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Covering-based Rough sets Based on the Refinement of Covering-element

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Abstract—Covering-based rough sets is an extension of rough sets and it is based on a covering instead of a partition of the universe. Therefore it is more powerful in describing some practical problems than rough sets. However, by extending the rough sets, covering-based rough sets can increase the roughness of each model in recognizing objects. How to obtain better approximations from the models of a covering-based rough sets is an important issue. In this paper, two concepts, determinate elements and indeterminate elements in a universe, are proposed and given precise definitions respectively. This research makes a reasonable refinement of the covering-element from a new viewpoint. And the refinement may generate better approximations of covering-based rough sets models. To prove the theory above, it is applied to eight major covering-based rough sets models which are adapted from other literature. The result is, in all these models, the lower approximation increases effectively. Correspondingly, in all models, the upper approximation decreases with exceptions of two models in some special situations. Therefore, the roughness of recognizing objects is reduced. This research provides a new approach to the study and application of covering-based rough sets.

Keywords—Determinate element, indeterminate element, refinement of covering-element, refinement of covering, covering-based rough sets.

I. Introduction

ROUGH sets theory proposed by Pawlak [1] is a mathematical tool which is used to deal with the uncertain, inaccurate and vague data. It approximately describes a target set via a pair of lower and upper approximations. In this way, it gives a good description of the fuzzy idea proposed by G. Frege. The rough sets has a powerful objectivity in recognizing the target set through a partition that is gotten from the equivalence relation between elements of universe and is independent of any priori knowledge. Therefore, since it was proposed, the rough sets theory has drawn much attention of many scholars and has been widely applied into many fields in both academia and industry such as data mining, machine learning, pattern recognition, and so on [2]–[10]. However, due to the rigid binary relation of the equivalence relation in rough sets, it limits the development of the rough sets itself and its application. So, plenty of extensive studies on generalized rough sets have been done by many researchers. For examples, the equivalence relations of rough sets were extended to such generalized binary relations as compatibility relations [11], [12], similar relations [13], [14]. Correspondingly, a partition of universe in rough sets was extended to a covering [15]–[17].

Covering-based rough sets is an extensive study of Pawlak's rough sets. It extends a partition in rough sets to a covering of a universe. Becauseunlike a parititiona covering does not results from a rigid equivalence relation, so it is more consistent with reality than partition is when a judgement and a description is given to an object. But it also enlarges the boundary set between lower and upper approximations at the same time. In addition, the problem of the redundancy of covering-element arises. In order to narrow the boundary set, some new models of covering-based rough sets have been proposed [15], [16], [18]–[27] by many scholars after they have made lots of studies about this field . Moreover, Moreover, a very important work in solving the redundancy of covering-element has been done by Zhu [28]. This work stimulates the development and application of covering-based rough sets. Nowadays, the covering-based rough sets models are usually studied through defining a new one by many scholars. Actually, different models may be applicable to different situations. And in different coverings, the results of comparisons between the lower and upper approximation generated from these models may be different, so it is difficult to judge which model is better than others. Hence, different from the research done previously, this paper, from a new point of view, studies how to get a pair of preferable lower and upper approximations in each model. By refining the covering-elements in a covering-based rough sets, the lower approximation can be increased and the upper approximation can be decreased. In this way, the object recognition capability of each model is fundamentally improved. Basing on the refinement of covering-element, we have studied and analyzed eight main models of covering-based rough sets. We found that the size of the lower approximation of each model after the refinement of covering-element is not smaller than the one which covering-element has not been refined. As for the upper approximation, every model except the first and the third ones can get a smaller upper approximation after the refinement of covering-element. In this paper, we also present an algorithm of refinement of covering-element.

The following content of this paper is organized as follows: section 2 is an introduction of some basic knowledge of the rough sets and the covering-based rough sets. In section 3, the reason and principle of refinement of covering-element

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are analyzed; some basic concepts of refinement of covering-element are defined; After a study of the relation between refinement and reduction has been conducted, some significant conclusions are drawn; And an algorithm of refinement of covering-element is also presented in this section. In section 4, several major models of covering-based rough sets are introduced and a comparative study on each model's lower and upper approximations which arise from a covering and refinement of the covering is made. Finally, the study is concluded in section 5 with remarks for future works.

II. Background

To better understand the content of the following section, we will introduce some fundamental concepts of rough sets.

A. The basic concepts of rough sets

Let U be a nonempty finite set which is called universe U. R is a cluster of equivalence relation of universe U. A pair $S = (U, R)$ is called approximation space of universe U [1], [29]. If $P \subseteq R$ and $P \neq \emptyset$, then P is still an equivalence relation of universe U and is called indiscernible relation, which denoted by $IND(P)$ [30]. $U/IND(P)$ is a partition of equivalence relation $IND(P)$ to universe U and is a basic knowledge of universe U in the approximation space $S = (U, R)$. Each element of partition is called a equivalence class about $IND(P)$. The elements of the same equivalence class are indiscernible. We denote $IND(P)$ as P simply.

An equivalence relation P can produce a partition of universe U and is considered as knowledge we master. For all $X \subseteq U$, it is hard to precisely describe X according to the knowledge. Then, for any target set X, we can employ a pair of approximation sets to approach to it and to describe it roughly. The pair of approximation sets is defined as follows:

$\underline{apr}(X) = \cup\{K|K \in U/P \wedge K \subseteq X\}$

$\overline{apr}(X) = \cup\{K|K \in U/P \wedge (K \cap X \neq \emptyset)\}$

We call them the lower and upper approximations of X, respectively. And the subtraction of upper and lower approximations is called the boundary region of X [1], and it is denoted as $Bn_P(X)$, that is, $Bn_P(X)$=$\underline{apr}(X) - \overline{apr}(X)$.

