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Ten limit cycles in a quintic Lyapunov system

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Abstract—In this paper, center conditions and bifurcation of limit cycles at the nilpotent critical point in a class of quintic polynomial differential system are investigated. With the help of computer algebra system MATHEMATICA, the first 10 quasi Lyapunov constants are deduced. As a result, sufficient and necessary conditions in order to have a center are obtained. The fact that there exist 10 small amplitude limit cycles created from the three order nilpotent critical point is also proved. Henceforth we give a lower bound of cyclicity of three-order nilpotent critical point for quintic Lyapunov systems. At last, we give an system which could bifurcate 10 limit circles.

Keywords—Three-order nilpotent critical point, Center-focus problem, Bifurcation of limit cycles, Quasi-Lyapunov constant.

I. Introduction

THE nilpotent center problem was theoretically solved by Moussu [10] and Stróżyna [12]. Nevertheless, in fact, given an analytic system with a monodromic point, it is very difficult to know if it is a focus or a center, even in the case of polynomial systems of a given degree. In this paper, we consider an autonomous planar ordinary differential equation having a three–order nilpotent critical point with the form

$$\begin{aligned}\frac{dx}{dt} &= y + a_{12}xy^2 + a_{03}y^3 + a_{31}x^3y + a_{22}x^2y^2\\ &\quad - 4b_{04}xy^3 + a_{04}y^4 - y(x^2+y^2)^2,\\ \frac{dy}{dt} &= -2x^3 + b_{21}x^2y + b_{03}y^3 + b_{40}x^4 - \tfrac{3}{2}a_{31}x^2y^2\\ &\quad + b_{13}xy^3 + b_{04}y^4 + x(x^2+y^2)^2.\end{aligned} \tag{1}$$

where $\mu \neq 0$, and all parameters are real.

In some suitable coordinates, the Lyapunov system with the origin as a nilpotent critical point can be written as

$$\begin{aligned}\frac{dx}{dt} &= y + \sum_{i+j=2}^{\infty} a_{ij}x^iy^j = X(x,y),\\ \frac{dy}{dt} &= \sum_{i+j=2}^{\infty} b_{ij}x^iy^j = Y(x,y).\end{aligned} \tag{2}$$

Suppose that the function $y = y(x)$ satisfies $X(x,y) = 0, y(0) = 0$. Lyapunov proved (see for instance [3]) that the origin of system (2) is a monodromic critical point (i.e., a center or a focus) if and only if

$$\begin{aligned}&Y(x,y(x)) = \alpha x^{2n+1} + o(x^{2n+1}), \alpha < 0\\ &\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x}\right]_{y=y(x)} = \beta x^n + o(x^n),\\ &\beta^2 + 4(n+1)\alpha < 0,\end{aligned} \tag{3}$$

where n is a positive integer. The monodromy problem in this case was solved in [4] and the center problem in [10], see also in [12]. As far as we know there are essentially three differential ways of obtaining the Lyapunov constant: by using normal form theory [8], by computing the Poincaré return map [6] or by using Lyapunov functions [11]. Álvarez study the momodromy and stability for nilpotent critical points with the method of computing the Poincaré return map, see for instance [1]; Chavarriga study the local analytic integrability for nilpotent centers by using Lyapunov functions, see for instance [7]; Moussu study the center-focus problem of nilpotent critical points with the method of normal form theory, see for instance [10]. Takens proved in [13] that system (2) can be formally transformed into a generalized Liénard system. Álvarez proved in [2] that using a reparametrization of the time to simplify even more. Giacomini et al. in [14] prove that the analytic nilpotent systems with a center can be expressed as limit of systems non-degenerated with a center. therefore, any nilpotent center can be detected using the same methods that for a nondegenerate center, for instance the Poincaré-Lyapunov method can be used to find the nilpotent centers.

For a given family of polynomial differential equations, let $N(n)$ be the maximum possible number of limit cycles bifurcating from nilpotent critical points for analytic vector fields of degree n. In [5] it is found that $N(3) \geq 2, N(5) \geq 5, N(7) \geq 9$; In [1] it is found that $N(3) \geq 3, N(5) \geq 5$; For a family of Kukles system with 6 parameters, in [2] it is found taht $N(3) \geq 3$. Hence in this paper. Recently, Liu Yirong and Li Jibin in [15] proved that $N(3) \geq 8$. Hence in this paper, employing the integral factor method introduced in [9], we will prove $N(5) \geq 10$. To the best of our knowledge, our results on the lower bounds of cyclicity of three-order nilpotent critical points for quintic systems are new.

