Simulation of the Performance of Novel Nonlinear Optimal Control Technique on Two Cart-inverted Pendulum System

B. Baigzadeh, V.Nazarzehi, and H.Khaloozadeh

Abstract—The two cart inverted pendulum system is a good bench mark for testing the performance of system dynamics and control engineering principles. Devasia introduced this system to study the asymptotic tracking problem for nonlinear systems. In this paper the problem of asymptotic tracking of the two-cart with an inverted-pendulum system to a sinusoidal reference inputs via introducing a novel method for solving finite-horizon nonlinear optimal control problems is presented. In this method, an iterative method applied to state dependent Riccati equation (SDRE) to obtain a reliable algorithm. The superiority of this technique has been shown by simulation and comparison with the nonlinear approach.

Keywords—Nonlinear optimal control, State dependent Riccati equation, Asymptotic tracking, inverted pendulum

I. INTRODUCTION

THE inverted pendulum is a classic problem in dynamics and control theory and is widely used as a benchmark for testing control algorithms (Nonlinear controller, PID controllers, neural networks, fuzzy control, genetic algorithms, etc). The problem of achieving asymptotic tracking of this system has been well studied in literature [1].for an advance study the problem of asymptotic tracking in a nonlinear system with nonhyperbolic zero dynamics, or what is the same, nonhyperbolic internal dynamics, Devasia ([3], [4]) added an extra cart of the same mass (M) to the one-cart with an inverted-pendulum system, resulting in a two-cart with an inverted-pendulum system, as shown in Fig. 1. The system consists of two elastically connected carts. An inverted pendulum is placed on the first cart and it is freely hinged to the cart, which is free to move on a horizontal plane while the input is the force F acting on this cart. The equation of the motion of the two-cart with an inverted-pendulum system is as follows:

$$\begin{aligned} &(M+m)\ddot{\eta}_1 + ml\,(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + b\dot{\eta}_1 = u + K(\eta_2-\eta_1) \\ &m(\ddot{\eta}_1\cos\theta + l\ddot{\theta} - g\sin\theta) = 0 \\ &M\ddot{\eta}_2 + K(\eta_2-\eta_1) = 0 \end{aligned} \quad (1)$$

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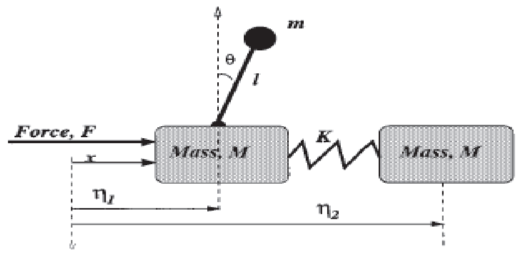


Fig. 1 A two-cart with an inverted-pendulum system [2]

where

M mass of the cart;
K is the spring constant
m mass of the block on the pendulum;
l length of the pendulum;
g acceleration because of gravity;
b coefficient of viscous friction for motion of the cart;
θ angle the pendulum makes with vertical;
η_1 position of the first cart cart;
η_2 is the position of the second cart
u applied force.

The problem of asymptotic tracking of the two-cart with an inverted-pendulum system to a class of sinusoidal reference inputs is actually solvable by the standard output regulation theory [1]. Moreover, an approximation method for calculating the center manifold equation associated with the output regulation problem for general nonlinear systems is given. This approach does not rely on the hyperbolicity condition and, hence, apply to a large class of nonlinear systems. Finally, an approximation control law is synthesized and its performance is illustrated through computer simulation [2]. A numerical method to solve the so-called regulator equation was presented in [5].The method is analyzed to obtain theoretical estimates of its convergence and it is tested on an example of the "two-cart with an inverted pendulum" system. In this paper for improving the problem of asymptotic tracking, a novel method for solving finite-horizon nonlinear optimal control problems is presented. The performance of this technique has been shown by simulation and comparison with nonlinear controller method. The problem of asymptotic tracking is to design a state feedback control law of the form u to locally stabilize the closed-loop system and to achieve zero tracking error asymptotically, i.e., to achieve