For any subset X of universe U, if $\underline{apr}(X) = \overline{apr}(X)$, then the partition of U generated by P can describe X accurately. On the contrary, the partition of U generated by P can describe X roughly, and the ordered pair of $(\underline{apr}(X), \overline{apr}(X))$ is called the rough set with respect to X.

B. The fundamental concepts of covering-based rough sets

Definition 2.1: (Covering, covering approximation space) [15] Let U be a universe, C is a family of subsets of U. If all subsets in C are non-empty and $\cup C = U$, then C is a covering. We call the ordered pair $< U, C >$ covering approximation space.

Definition 2.2: (Minimal description) [15] Let $< P, C >$ be a covering approximation space. $\forall x \in U$,

$Md(x) = \{K \in C|x \in K \wedge (\forall S \in C \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$

is called the minimal description of x.

In the following example, we can better understand the conception of minimal description.

Example 2.1: Let $U = \{a, b, c, d\}$, $K_1 = \{a, b\}$, $K_2 = \{b, c\}$, $K_3 = \{b, c, d\}$, and $C = \{K_1, K_2, K_3\}$ be the covering of U. The minimal description of b is $Md(b) = \{\{a, b\}, \{b, c\}\} = \{K_1, K_2\}$.

Definition 2.3: (Covering lower approximation set family, covering lower approximation and so on) [15] Let C be a covering of universe U and $X \subseteq U$, then:

Set family $C_*(x) = \{K \in C|K \subseteq X\}$ is called the covering lower approximation set family of X;

Set $X_* = \bigcup C_*(x)$ is called the covering lower approximation of X;

Set $X_*^* = X - X_*$ is called the covering boundary approximation of X;

Set family $Bn(X) = \{Md(x)|x \in X_*^*\}$ is called the covering boundary approximation set family of X;

Set family $C^*(X) = C_*(x) \bigcup Bn(X)$ is called the covering upper approximation set family of X;

Set $X^* = \bigcup C^*(X)$ is called the covering upper approximation of X;

Definition 2.4: (Reducible element, irreducible element) [28] Let C be a covering of a universe U, $K \in C$. If K is a union of some sets in $C - \{K\}$, we say K is a reducible element of C, otherwise K is an irreducible element of C.

Proposition 2.1: [28] Let C be a covering of a universe U. If K is a reducible element of C, then $C - \{K\}$ is still a covering of U.

Proposition 2.2: [28] Let C be a covering of a universe U, $K \in C$, K is a reducible element of C, and $K_1 \in C - \{K\}$, then K_1 is a reducible element of C if and only if it is a reducible element of $C - \{K\}$.

Definition 2.5: (Reduct of covering) [28] Let C be a covering of a universe U, the new covering come from the reducing process of proposition 2.1 and proposition 2.2 is called the reduct of C, and denoted by $reduct(C)$.

The definition of reducible element solves effectively the problem of redundant covering-element in covering rough sets. In the next section, we will explore the problem of refinement of covering-element by the concept of reduct of covering.

III. The refinement of covering-element

A. The origin and analysis of the refinement of covering-element

According to the definition of partition and covering in the same universe, we know that the similarity between partition and covering is that the union of all equivalent classes in partition is the same as the union of all covering-elements in covering, that is, the two union is equal to the universe U. And the difference between partition and covering is that join of any two equivalence classes in partition is empty, but, the join of any two covering-elements in covering maybe not empty. A covering is a partition when the join of all covering-elements is null set. In covering-based rough sets, an element x of universe could be from several covering-elements, that is, x belongs to several covering-elements. And this increases the difficulty of distinguishing x exactly. Certainly, there are also some elements of universe only appear in one covering-element, but some other elements of this covering-element maybe appear in another covering-elements, so it also increases the difficulty of distinguishing these elements which only appear in one covering-element. These will cause a too small lower approximation and a too large upper approximation when recognizing target set. And the recognition capability of covering-based rough sets is reduced.

For these reasons, we introduce a new method to refine covering-element. The main idea of the method is as follows:

According to the definition of lower and upper approximations in rough sets, we know that the less number of elements of equivalence class to a partition of universe, the larger lower approximation and the larger upper approximation may be generated. That is, this partition has more strong recognition capability to target set. This idea is also apply to covering-based rough sets. It is that the smaller the size of covering-element, the more strong recognition capability of the covering-based rough sets. So, if we can reduce the size of covering-element effectively, the recognition capability of the covering-based rough sets would be improved. How to reduce the size of covering-element?

Let universe $U = a, b, c, d$, $C = \{K_1, K_2, K_3, K_4\} = \{\{a, b, c\}, \{b, c\}, \{b, d\}, \{e\}\}$ is a covering of U. It is easy to see that element b appears in covering-element K_1, K_2 and K_3, element c appears in covering-element K_1 and K_2. And a, d and e appear in K_1, K_3 and K_4 respectively. Thus, we can consider b and c as indeterminate element and a, d, e as determinate element. Then take out the determinate element from each covering-element to form a new covering-element respectively, and combine every indeterminate element with determinate element of each covering-element to form new covering-element. If all elements of a covering-element are determinate element or indeterminate element, then we leave it as it is. By doing this, a covering-element is refined. In the covering C given above, for instance, a is a determinate element and b, c are indeterminate element to K_1. So, we take out a as a new covering-element $\{a\}$, and combine respectively with b and c to form two covering-element $\{a, b\}$, $\{a, c\}$. Similarly, K_3 can be refined as $\{b, c\}, \{c\}$. And leave K_2 and K_4 as them are because all the element of them are determinate or indeterminate. Finally, we get a new covering $\{\{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{d\}, \{e\}\}$. As shown in Fig.1, the upper level are the covering-elements of the primary covering and the lower level are the refinement of covering-elements of the primary covering-elements. The bold italic letters of each covering-element are determinate elements.