We will organize this paper as follows. In Section 2, using the linear recursive formulae in [15] to do direct computation, we obtain with relative ease the first 10 quasi-Lyapunov constants and the sufficient and necessary conditions of center. This paper is ended with Section 3 in which the 10-order weak focus conditions and the fact that there exist 10 limit cycles in the neighborhood of the three-order nilpotent critical point are proved.

II. Quasi–Lyapunov constants and center conditions

According to Theorem in [9], for system (1). Carrying out calculations in MATHEMATICA, we have

$$\begin{aligned}&\omega_3 = \omega_4 = \omega_5 = 0,\\ &\omega_6 = -\tfrac{1}{3}b_{21}(-1+4s),\\ &\omega_7 \sim 3(s+1)c_{03},\\ &\omega_8 \sim -\tfrac{2(a_{12}+3b_{03})}{5}(-3+4s),\\ &\omega_9 \sim -\tfrac{2(2a_{22}+3b_{13})}{3}(-1+s).\end{aligned} \tag{4}$$

From (3.1), we obtain the first two quasi-Lyapunov constants of system (1):

$$\begin{aligned}\lambda_1 &= \tfrac{\omega_6}{1-4s} = \tfrac{b_{21}}{3},\\ \lambda_2 &\sim \tfrac{\omega_8}{3-4s} = \tfrac{2(a_{12}+3b_{03})}{5}.\end{aligned} \tag{5}$$

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we see from $\omega_7 = \omega_9 = 0$ that

$$c_{03} = 0, s = 1. \tag{6}$$

Furthermore, take $s = 1$, we obtain the following conclusion.

Proposition 2.1: For system (1), one can determine successively the terms of the formal series $M(x, y) = x^4+y^2+o(r^4)$, such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - 2\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=1}^{11} \lambda_m[(2m-5)x^{2m+4} + o(r^{28})], \tag{7}$$

where λ_m is the m-th quasi-Lyapunov constant at the origin of system (1), $m = 1, 2, \cdots, 12$.

Theorem 2.1: For system (1), the first 12 quasi-Lyapunov constants at the origin are given by

$$\begin{aligned}
&\lambda_1 = \tfrac{b_{21}}{3},\\
&\lambda_2 = \tfrac{2(a_{12}+3b_{03})}{5},\\
&\lambda_3 = \tfrac{b_{40}(2a_{22}+3b_{13})}{35},\\
&\lambda_4 = -\tfrac{(2a_{22}+3b_{13})a_{31}}{15},\\
&\lambda_5 = \tfrac{20b_{04}(2a_{22}+3b_{13})}{77},\\
&\lambda_6 = \tfrac{-4b_{03}(172a_{22}-13b_{13})(2a_{22}+3b_{13})}{3003},\\
&\lambda_7 = \tfrac{8b_{03}(41067a_{04}-7658a_{22})(2a_{22}+3b_{13})}{405405},\\
&\lambda_8 = \tfrac{112(160681+733941a_{03})a_{22}b_{03}(2a_{22}+3b_{13})}{45379035},\\
&\lambda_9 = \tfrac{4a_{22}b_{03}(2a_{22}+3b_{13})}{6240681974475}(-9539331965897\\
&\quad + 20127128261760b_{03}^2),\\
&\lambda_{10} = \tfrac{-a_{22}b_{03}(2a_{22}+3b_{13})}{188992023730839771840450}(632226312156980494004945\\
&\quad + 815899547527119916257024a_{22}^2)
\end{aligned} \tag{8}$$

In the above expression of λ_k, we have already let $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0,\ k = 2, \cdots, 10$.

From Theorem 2.1, we obtain the following assertion.

Proposition 2.2: The first 10quasi-Lyapunov constants at the origin of system (1) are zero if and only if the following condition is satisfied:

$$b_{21} = a_{31} = b_{03} = b_{40} = b_{04} = a_{12} = 0; \tag{9}$$

$$b_{21} = 0,\ a_{12} = -3b_{03},\ a_{22} = -\tfrac{3}{2}b_{13}. \tag{10}$$

Proof. When condition (9) of Proposition 3.2 holds, system (1) can be brought to

$$\begin{aligned}
\tfrac{dx}{dt} &= y + a_{03}y^3 + a_{22}x^2y^2 + a_{04}y^4 - y(x^2 + y^2)^2,\\
\tfrac{dy}{dt} &= -2x^3 + b_{13}xy^3 + x(x^2 + y^2)^2.
\end{aligned} \tag{11}$$

whose vector field is symmetric with respect to the y-axis.