$$\lim_{t\to\infty} (y(t) - y_d(t) = 0) \tag{2}$$

The works have been done in nonlinear optimal control were based to simplified assumption or based on usage of series with side iterative process such as Chebyshev series([6], [7]) power series[8] and Valesh series[9] or successive approximation method ([10], [11]). These techniques are approximated and based on initial guess with the linear view to the problem. The idea of initial condition estimation for co-states introduce another method for solving Hamiltonian systems([12]) that because of high sensitivity of co-states to initial condition, its application has been limited to stable nonlinear systems.

Besides using direct optimization and estimation methods based on well-known series, there is a different group of techniques derived from indirect optimization and using Riccati equation besides iterative process ([13], [14], [15]). The idea has been used in this paper is based on the previous procedure.

In this research unlike the previously work that use Riccati equations that is validated for linear systems, we used a nonlinear Riccati structure without approximation.

The paper is organizes as follows, in first section to explain the above ideas precisely a novel method for solving finite horizon nonlinear optimal control problems is presented. In section 3 the performance of method is illustrated through computer simulation. Comparison between the standard output regulation theory method and our new method is given in Section 4. Conclusions are drawn in the final section.

II. FINITE HORIZON NONLINEAR OPTIMAL CONTROL

Optimal tracking problem is finding an optimal control law that forces the plant to maintain the system output as close as possible to the desired output. Suppose we have a plant with the following affine dynamics:

$$\dot{x} = f(x) + g(x)u \tag{3}$$

$$y = Cx \tag{4}$$

In which $f \in R^n$ and $g \in R^{n\times m}$ are supposed to be continuous and differentiable functions satisfying the Lipschitz conditions. $y \in R^r$ and $C \in R^{r\times n}$ are in sequence the output and the output matrix of the system. Quadratic cost functions are usually used for such problems:

$$J = \frac{1}{2}(Cx(t_f) - r(t_f))^T P(t_f)(Cx(t_f) - r(t_f)) + \frac{1}{2}\int_{t_0}^{t_f} \left((Cx(t) - r(t))^T Q(Cx(t) - r(t)) + u^T(t)Ru(t)\right) dt \tag{5}$$

Where $P(t_f) \in R^{r\times r}$ ، $Q \in R^{r\times r}$ are positive semi definite matrices and $R \in R^{m\times m}$ is positive definite matrixes, which are determined by the designer.

According to necessary conditions for optimality theorem [12] and the cost function of (5), the optimal control problem of such systems is found as:

$$\dot{x} = f(x) + g(x)u \qquad t \ge t_o \tag{6}$$

$$-\dot{\lambda} = C^T Q(Cx - r) + \frac{\partial (f(x) + g(x)u)^T}{\partial x}\lambda \qquad t \le t_f \tag{7}$$

$$0 = Ru + g^T(x)\lambda \tag{8}$$

In the cases of free final state and fix final time, given t_o and $x(t_o)$, the boundary conditions are:

$$x(t_o) = x_o \tag{9}$$

$$\lambda(t_f) = \left.\frac{\partial \phi(x)}{\partial x}\right|_{t=t_f} = C^T P(t_f)(Cx(t_f) - r(t_f)) \tag{10}$$

