)^{-1} = A_0 \sum_{\nu=0}^{\nu=\infty} (-1)^{\nu} (K_r A_1 A_0)^{\nu} \tag{13}$$

where $\overline{K} = M[K]$ is the mean of random gain, and K_r is random central component of uniform variable K. Matrix in (13) is called stochastic operational matrix of system.

Assume that all eigenvalues of random matrix variable $K_r A_1 A_0 = A_r$ are inside of the unit circle or $|\lambda_j|_{A_r} < 1$ for the convergences of series by (13) [4]. If this condition is not satisfied, we can apply the precondition technique in [2].

2. Statistical analysis using operational matrix

Let consider the output and input signals in the form of Fourier series expansions

$$\begin{cases} D(t) \simeq D_l(t) = \sum_{k=1}^{l} c^D{}_k \varphi_k(t) = \Phi^T(t)C^D \\ Y(t) \simeq Y_l(t) = \sum_{k=1}^{l} c^Y{}_k \varphi_k(t) = \Phi^T(t)C^Y \end{cases} \tag{14}$$

and the spectral characteristics of output and input are linked by $C^Y = AC^D$.

Thus, an equation for the output of stochastic systems is

$$Y_l(t) = \Phi^T(t)C^Y = \Phi^T(t)AC^D \tag{15}$$

where A is stochastic matrix operator defined by (13).

The mean of (15) can be calculated as

$$\begin{aligned} m^l{}_Y(t) &= M[Y_l(t)] = M[\Phi^T(t)C^{m_Y}] = \Phi^T(t)M[C^Y] \\ &= \Phi^T(t)M[AC^D] \end{aligned} \tag{16}$$

From statistical independence of matrix A and vector-column of coefficient expansion of input C^y,

$$\begin{aligned} m^l{}_Y(t) &= \Phi^T(t)C^{m_Y} = \Phi^T(t)M[A]M[C^D] \\ &= \Phi^T(t)\overline{A}C^{m_D} \end{aligned} \tag{17}$$

Thus, the spectral characteristics of mathematical expectations of output and input signals of the stochastic system are related by

$$C^{m_Y} = \overline{A}C^{m_D} \tag{18}$$

According to which the spectral characteristics of the mathematical expectation of output signal is defined as a linear transformation of the spectral characteristics of the mathematical expectation input.

Deterministic matrix operator $\overline{A}$ is the expectation of a random stochastic matrix operator A. To determine the deterministic matrix operator $\overline{A}$, we calculate expectation of stochastic matrix operator, defined by (13)

$$\overline{A} = M[A] = M\{A_0 \sum_{\nu=0}^{\nu=\infty} (-1)^{\nu} (K_r A_1 A_0)^{\nu}\} = A_0 \sum_{\nu=0}^{\nu=\infty} (-1)^{\nu} M\{(K_r)^{\nu}\}(A_1 A_0)^{\nu} \quad (19)$$

The arbitrary-order stochastic moments $M\{(K_r)^{\nu}\}$ in (20) is calculated for each ν using the method mentioned in II.

$$A_0 \sum_{\nu=0}^{\nu=\infty} (-1)^{\nu} M\{(K_r)^{\nu}\}(A_1 A_0)^{\nu} \quad (20)$$

Equation (20) gives a relation between mathematical expectations of a stochastic matrix operator A and the statistical characteristics of random parameter in the stochastic system. The mathematical expectation of the output system, as determined by (16), (18) and (19), can be calculated with the desired accuracy. The accuracy depends on expectation stochastic matrix operator which is defined by ν-the number of approximation (20).

Let define the correlation function of the output stochastic system and represent its second central moment. By introducing the signal system in the form of (14), an equation to define the second moment of output can be written as

$$\theta^{l}_{YY}(t_1,t_2) = M[Y_l(t_1)Y_l(t_2)] = M[\Phi^T(t_1)C^Y(C^Y)^T\Phi(t_2)] = \Phi^T(t_1)M[C^Y(C^Y)^T]\Phi(t_2) = \Phi^T(t_1)M[AC^D(C^D)^T A^T]\Phi(t_2) \quad (21)$$

Random matrix $C^D(C^D)^T$ of (21) is statistically independent. Thus, equation (21) can take the form

$$\theta^{l}_{YY}(t_1,t_2) = \Phi^T(t_1)M[AC^{\theta_{DD}}A^T]\Phi(t_2) \quad (22)$$

where $C^{\theta_{DD}}$ is square matrix of the spectral characteristics of the second moment of the input of system which is determined by using (14), as follows

$$\theta^{l}_{DD}(t_1,t_2) = M[D_l(t_1)D_l(t_2)] = \Phi^T(t_1)M[C^D(C^D)^T]\Phi^T(t_2) = \Phi^T(t_1)M[C^{\theta_{DD}}]\Phi^T(t_2) \quad (23)$$

Covariance function or the second central moment of the output system is defined as

$$R^{l}_{YY}(t_1,t_2) = M\{[Y_l(t_1) - m^{l}_{Y}(t_1)][Y_l(t_2) - m^{l}_{Y}(t_2)]\} = M[Y_l(t_1)Y_l(t_2)] - m^{l}_{Y}(t_1)m^{l}_{Y}(t_2) = \theta^{l}_{YY}(t_1,t_2) - m^{l}_{Y}(t_1)m^{l}_{Y}(t_2) \quad (24)$$

where the first order moment $m^{l}_{Y}(t_1)$ is determined by (18) and the second moment by (22).