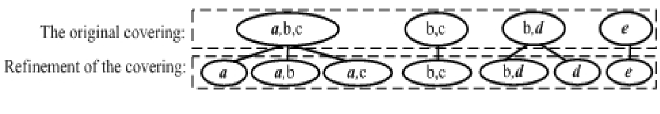

Fig. 1. The refinement of covering-element

B. The concepts of the refinement of covering-element

In this subsection, we will define some new concepts. Through these concepts, we will propose the definition of the refinement of covering-element. Meanwhile, we will discuss some interesting results after refining covering-element. Paper [31] studies a special covering which is called the fined covering. In a fined covering, the join of any two covering-elements is equal to the union of some covering-elements. In paper [31], the author defines a concept named neighbor family. We will borrow the concept in this paper. But, in order to vividly describe the idea of the refinement of covering-element, we will call this concept family of membership and the definition as follows:

Definition 3.1: (Family of membership) Let $< U, C >$ be a covering approximation space, $x \in U$, we call $\{K | x \in K \wedge K \in C\}$ the family of membership of x to covering C, and denote as $FM(x)$, namely, $FM(x) = \{K | x \in K \wedge K \in C\}$.

Example 3.1: Let $U = \{a, b, c, d, e\}$ be a universe, $C = \{\{a, b, c\}, \{b, c\}, \{b, d\}, \{e\}\}$ is a covering of U, then:

$FM(a) = \{\{a, b, c\}\}$, $FM(b) = \{\{a, b, c\}, \{b, c\}, \{b, d\}\}$, $FM(c) = \{\{a, b, c\}, \{b, c\}\}$, $FM(d) = \{\{b, d\}\}$, $FM(e) = \{\{e\}\}$.

Proposition 3.1: Let U be a universe. C is a covering of universe U. For any $x \in U$, we have $Md(x) \subseteq FM(x)$.

Proof: Let $C = \{K_1, K_2, \ldots, K_m\}$, $FM(x) = \{K_1, K_2, \ldots, K_p\}$, where $1 \leq p \leq m$. For any $K_i \in FM(x)$, if there is not exist $K_j \in FM(x) - \{K_i\}$ such that $K_i \subseteq K_j$, then $Md(x) = FM(x)$. On the contrary, if there exists $K_j \in FM(x) - \{K_i\}$ such that $K_i \subseteq K_j$, then $K_j \notin Md(x)$, that is, $Md(x) \subset FM(x)$. According to the above results, we get that $Md(x) \subseteq FM(x)$. ■

Definition 3.2: (Determinate element, indeterminate element) Let U be a universe. C is a covering of U. For any $x \in U$, x is a determinate element if and only if $|FM(x)| = 1$. Otherwise, x is indeterminate element.

In example 2, a, d, e are determinate elements and b, c are indeterminate elements.

Definition 3.3: (Determinate element set, indeterminate element set) Let U be a universe. $C = K_1, K_2, \ldots, K_m$ is a covering of U. We call $DS(K_i) = \{x | x \in K_i \wedge (|FM(x)| = 1)\}$ the determinate element set of K_i, and $IDS(K_i) = \{x | x \in K_i \wedge (|FM(x)| > 1)\}$ the indeterminate element set of K_i. We suppose that $DS(K_i) = \emptyset$, $|DS(K_i)| = 0$.

Example 3.2: Let $U = \{a, b, c, d, e\}$ be a universe, $C = \{\{a, b, c\}, \{b, c\}, \{b, d\}, \{e\}\}$ is a covering of U, then,

$DS(K_1) = \{a\}$, $DS(K_2) = \emptyset$, $DS(K_3) = \{d\}$, $DS(K_4) = \{e\}$;

$IDS(K_1) = \{b, c\}$, $IDS(K_2) = \{b, c\}$, $IDS(K_3) = \{b\}$, $IDS(K_4) = \emptyset$.

Definition 3.4: (Combination of covering-element) Let $U = \{x_1, x_2, \ldots, xn\}$ be a universe. $C = \{K_1, K_2, \ldots, K_m\}$ is a covering, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. If $x_j \in K_i$, then,

$$CCE(x_j) = \begin{cases} DS(K_i) \cup \{x_j\}, & |DS(K_i)| > 0 \\ IDS(K_i), & |DS(K_i)| = 0 \end{cases}$$

is called the combination of covering-element about covering-element K_i.

In example 3, for covering-element K_1, we can get that $CCE(a) = \{a\} \bigcup \{a\} = \{a\}$, $CCE(b) = \{a\} \bigcup \{b\} = \{a, b\}$, and $CCE(c) = \{a\} \bigcup \{c\} = \{a, c\}$. For covering-element K_2, because b and c are both indeterminate elements, then $|DS(K_2)| = 0$, $CCE(b) = CCE(c) = \{b, c\}$.

Definition 3.5: (Refinement of covering-element) Let U be a universe, $C = \{K_1, K_2, \ldots, K_m\}$ is a covering of U. For any $K_i \in C (i = 1, 2, \ldots, m)$, we call $RCE(K_i) = \{CCE(x)| \; x \in K_i\}$ the refinement of covering-element K_i.

Example 3.3: Let $U = \{a, b, c, d, e\}$. $C = \{\{a, b, c\}, \{b, c\}, \{b, d\}, \{e\}\}$ is a covering of universe U, where K_1=$\{a, b, c\}$, K_2=$\{b, c\}$, K_3=$\{b, d\}$, K_4=$\{e\}$. Then,

$RCE(K_1)$= $\{\{a\}, \{a, b\}, \{a, c\}\}$, $RCE(K_2)$=$\{\{b, c\}\}$, $RCE(K_3)$=$\{\{b, d\}, \{d\}\}$, $RCE(K_4)$=$\{\{e\}\}$.