In (10), the boundary conditions of co-states are determined according to the cost function. To solve the two-point boundary value problem specified by (9) and (10), the sweep method shall be used [16], i.e. the following linear condition is supposed to hold between $x(t)$ and $\lambda(t)$ in the interval $([t_0, t_f])$:

$$\lambda(t) = C^T P(t)(Cx(t) - r(t)) \tag{11}$$

where $P(t) \in R^{n\times n}$ is positive definite. To obtain the known form of Riccati equation $f(x)$ could be rewritten without the loss of generality in the following form:

$$f(x) = A(x)x \tag{12}$$

where $A(x) \in R^{n\times n}$ is a continuous and differentiable matrix function. Differentiating (11) and substituting (6), (7) and (8) by defining $S(t)$ and $\nu(t)$ as follow:

$$S(t) = C^T PC \tag{13}$$

$$\nu(t) = C^T P(t)r(t) \tag{14}$$

the optimal tracking problem leads to the numerical solution of the following equations:

$$\dot{x} = f(x) + g(x)u \qquad t \ge t_o \tag{15}$$

$$-\dot{S} = \frac{\partial (f(x) + g(x)u)^T}{\partial x} S + SA(x) - Sg(x)R^{-1}g^T(x)S + C^T QC \qquad t \le t_f \tag{16}$$

$$-\dot{\nu} = \frac{\partial (f(x) + g(x)u)^T}{\partial x}\nu - Sg(x)R^{-1}g^T(x)\nu + C^T Qr \qquad t \le t_f \tag{17}$$

$$u = -R^{-1}g^T(x)Sx + R^{-1}g^T(x)\nu \tag{18}$$

There is a difficulty to solve the equations (16) and (17). Because These equations are dependent on states, but we have not the state values in the whole interval (i.e. $t \in [t_0, t_f]$). To overcome this difficulty we solve the above equations by an iterative procedure. At first we obtain the state values in the whole interval for the unforced system:

$$\dot{x} = f(x) \qquad (t \ge t_o, x(t_0) = x_0) \tag{19}$$

Now we can solve the equations (16) and (17), thus we have $u(t)$ from (18). Applying this control signal to the system (i.e. the equation (15)), we obtain new state values and again solving the equations (16) and (17). This iterative procedure will be continued until the expected performances will be achieved.

In this technique each iteration is completely based on the previous one that was not considered in last algorithm. In reality according to this method there is dynamic in both aspects, index (which is related to iterative process) and time aspect.

The method presented in ([13], [15]) parameters $S(t)$ and $\nu(t)$ in each iteration made based on $S(t)$ and $\nu(t)$ that on iteration and system state in last iteration whereas in new algorithm parameters $S(t)$ and $\nu(t)$ in each iteration derived from $S(t)$ and $\nu(t)$ and system states in preceding iteration ,thus this changes causes dynamic in both aspects (iteration and time) although in earlier method there is a dynamic only in time dimension likewise in this technique the Riccati equation is solvable in nonlinear form.

III. SIMULATION

We illustrate our approach performance on a two-cart system with an inverted pendulum. With the choice of the state variables as $x_1 = \eta_1$, $x_2 = \dot{\eta}_1$, $x_3 = \theta$, $x_4 = \dot{\theta}$, $x_4 = \eta_2$, $x_5 = \dot{\eta}_2$ and the control input $u(t) = F(t)$, the state space equations of system are :

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{M + m(\sin x_3)^2}\Big(u + mlx_4^2 \sin x_3 - bx_2 \\
&\quad - mg\cos x_3 \sin x_3 + K(x_5 - x_1)\Big) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{l(M + m(\sin x_3)^2)}\Big((M+m)g\sin x_3 \\
&\quad + \left[bx_2 - u - mlx_4^2 \sin x_3 - K(x_5 - x_1)\right]\cos x_3\Big) \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \frac{K}{M}(x_1 - x_5)
\end{aligned} \tag{20}$$