The covariance function of the input signal is similarly associated with the second order moment

$$R^{l}_{DD}(t_1,t_2) = \theta^{l}_{DD}(t_1,t_2) - m^{l}_{D}(t_1)m^{l}_{D}(t_2) \quad (25)$$

where $m^{l}_{D}(t_2)$ is the mathematical expectation of the input signal.

Furthermore, the covariance function of the input signal can be in the form of expansions in orthogonal basis

$$R^{l}_{DD}(t_1,t_2) = \Phi^T(t_1)C^{R_{DD}}\Phi(t_2) = \Phi^T(t_1)C^{\theta_{DD}}\Phi(t_2) - \Phi^T(t_1)C^{m_D}(C^{m_D})^T\Phi(t_2) \quad (26)$$

Thus, the spectral characteristics of the moments of input signal are related by

$$C^{R_{DD}} = C^{\theta_{DD}} - C^{m_D}(C^{m_D})^T \quad (27)$$

Equation (22) can be rewritten as follows

$$\theta^{l}_{YY}(t_1,t_2) = \Phi^T(t_1)M\{A[C^{R_{DD}} + C^{m_D}(C^{m_D})^T]A^T\}\Phi(t_2) \quad (28)$$

Taking into account (24) and (28), we have the following equations for the covariance function of the output stochastic system

$$R^{l}_{YY}(t_1,t_2) = \Phi^T(t_1)C^{R_{YY}}\Phi(t_2) = \Phi^T(t_1)M\{A[C^{R_{DD}} + C^{m_D}(C^{m_D})^T]A^T\}\Phi(t_2) - \Phi^T(t_1)C^{m_Y}(C^{m_Y})^T\Phi(t_2) \quad (29)$$

or

$$C^{R_{YY}} = C^{\theta_{YY}} - C^{m_Y}(C^{m_Y})^T = M\{A[C^{R_{DD}} + C^{m_D}(C^{m_D})^T]A^T\} - C^{m_Y}(C^{m_Y})^T \quad (30)$$

where A is random stochastic matrix operator, defined by (13).

Equation (30) gives the relation between the spectral characteristics of the covariance function of the output and the input signal, and the mathematical expectations of the output and the input signal.

IV. Case Studies

Consider regulatory problem for the following process

$$\frac{K}{s+1}e^{(-s)}$$

where K is uniform random variable in interval [0.5, 1.5]

Case 1. $K_p = 1; K_i = 0;$ Statistical analysis with disturbance is deterministic step signal using Walsh functions [1].

The number of Walsh basis is 128. Statistical characteristic of output signal using operational method determined is compared with the traditional Monte- Carlo analysis. The number of samples for random variable K is 2000. Some guide line for number of samples for the Monte-Carlo method can be found in [5]. Figs.2 and 3 show the statistical characteristics of output.

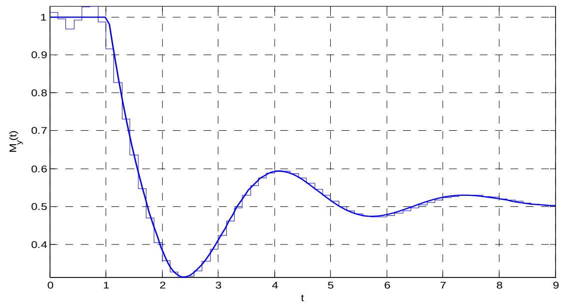

Fig. 2. Mean of output signal. Solid line: Monte-Carlo; Stair case line: method by operational matrix.

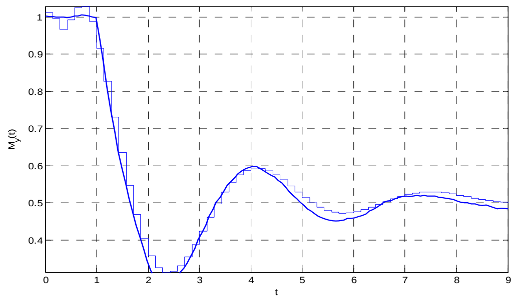

Fig. 4. Mean of output signal. Solid line: Monte-Carlo; Stair case line: operational matrix method.