Proposition 3.2: Let U be a universe, $C = \{K_1, K_2, \ldots, K_m\}$ is a covering of U. For any $K_i \in C$, $K_i = \cup RCE(K_i)$.

Proof: According to definition 3.4 and 3.5, we can easily get that $\cup\{CCE(x)|x \in K_i\} = K_i$, that is, $K_i = \cup RCE(K_i)$. ■

Definition 3.6: (Refinement of covering). Let U be a universe, $C = \{K_1, K_2, \ldots, K_m\}$ is a covering of U. $RC(C) = \cup\{RCE(K_i)|K_i \in C\}$ is called the refinement of covering C. For any $x \in U$, we call $RMd(x) = \{K \in RC(C)|x \in K \wedge (\forall S \in RC(C) \wedge \in S \wedge S \subseteq K \Rightarrow K = S)\}$ the minimal description of x to $RC(C)$.

In example 4, according to definition 3.6, we can get that:

$RC(C) = \{\{a\}, \{a, b\}, \{a, c\}\} \cup \{\{b, c\}\} \cup \{\{b, d\}, \{d\}\} \cup \{\{e\}\} = \{\{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{d\}, \{e\}\}$;

$RMd(a) = \{\{a, b\}, \{a, c\}\}$, $RMd(b) = \{\{a, b\}, \{b, c\}, \{b, d\}\}$, $RMd(c) = \{\{a, c\}, \{b, c\}\}$, $RMd(d) = \{\{b, d\}\}$, $RMd(e) = \{\{e\}\}$.

Proposition 3.3: Let U be a universe and C be a covering of U. Then $RC(C)$ is a covering of universe U.

Proof: Let U be a universe and $C = \{K_1, K_2, \ldots, K_m\}$ be a covering of U. According to proposition 3.2, we get that $RCE(K_i) = K_i$. Meanwhile $\cup K_1 \cup K_2 \cup \ldots \cup K_m = U$. So, $(\cup RCE(K_1)) \cup (\cup RCE(K_2)) \cup \ldots \cup (\cup RCE(K_m)) = U$, that is , $\cup RC(C) = U$. ■

Proposition 3.4: Let U be a universe and $C = \{K_1, K_2, \ldots, K_n\}$ be a covering of U. $RC(C) = \{T_1, T_2, \ldots, T_m\}$ is a refinement of C. For any $T_i \in RC(C)$, there exists $K_j \in C$ such that $T_i \subseteq K_j$.

Proof: According to definition 3.3 and 3.4, we get that, for any $T_i \in RC(C)$, there exists $K_j \in C$ such that $T_i \in RCE(K_j)$. From proposition 3.2, we get that $\cup RCE(K_j) = K_j$. So, $T_i \subseteq K_j$. ■

Proposition 3.5: Let U be a universe and C be a covering of U. If C is a partition, then the refinement of covering C is itself.

Proof: When C is a partition of U, namely, $x \in U$, x only appears in one covering-element. According to definition 3.1, we can know that $FM(x)$ has only one element. Then $|FM(x)| = 1$. According to definition 3.2, we know that every element of covering-element is determinate element. From definition 3.3, we get that the determinate element set of every covering-element in covering is itself and the indeterminate element set is empty. From the definition 3.4 and 3.5, we can know that the refinement of every covering-element in covering is itself. Finally, we get that the refinement of covering C is itself according to definition 3.6. ■

Proposition 3.6: Let U is a universe and C is a covering of U. For any $x \in U$, if $|FM(x)| > 1$, then, the refinement of covering C is itself.

Proof: If $\forall x \in U$, $|FM(x)| > 1$. Then, according to definition 3.1 and 3.2, we can know that all element of universe U are indeterminate element. From definition 3.3, we get that the indeterminate element set of each covering-element in covering is itself and determinate element set is empty. According to definition 3.4 and 3.5, we get that the refinement of each covering-element is itself. Lastly, according to 11, we can know that the refinement of covering C is itself. ■

For convenience, Let $Md(x)$ and $RMd(x)$ represent the minimal description of x on $reduct(C)$ and $RC(reduct(C))$ respectively. The proposition 3.7 can be obtained as follows:

Proposition 3.7: Let U is a universe and C is a covering of U. For any $x \in U$, $|RMd(x)| \geq |Md(x)|$.

Proof: Let $C = \{K_1, K_2, \ldots, K_m\}, Md(x) = \{K_1, K_2, \ldots, K_p\}$, where $1 \leq p \leq m$. For any $K_i \in Md(x)$, we suppose that $RCE(K_i) = \{T_{i1}, T_{i2}, \ldots, T_{iq}\}$, where $q \geq 1$. According to definition 3.3and 3.4, we can get that there at least exists $T_{ij} \in RCE(K_i) (1 \leq j \leq q)$ such that $x \in T_{ij}$. So, $|RMD(x)| \geq |Md(x)|$. ■

Let U be a universe, and $C = \{K_1, K_2, \ldots, K_m\}$ is a covering of U. I is an index set and $i, j, p, q, r, s \in I$. Let the reduction of X is $reduct(C) = \{K_1, K_2, \ldots, K_p\}$, $Md(x)$ and $RMd(x)$ are respectively the minimal description of x in $reduct(C)$ and $RC(reduct(C)$. Then, we can get proposition 3.8, 3.9 and corollary 3.1.

Proposition 3.8: Let U be a universe, C is a covering of U, $K_i \in reduct(C)$, $x \in K_i$, $K_i \in Md(x)$, if there exists $K_j \in reduct(C)$ such that $K_i \subseteq K_j$, then there exists $y \in K_j$ such that $|FM(y)| = 1$ if and only if $\bigcup RMd(x) \subseteq \bigcup Md(x)$.