The output of the system is $y(t) = x_1(t)$ (or $\eta_1(t)$) .We will consider asymptotic tracking of the output $y(t)$ to a sinusoid input ($y_d(t) = A\sin \omega t$ with values of the amplitude $A = 1$ and the frequency $\omega = 1$). To achieve this goal, the choice of cost function could be:

$$J = \frac{1}{2}\int_0^{20}\Big(20000(x_1 - \sin(t))^2 + 500000x_3^2 + 7u^2\Big)dt. \tag{21}$$

$$A(x) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{-K}{\gamma(x_3)} & \frac{-b}{\gamma(x_3)} & \frac{-mg\cos x_3 \operatorname{sinc}(\frac{x_3}{\pi})}{\gamma(x_3)} & \frac{mlx_4 \sin x_3}{\gamma(x_3)} & \frac{K}{\gamma(x_3)} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{K\cos x_3}{l\gamma(x_3)} & \frac{b\cos x_3}{l\gamma(x_3)} & \frac{(M+m)g\operatorname{sinc}(\frac{x_3}{\pi})}{l\gamma(x_3)} & \frac{-mx_4\cos x_3 \sin x_3}{\gamma(x_3)} & \frac{-K\cos x_3}{l\gamma(x_3)} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{K}{M} & 0 & 0 & 0 & \frac{-K}{M} & 0
\end{bmatrix} \tag{22}$$

$$g(x) = \begin{bmatrix}
0 \\
\frac{1}{\gamma(x_3)} \\
0 \\
\frac{-\cos x_3}{l\gamma(x_3)} \\
0 \\
0
\end{bmatrix} \tag{23}$$

$A(x)$ and $g(x)$ are selected as shown in (22) and (23), at the bottom of the page, Where $\text{sinc}\,(\frac{x_3}{\pi}) = \frac{\sin x_3}{x_3}$ and $\gamma(x_3) = M + m\sin^2 x_3$.

System parameters were chosen, for the simulation, as $K = 10\frac{N}{m}$, $m = 0.051Kg$, $M = 1.378Kg$, $l = 0.325m$, $b = 12.98\frac{Kg}{s}$ and $g = 9.8\frac{m}{s^2}$. Performance of our control law has been shown in figure 2.

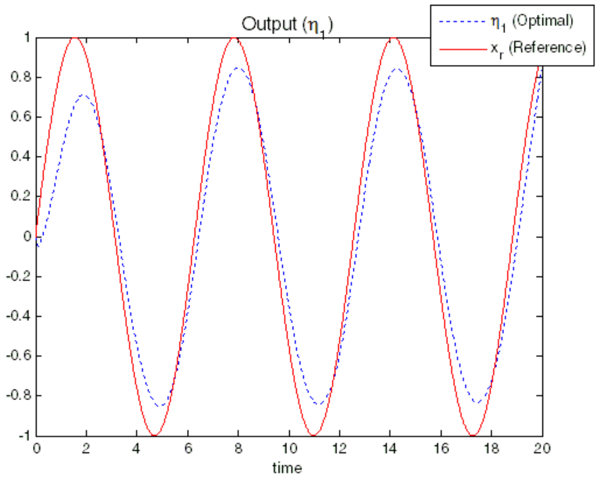


Fig. 2 tracking performance of optimal control

As shown in this figure the tracking signal has not good performance. Since Finite horizon optimal control is designed offline, hence for improving the performance of tracking signal its cost function is chosen as:

$$J = \frac{1}{2}\int_0^{20} \Big(15000\,(x_1 - \sin(t))^2 + 300000 x_3^{\,2} + 15000\,(x_5 - 1.7\sin(t))^2 + 7u^2\Big)dt \qquad (24)$$

Because of closed relationship between η_1 and η_2, to reach a better tracking a part of weight of cost function is allocated to x_5 with this work in addition to maintain the inverted pendulum in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Its permitted region) a better result can achieved, As shown in figure 3.

IV. COMPARISON OF RESULTS

The purpose of this section is to provide a comparative study of the new method proposed by this paper and the well-known method based on computation of undetermined coefficients of the Taylor expansion. In order to compare the performance of novel optimal control approach with a Huang nonlinear control the output of model is presented in figure 3 which the optimal control method has better performance in transient response.