When condition (10) of Proposition 3.2 holds, system (1) can be brought to

$$\begin{aligned}
\tfrac{dx}{dt} &= y + -3b_{03}xy^2 + a_{03}y^3 + a_{31}x^3y - \tfrac{3}{2}b_{13}x^2y^2\\
&\quad - 4b_{04}xy^3 + a_{04}y^4 - y(x^2 + y^2)^2,\\
\tfrac{dy}{dt} &= -2x^3 + b_{03}y^3 + b_{40}x^4 - \tfrac{3}{2}a_{31}x^2y^2 + b_{13}xy^3\\
&\quad + b_{04}y^4 + x(x^2 + y^2)^2.
\end{aligned} \tag{12}$$

the system (12) has an analytic first integral

$$\begin{aligned}
H(x, y) = &-\tfrac{1}{2}y^2 - \tfrac{1}{2}x^4 - \tfrac{1}{2}a_{31}x^3y^2 + \tfrac{1}{2}b_{13}x^2y^3 + b_{04}xy^4\\
&- \tfrac{1}{4}a_{03}y^4 - \tfrac{1}{5}a_{04}y^5 + b_{03}xy^3 + \tfrac{1}{3}(x^2 + y^2)^3.
\end{aligned}$$

□

We see from Propositions 2.2 that

Theorem 2.2: The origin of system (1) is a center if and only if the first 10 quasi–Lyapunov constants are zero, that is, one of the conditions in Proposition 2.2 is satisfied.

III. Multiple bifurcation of limit cycles

This section is devoted proving that when the three–order nilpotent critical point $O(0, 0)$ is a 10-order weak focus, the perturbed system of (1) can generate 10 limit cycles enclosing an elementary node at the origin of perturbation system (1).

Using the fact $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 == 0,\ \lambda_{10} \neq 0$, we obtain

Theorem 3.1: The origin of system (1) is a 10–order weak focus if and only if

$$\begin{aligned}
&b_{21} = b_{40} = a_{31} = b_{04} = 0,\\
&a_{12} = -3b_{03}, b_{13} = \tfrac{171}{13}a_{22},\\
&a_{04} = \tfrac{7658}{41067}a_{22},\\
&a_{03} = -\tfrac{160681}{733941},\\
&b_03 = \pm\frac{\sqrt{\frac{9539331965897}{90610}}}{14904}, a_{22} \neq 0.
\end{aligned} \tag{13}$$

Proof. By letting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = 0$, we obtain the relations of $b_{21},\ b_{40},\ a_{31},\ b_{04},\ a_{12},\ b_{13},\ a_{22},\ a_{04},\ a_{03},\ b_{03}$. Because $a_{22} \neq 0$, the origin of system (1) is a 10–order weak focus. □

We next study the perturbed system of (1) as follows:

$$\begin{aligned}
\tfrac{dx}{dt} &= \delta x + y + a_{12}xy^2 + a_{03}y^3 + a_{31}x^3y + a_{22}x^2y^2\\
&\quad - 4b_{04}xy^3 + a_{04}y^4 - y(x^2 + y^2)^2,\\
\tfrac{dy}{dt} &= 2\delta y - 2x^3 + b_{21}x^2y + b_{03}y^3 + b_{40}x^4 - \tfrac{3}{2}a_{31}x^2y^2\\
&\quad + b_{13}xy^3 + b_{04}y^4 + x(x^2 + y^2)^2.
\end{aligned} \tag{14}$$

When conditions in (13) hold, we have

$$\begin{aligned}
J &= \frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9)}{\partial(b_{21},\ a_{12},\ b_{40},\ a_{31},\ b_{04},\ b_{13},\ a_{04},\ a_{03},\ b_{03})}\\
&= \frac{\partial\lambda_1}{\partial b_{21}} \frac{\partial\lambda_2}{\partial a_{12}} \frac{\partial\lambda_3}{\partial b_{40}} \frac{\partial\lambda_4}{\partial a_{31}} \frac{\partial\lambda_5}{\partial b_{04}} \frac{\partial\lambda_6}{\partial b_{13}} \frac{\partial\lambda_7}{\partial a_{04}} \frac{\partial\lambda_8}{\partial a_{03}} \frac{\partial\lambda_9}{\partial b_{03}}\\
&= \frac{47208816262725486071852277932329645739952 64a_{22}^9b_{03}}{428305835520197903912986777109695312 5}\\
&\neq 0.
\end{aligned} \tag{15}$$

The statement mentioned above follows that

Theorem 3.2: If the origin of system (1) is a 10-order weak focus, for $0 < \delta \ll 1$, making a small perturbation to the coefficients of system (1), then, for system (14), in a small neighborhood of the origin, there exist exactly 10 small amplitude limit cycles enclosing the origin $O(0,0)$, which is an elementary node.