Proof: Sufficiency. Because $K_i \subseteq K_j$, according to definition 2.2, we can know that $K_j \notin Md(x)$. While there exists $y \in K_j$, and $|FM(x)| = 1$, according to definition 3.2 we can know that y is a determinate element in $reduct(C)$. So, $y \notin \cup Md(x)$, We can get that $K_j \notin RC(reduct(C))$ according to definition 3.3, 3.4, 3.5. Let $RCE(K_j) = T_{j1}, T_{j2}, \ldots, T_{jq} (q \geq 2)$. Then, there at least exists an element T_{jr} in $RCE(K_j) (1 \leq r \leq q)$ such that $x \in T_{jq}$ and $y \in T_{jq}$. According to definition 3.6, we can know that $T_{jr} \in RMd(x)$. So, $y \in \cup RMd(x)$, that is, $y \notin \cup Md(x)$ and $y \in \cup RMd(x)$. Therefore, $\cup RMd(x)$

Necessity. Because $\cup RMd(x)$and there exists $y \in RMd(x)$ and $y \notin Md(x)$. Let $y \notin K_j - K_i$, for any $K_p \in C$, if $x, y \in K_p$, then these surely exists $K_s \in C$ such that $K_s \subseteq K_p$, $x \in K_s$ and $y \notin K_s$. Then, $y \notin \cup RMd(x)$. This is contract to $y \in RMd(x)$. The hypothesis is not hold. Thus, $y \in K_j - K_i$, that is, y is only in K_j. Then $|FM(y)| = 1$. ■

Proposition 3.9: Let U be a universe, C is a covering of U, $K_i \in reduct(C)$, $x \in K_i$, $K_i \in Md(x)$, if there exists $K_j \in reduct(X)$ such that $K_i \subseteq K_j$, then there exists $y \in K_j$ such that $|FM(y)| > 1$ if and only if $RMd(x) \subseteq Md(x)$.

Proof: Sufficiency. For any $y \in K_j, |FM(y)| > 1$, according to definition 3.2 we can know that all elements of K_j are indeterminate elements. We get that $K_j = RCE(K_j)$ from definition 3.3, 3.4 and 3.5. Thus, $K_j \in RC(reduct(C))$. For $K_i \subseteq K_j$, if $K_i \in Md(x)$, then $K_j \notin Md(x)$, that is, $K_i \in RMd(x)$ and $K_j \notin RMd(x)$. For any $K_p \in Md(x)$, we suppose that $RCE(K_p) = \{T_{p1}, T_{p2}, \ldots, T_{pq}\}$, where $q \geq 1$. If $x \in T_{p1}, T_{p2}, \ldots, T_{ps} (1 \leq s \leq q)$, then $T_{p1}, T_{p2}, \ldots, T_{ps} \in RMd(x)$. Let $A_p = \{T_{p1}, T_{p2}, \ldots, T_{ps}\}$, then $\cup A_p \subseteq K_p$. Thus, if $Md(x) = \{K_1, K_2, \ldots, K_r\}$, then there correspondingly exists $RMd(x) = \{A_1 \cup A_2 \cup \ldots \cup A_r\}$ such that $\cup A_1 \subseteq K_1$, $\cup A_2 \subseteq K_2$, $\ldots$, $\cup A_r \subseteq K_r$. Accordingly, $\cup RMd(x) \subseteq \cup Md(x)$.

Necessity. For $\cup RMd(x) \subseteq \cup Md(x)$, if $\forall y \in RMd(x)$, then $y \in \cup Md(x)$. Let $y \in K_j - K_i$, if $y \in \cup Md(x)$, then $K_j \notin Md(x)$, for $K_i \subseteq K_j$. There surely exists $K_p \in C$ such that $x, y \in K_p$, and $K_p \in Md(x)$, that is, $y \in K_j$ and $y \in K_p$. So, $|FM(x)| > 1$. Of course, if $y \notin Md(x)$, assume that y is a determinate element, according to definition 3.4, we get that $y \cup RMd(x)$. This is contract to $\cup RMd(x) \subseteq \cup Md(x)$. So, y is an indeterminate element. Then $|FM(x)| > 1$. If $y \notin K_j - K_i$ and $y \in K_i$, then $y \in K_j$, that is, $|FM(x)| > 1$. Of course, if $y \in K_i$, then $y \in K_j$. Therefore, the above results hold. ■

Corollary 3.1: Let U be a universe, C is a covering of U, $K_i \in reduct(C)$, $x \in K_i$ and $K_i \in Md(x)$, if there is not exist $K_j \in reduct(X)$ such that $K_i \subseteq K_j$, then $\cup RMd(x) \subseteq \cup Md(x)$.

Proof: Because there is not exist $K_j \in reduct(C)$ such that $K_i \subseteq K_j$. Then, for any $K_p \in reduct(C)$, if $x \in K_p$, then $K_p \in Md(x)$. Let $RCE(K_p) = \{T_{p1}, T_{p2}, \ldots, T_{pq}\}$, where $p \geq 1$. If $x \in T_{p1}, T_{p2}, \ldots, T_{ps} (1 \leq s \leq q)$, then $T_{p1}, T_{p2}, \ldots, T_{ps} \in RMd(x)$. Let $A_p = \{T_{p1}, T_{p2}, T_{ps}\}$, then, $\cup A_p \subseteq K_p$. So, if $Md(x) = \{K_1, K_2, \ldots, K_r\}$, then $RMd(x) = \{A_1 \cup A_2 \cup \ldots \cup A_r\}$, correspondingly. So, $\cup A_1 \subseteq K_1, \cup A_2 \subseteq K_2, \ldots, \cup A_r \subseteq K_r$. We get that $\cup RMd(x) \subseteq \cup Md(x)$. ■

We maybe consider that two different coverings of the same universe whether produce the same refinement. Let us see the following example.