Fig. 4 shows the inverted pendulum position signal. As shown in this figure for both controllers, its position is in allowed interval ($\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$). For having a better tracking in transient state in optimal case the inverted pendulum's deviation related to a vertical position is more than nonlinear one. Moreover big deviation in transient response decrease the system nonminimum phase tracking response in comparison with nonlinear case.

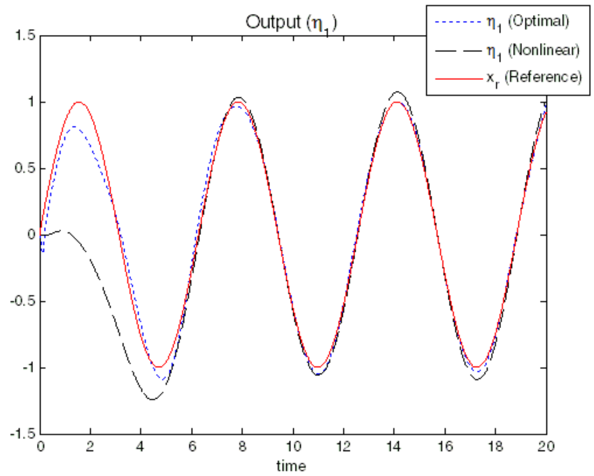
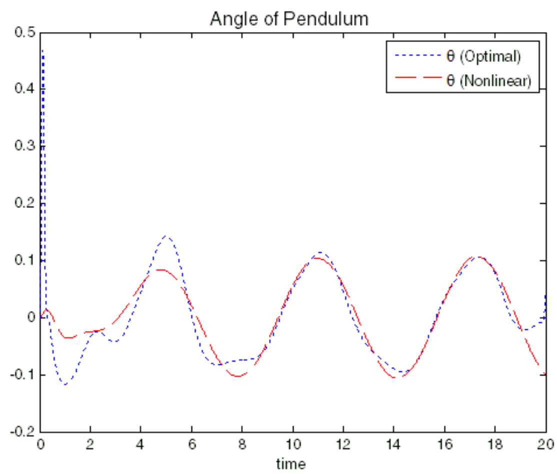


Fig. 3 Comparison of the tracking performance of the optimal and nonlinear controllers



.Fig. 4 Inverted pendulum position

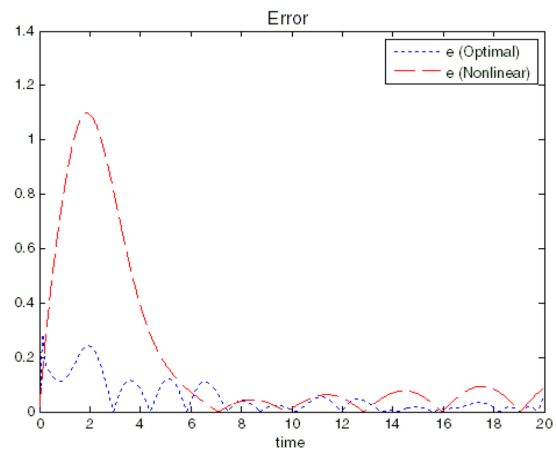


Fig. 5 Error signals

Figure 5 shows the error signal between reference values and actual values for both cases. The error signal in optimal case is smaller than nonlinear case especially in transient response. Finally, to have an exact comparison between two techniques, the error norms are presented in table 1.

TABLE I COMPARISION THE PERFORMANCE OF TWO METHODS

	Optimal control	Nonlinear control
$\|e\|_1$	0.3527	1.6524
$\|e\|_2$	1.0562	4.0332
$\|e\|_\infty$	0.2862	1.0982
Transient response	good	bad

V. CONCLUSIONS

In this paper for improving the problem of asymptotic tracking of the two-cart system with an inverted pendulum a novel optimal control technique has been presented. The performance of this technique in tracking a sinusoid reference signal has been shown by simulation. Its superiority is presented by comparison with nonlinear controller by comparing their tracking performance based on different performance indices.

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