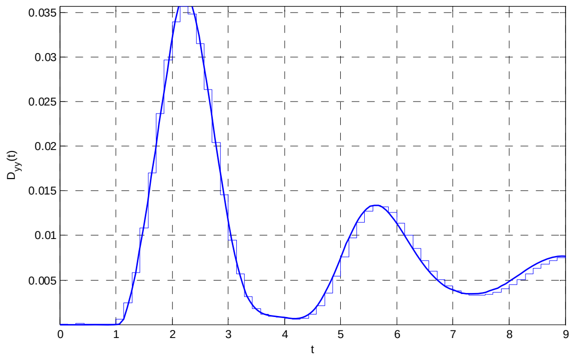

Fig. 3. Variance of output signal. Solid line Monte-Carlo; Stair case line: method by operational matrix.

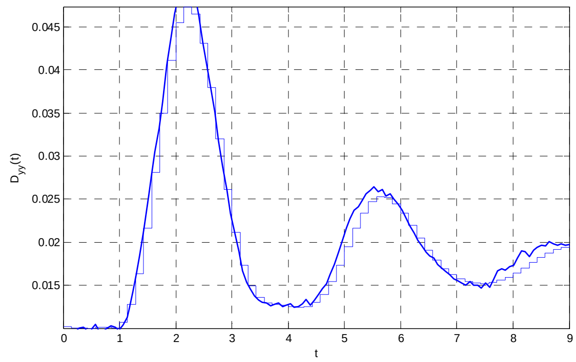

Fig. 5. Variance of output signal. Solid thick line Monte-Carlo; Stair case line: operational matrix method.

Computation time for operational matrix was 0.2 sec while that for Monte-Carlo was 198 sec. The computer with AMD Phenom II X3 2.81 GHz 2GB RAM was used for the test. Calculations were made using the library SML [1].

Case 2. $K_p = 1; K_i = 0;$ Statistical analysis with disturbance is random Gaussian process with $M_D(t) = 1$ and correlation function of $R_{DD} = 0.01e^{-5(t_1 - t_2)}$

The number of samples in ensemble of random disturbance signal for Monte Carlo analysis is 2000. The number of sample in random variable K is 200.The number of Walsh basis is 64.

Figs.4 and 5 show the statistical characteristics of output. Computation time for operational matrix was 0.3 sec while that for Monte-Carlo was 34 min. The computer with AMD Phenom II X3 2.81 GHz 2GB RAM was used for the test.

Case 3. Operational method is applied to design PI controller with constraint on variance of output for the deterministic step disturbance.

The performance objective function [1]:

$$\min_{K_p, K_i} J = \int_0^T M_Y(t, K_p, K_i)^2 \, dt$$

$$= \int_0^T (C^{m_y}(K_p, K_i))^T \Phi(t) \Phi^T(t) (C^{m_y}(K_p, K_i)) dt$$

$$= (C^{m_y}(K_p, K_i))^T (C^{m_y}(K_p, K_i))$$

subject to

$$\max_{0 \le t \le T} D_Y(t, K_p, K_i) \le D_{\max} = 5 * 10^{-2}$$

Figs.6 and 7 show the statistical characteristics of controlled output with optimal controller $K_p = 0.418$ $K_i = 0.451$.

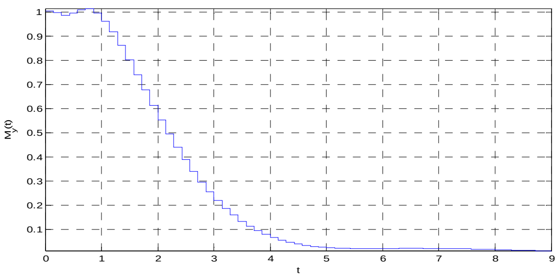

Fig. 6. Mean of output signal of controlled system

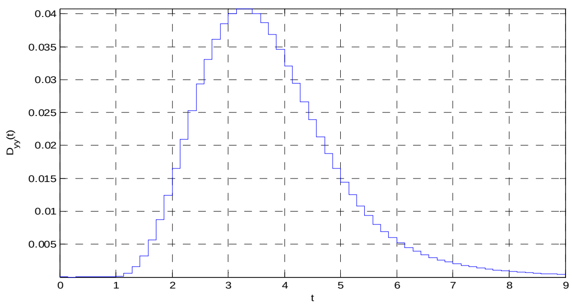

Fig. 7. Variance of output signal of controlled system

V. CONCLUSIONS

In this work, a statistical analysis for a first order plus dead-time system with random gain and random input was studied. It is shown that the use of operational matrix drastically reduces a computation time with a desired accuracy over that by the traditional Monte-Carlo method. Although the analysis was done only for a regulating problem of the FOPDT system, the proposed method can also be applied to other types of processes such as an integrator plus dead-time process.

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