Example 3.4: Let U=$\{a, b, c, d\}$, C_1 and C_2 are two covering of U, and C_1=$\{\{a, b, c\}, \{b\}, \{b, c\}, \{b, d\}, \{c, d\}\}$, $C_2 = \{\{a\}, \{a, b, c, d\}, \{a, c\}, \{c, d\}\}$, please computer the refinement of C_1 and C_2.

Solve: According to the definition 3.6, $RC(C_1) = \{\{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}, \{b, d\}, \{c, d\}\}$, $RC(C_2) = \{\{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}, \{b, d\}, \{c, d\}\}$.

Accordingly, $RC(C_1) = RC(C_2)$.Therefore, two different coverings of the same universe U maybe produce the same refinement.

Whether we can continue to refine the covering after the refinement of covering? Or, what we do is meaning? Through studying and analyzing, we discover that it will not produce a covering when refine a covering has been refined. Namely, $RC(C) = RC(RC(C))$. Therefore, we can get the following theorem.

Theorem 3.2: Let U be a universe and C be a covering of universe U, then $RC(C) = RC(RC(C))$.

Proof: Let U be a universe and C is a covering of U. I is an index set and $m, p, n, i, j, r \in I$. Let $C = \{D_1, D_2, \ldots, D_m, H_1, H_2, \ldots, H_p, K_1, K_2, \ldots K_n\}$, where $m \geq 0, p \geq 0, n \geq 0$ and $m+n+p \geq 1$. $D_i (1 \leq i \leq m)$ is composed by all determinate elements of covering C, that is, $\forall x \in D_i (1 \leq i \leq m)$, $|FM(x)| = 1$. $H_j (1 \leq j \leq p)$ is composed by all indeterminate elements of covering C. Then, $\forall x \in H_j$, $|FM(x)| > 1$. $K_r (1 \leq r \leq n)$ is composed by such elements of covering C that there at least one determinate element and indeterminate element in the same covering element, that is, there exists $x, y \in K_r$ such that $|FM(x)| > 1$ and $|FM(y)| = 1$. We suppose that $D = \{D_1, D_2, \ldots, D_m\}$, $H = \{H_1, H_2, \ldots, H_p\}$, $K = \{K_1, K_2, \ldots, K_n\}$. For own convenience, we regard $RFM(x)$ as the family of membership x corresponding to covering $RC(C)$.

According to definition 3.4 and 3.6, we can know that $DRC(C)$, $HRC(C)$. From definition 3.5 we get $RCE(K_1)$, $RCE(K_2)$, $\ldots$, $RCE(K_n)$. So, $RC(C) = D \cup H \cup RCE(K_1) \cup RCE(K_2) \cup \ldots \cup RCE(K_n)$. For any element x of K_r, if $|FM(x)| = 1$, then, according to definition 3.5, we get that $|RFM(x) > 1|$. If $|FM(x)| > 1$. Similarly, we get that $|RFM(x) > 1|$. That is, for any $x \in \cup K$ in $RC(C)$, $|RFM(x)| > 1$. From definition 3.4 and 3.5 again, we get that $D \subseteq RC(RC(C))$, $H \subseteq RC(RC(C))$, $RCE(K_r) \subseteq RC(RC(C))$. Therefore, $RC(RC(C)) = D \cup H \cup RCE(K_1) \cup \ldots \cup RCE(K_n)$, namely, $RC(C) = RC(RC(C))$.

■

Accordingly, it will not produce a new covering to refine a refined covering. Hence, it is not necessary to refine a refined covering.

C. Reduction

Through reducing of a covering, we can reduce redundant covering-element. It is necessary to reduce before or after the refinement of covering-element. Sine we get a new covering after the refinement of covering-element, then this new covering satisfies all the properties of covering and reduction [28]. Now, the problem is that whether the new covering reduce before refine is the same as it is reduced after refined. Or, under what conditions does they are the same under what conditions.

Please read an example first.

Example 3.5: Let $U = \{a, b, c, d\}$, be a universe, $C = \{K_1, K_2, K_3, K_4, K_5\} = \{\{a, b, c\}, \{a, b\}, \{b, c\}, \{b, c, d\}, \{a, b, c, d\}\}$ is a covering of U. Please computer the refinement of C in a different order of reduct and refinement.

Solve: (1) Reduct before refinement.

Because $K_1 = K_2 \bigcup K_3$, $K_5 = K_1 \bigcup K_2 \bigcup K_3 \bigcup K_4$, according to the definition 2.4, we know that K_1 and K_5 are two reducible element. With definition 2.5 shows that K_1 and K_5 can be reduct, so $reduct(C) = \{K_2, K_3, K_4\}$. Then according to definition 3.5, we know that $RCE(K_2) = \{\{a\}, \{a, b\}\}$, $RCE(K_3) = \{\{b, c\}\}$, $RCE(K_4) = \{\{b, d\}, \{c, d\}, \{d\}\}$. At last, according to definition 3.6, we can work out $RC(reduct(C))$, that is, $RC(reduct(C)) = \{\{a\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d\}\}$.

(2) Refinement before reduct.

According to definition 3.1, we can know that, for any $x \in U, |FM(x)| > 1$, that is to say, all elements of U are indeterminate elements. From definition 3.3 we get that the refinement of C is itself. That is, $RC(C) = \{K_1, K_2, K_3, K_4, K_5\}$. After reducing of $RC(C)$, we get that $reduct(RC(C)) = \{K_2, K_3, K_4\}$. Thus, $RC(reduct(C)) \neq reduct(RC(C))$.

Here, we can see that if the order of reduction and refinement is different, then the new covering is different.