IV. Example of bifurcation of limit cycles at origin

Now we consider bifurcation of limit cycles at the origin for perturbed system (14).

Theorem 4.1: Suppose that the coefficients of system (14) satisfy

$$\begin{aligned}
&\delta = \tfrac{1}{2}\varepsilon^{55}, b_{21} = 3\varepsilon^{45},\\
&a_{12} = -\tfrac{C}{4968} - \tfrac{243386597004525\varepsilon}{41133599436947864} + \tfrac{5}{2}\varepsilon^{36},\\
&b_{40} = \tfrac{65}{77}\varepsilon^{28}, a_{31} = \tfrac{195}{539}\varepsilon^{21},\\
&b_{03} = \tfrac{13}{140}\varepsilon^{15}, a_{22} = 1,\\
&b_{13} = \tfrac{171}{13} - \tfrac{145314C}{7}\varepsilon^{10},\\
&a_{04} = \tfrac{7658}{41067} - \tfrac{40365C}{49}\varepsilon^{6},\\
&a_{03} = -\tfrac{160681}{733941} - \tfrac{61395165C}{3773}\varepsilon^{3},\\
&b_{03} = \tfrac{1}{1914C} + \tfrac{81128865668175}{41133599436947864}\varepsilon,
\end{aligned} \tag{16}$$

where $C = \sqrt{\frac{90610}{9539331965897}}$. Then, if $\varepsilon = 0$, the origin of system (14) is an tenth fine focus with stability. If $0 < \varepsilon \ll 1$, there exist ten limit cycles in a small enough neighborhood of the origin of system (14).

Proof. According to Theorem 2.1, we have

$$\begin{aligned}
&v_1(2\pi,\delta) = -\varepsilon^{55} + O(\varepsilon^{55}),\\
&v_2(2\pi,\delta) = \varepsilon^{45} + O(\varepsilon^{45})\\
&v_3(2\pi,\delta) = -\varepsilon^{36} + O(\varepsilon^{36}),\\
&v_4(2\pi,\delta) = \varepsilon^{28} - \tfrac{5667246C}{3773}\varepsilon^{38} + O(\varepsilon^{38}),\\
&v_5(2\pi,\delta) = -\varepsilon^{21} + \tfrac{5667246C}{3773}\varepsilon^{31} + O(\varepsilon^{31}),\\
&v_6(2\pi,\delta) = \varepsilon^{15} - \tfrac{5667246C}{3773}\varepsilon^{25} + O(\varepsilon^{25}),\\
&v_7(2\pi,\delta) = -\varepsilon^{10} - \tfrac{151143076739810025C}{5141699929618483}\varepsilon^{11} + O(\varepsilon^{11}),\\
&v_8(2\pi,\delta) = \varepsilon^{6} + \tfrac{151143076739810025C}{5141699929618483}\varepsilon^{7} + O(\varepsilon^{7}),\\
&v_9(2\pi,\delta) = -\varepsilon^{3} - \tfrac{151143076739810025C}{5141699929618483}\varepsilon^{4}\\
&\qquad + \tfrac{5667246C}{3773}\varepsilon^{13} + O(\varepsilon^{13}),\\
&v_{10}(2\pi,\delta) = \varepsilon + \tfrac{151143076739810025C}{5141699929618483}\varepsilon^{2}\\
&\qquad - \tfrac{5667246C}{3773}\varepsilon^{11} + O(\varepsilon^{11}),\\
&v_{11}(2\pi,\delta) = -\tfrac{78053983836973012113120129 1C}{366175825818976674736308684 00}\\
&\qquad - \tfrac{1448125859684100410261969}{2311103383443064036777392}\varepsilon + O(\varepsilon),
\end{aligned} \tag{17}$$

Because the sign of the focal values of the origin has reversed eleven times, from Theorem in [15] there exist ten limit cycles in a small enough neighborhood of the origin of system (14). □

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