The result in the above is different, for the reason is that we produce new determinate elements a and d after reduce C, which bring about the changing of refinement. If the covering is $C = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, c, d\}\}$ and the order is refinement after reduction, then, $RC(reduct(C)) = \{\{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d\}\}$. If the order is reduction after refinement, then $reduct(RC(C)) = \{\{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d\}\}$. Hence, $RC(reduct(C)) = reduct(RC(C))$. The appearance of this result is that the number of determinate element after reduce K_1 is not change in the process of refinement after reduction. Accordingly, we can get proposition 3.10.

Proposition 3.10: Let U be a universe and C be a covering of U. C_1 is the new covering of refinement after reduction. C_2 is the new covering of reduction after refinement. If C and $reduce(C)$ has the same number of determinate element, then $C_1 = C_2$. Otherwise, $C_1 \neq C_2$.

Proof: Let $U = x1, x2, , xn$ be a universe. $C = K1, K2, , Km$ is a covering of U. I is an index set and $i, j, p, q, r \in I$.

(1) C and $reduce(C)$ have the same number of determinate elements. Firstly, we analyze the condition of refinement after reduction. For any $K_i \in C$, if K_i is a reducible element , then there at least exists more than two covering-elements $K_1, K_2, \ldots, K_r (r \geq 2)$ in $C - \{K_i\}$ such that $K_i = K_1 \cup K_2 \cup K_r$. K_i will be deleted after reduction C. If there still exists reducible element K_p in C, we will delete it according to definition 2.5 until there is not reducible element in C. Here, we get the reduction $reduct(C)$ of C. After refining of $reduct(C)$, we get a new covering $C_1 = RC(reduct(C))$.

On the contrary, if we reduce after refine C, according to some related definition, we can get $RCE(K_i) = K_i$ for K_i is a reducible element. Similarly, if there still exists other reducible element K_p, then $RCE(K_p) = K_p$. From definition 3.6, we can know that $K_i, K_p \in RC(C)$. In the following, we can get $RCE(K_1), RCE(K_2), \ldots, RCE(K_r)$ after refining $K_1, K_2, \ldots, K_r$. And $RCE(K_1) \subseteq RC(C), RCE(K_2) \subseteq RC(C), \ldots, RCE(K_r) \subseteq RC(C)$. So, $K_i = RCE(K_i) = RCE(K_1) \cup RCE(K_2) \cup \ldots \cup RCE(K_r)$. Then, according to the definition of reducible element and reduction , we delete K_i from $RC(C)$. Similarly, K_p can also be deleted from $RC(C)$. Here, $reduct(RC(C)) = C_2$.

Because C and $reduct(C)$ have the dame determinate element, that is to say, for $K_i \in C, K_j \in reduct(C)$, if $K_i = K_j$, then $DS(K_i) = DS(K_j)$, $IDS(K_i) = IDS(K_j)$. From definition 3.4, we get that $C_1 = C_2$.

(2) The number of determinate elements in C and $reduce(C)$ are different.

When the number of determinate elements in C and $reduce(C)$ are different, that is to say, there at least exists $K_i \in C$ and $K_j \in reduct(C)$ such that $K_i = K_j$ and $DS(K_i) \neq DS(K_j)$. Then $REC(K_i) \neq REC(K_j)$. According to definition 3.6, we get that $C_1 \neq C_2$. ■

Now, let we think about a problem. For a covering we know that the redundant covering-element of the refinement of the covering would be reduced in the order of refinement before reduct. Then, what result would be in the order of reduct before refinement?

Proposition 3.11: Let U be a universe and C be a covering of U. The refinement after reduction of covering C will not produce new reducible element, that is to say, $RC(reduct(C))$ has not reducible element.

Proof: Let $reduct(C) = K_1, K_2, \ldots, K_m$. For any $K_i \in reduct(C)$, assume $RCE(K_i) = \{T_{i1}, T_{i2}, \ldots, T_{ip}\}$, where $p \geq 1$. Then $T_{ij} \in RCE(K_i), 1 \leq j \leq p$. According to definition 3.3, 3.4, 3.5, we get that there is not exist $T_{iq} \in RCE(K_i) - \{T_{ij}\}$ such that $T_{iq} \subseteq T_{ij}$. So, T_{ij} is irreducible in $RCE(K_i)$. Let $RC(reduct(C)) = \{T_{11}, T_{12}, \ldots, T_{1a}, T_{21}, T_{22}, \ldots, T_{2b}, \ldots, T_{m1}, T_{m2}, \ldots, T_{mc}\}(a \geq 1, b \geq 1, c \geq 1)$. If there at least exists a determinate element y in K_i, according to definition 3.1 and 3.2, we get that $y \notin \cup(reduct(C) - \{K_i\})$. So, T_{ij} is irreducible in $RC(reduct(C))$. If there exists indeterminate element in K_i, according to definition 3.3, 3.4, 3.5 and 3.6, we get that $K_i \in RC(reduct(C))$. Let there exist $T_{de}, T_{fg} \in RC(reduct(C))(1 \leq d \leq m, 1 \leq f \leq m, e \geq 1, g \geq 1)$ such that $K_i = T_{de} \cup T_{fg}$, then T_{de} and T_{fg} are composed by indeterminate elements. According to definition 3.3, 3.4, 3.5 and proposition 3.2, we get that $T_{de}, T_{fg} \in reduct(C)$. This is contract to $K_i \in reduct(C)$. So, there are not exist $T_{de}, T_{fg} \in RC(reduct(C))$ such that $K_i = T_{de} \cup T_{fg}$. Accordingly, the refinement after reduction of covering C will not produce new reducible element,. ■

Proposition 3.12: Let U be a universe and C is a covering of U. If each covering-element of C at least has one determinate element, then C is irreducible.

Proof: Let $C = \{K_1, K_2, \ldots, K_m\}$. For any $K_i \in C$, if $x \in K_i$ and $|FM(x)| = 1$, according to definition 3.1 and 3.2, we get that $x \notin (C - \{K_i\})$. So, there are not exist two or more covering-element in $C - \{K_i\}$ such that the union of them equal to K_i. That is to say, K_i is a irreducible element. Similarly, we get that each covering-element of C is irreducible. Then, C is irreducible. ■

Proposition 3.13: Let U be a universe and C be a covering of U. $K \in C$, if K is a reducible element, then for any $x \in K$, $|FM(x)| > 1$.

Proof: Let $C = K_1, K_2, \ldots, K_m$. For any $K_i \in C$, if K_i is an irreducible element, then there exist $K_1, K_2, \ldots, K_p \in C - \{K_i\}(P \geq 2)$ such that $K_i = K_1 \cup K_2 \cup \ldots \cup K_p$. According to definition 3.1, we get that, for any xKi, $|FM(x)| > 1$. ■

D. The algorithm of the refinement of covering-element

According to proposition 3.10, we get that the result that reduce a covering which has been refined is different from the refined covering of reduction. When the two results are different, according to the process of proofing in proposition 3.10, we can know that the number of covering elements of the former result is not greater than the later. This means that the judgment of later is stronger than former. When the two results are the same, according to definition 3.6 we can know that the number of covering elements is not greater than the number of covering-element after refined covering. From this point we can say that reducing a covering is convenient than a reduction of refined covering. For this reason, we will give the algorithm of refinement after reducing a covering.

The algorithm we give includes two parts: reduction and refinement. Zhu [32], [33] give a algorithm of reduction. After we analyze the properties of reduction: (1) for any $K_i \in C$, if K_i is a reducible element, then there at least exists two or more covering-elements $K_1, K_2, \ldots, K_r$ such that $K_i = K_1 \cup K_2 \cup \ldots \cup K_r$; (2) if $K_i = K_1 \cup K_2 \cup \ldots \cup K_r$, then $|K_i| > |K_1|, |K_i| > |K_2|, \ldots, |K_i| > |K_r|$, that is to say, the number of element of reducible element to K_i is surely greater than the number of any proper subset. We improve the algorithm of reduction about Zhu [32], [33], on which we give the refined algorithm.

Part I: reduction

Input: universe U and covering $C = \{K_1, K_2, \ldots, K_m\}$.

Output: $reduct(C)$

S1 initialization: $reduct(C) = C$, $i = 3$;

S2 $temp = \emptyset$, $j = 1$;

S3 we sort the number of element in covering C into nondecreasing order and get $C = \{P_1, P_2, \ldots, P_m\}$;

S4 if $m < 3$, then jump to S14;

S5 if $P_j \subset P_i$, then $temp = temp \cup P_j$;

S6 $j = j + 1$;

S7 if $j < I$, then jump to S5;

S8 if $temp \neq P_i$, then jump to S12

S9 $reduct(C) = reduct(C) - \{P_i\}$

S10 we rearrange the number from i: $P_i = P_{i+1}, P_{i+1} = P_{i+2}, \ldots, P_{m-1} = P_m$;

S11 $j = 1$, jump to S5

S12 $i = i + 1$;

S13 if $i \leq m$, jump to S11

S14 the end.

Part II refinement

Input: a reduction $reduct(C)$ of universe U;

Output: the refinement $RC(reduct(C))$ of $reduct(C)$;

S1 initialization: $RC(reduct(C)) = \emptyset, i = 1, h = 1$;

S2 if $i > r$, then jump to S13;

S3 $DS(T_i) = \emptyset, IDS(T_i) = \emptyset, j = 1$;

S4 $m = |T_i|$;

S5 compute $FM(x_j)$;

S6 if $|FM(x_j)| = 1$, then $DS(T_i) = x_j$, $j = j + 1$, jump to S8;

S7 $IDS(T_i) = x_j, j = j + 1$;

S8 if $j < m$, then jump to S5;

S9 if $|DS(T_i)| = 0$, then $RC(reduct(C)) = RC(reduct(C)) \cup IDS(T_i), i = i + 1$, jump to S2;

S10 $RC(reduct(C)) = RC(reduct(C)) \cup \{IDS(T_i)\{x_h\}\}, h = h + 1$;

S11 if $h < m$, jump to S10;

S12 $h = 1, i = i + 1$;

S13 the end.

(x_j is the jth element of T_i)

IV. The comparison of covering-based rough sets model

In this section, we will compare some mainly covering-based rough sets based on the refinement of covering-element. By comparing, we discover that the lower approximation of all models in original covering are not greater than the lower approximation in the refinement of covering-element. And the upper approximation of all models in original covering are not less than the upper approximation in the refinement of covering-element. This means that the judgment of each model to object is stronger on the basis of refinement of covering-element.

A. The model of covering-based rough sets

In this section, we will mainly introduce eight main models of covering-based rough sets. In order to better understand some model, we introduce some new concepts.

Definition 4.1: (Neighbor [25], [34], friend [27], enemy [18]) Let $< U, C >$ be a covering approximation space, for any $x \in U$, $Neighbor(x) = \bigcap\{K|x \in K \wedge K \in C\}$ is called the neighbor of x and denote as $N(x)$; $\bigcup\{K|x \in K \wedge K \in C\}$ is called the friend of x and denote as $Friends(x)$; $U - Friends(x)$ is called the enemy of x and denote as $e.f(x)$.

Definition 4.2: (Eight models of covering-based rough sets) [15], [16], [18]–[25] Let C be a covering of a universe U, for any set $X \subseteq U$, define:

The lower approximation of X in eight models of covering-based rough sets from the first to the eighth are $X_*, \underline{X}, X_\#, X_@, X_+, X_\$, X_\%, X_\&$. And $X_* = \underline{X} = X_\# = X_@ = X_+ = X_\% = X_\& = \bigcup\{K|K \in C \wedge K \subseteq X\}$, $X_\$ = \{x|N(x) \subseteq X\}$. The upper approximation of X in eight models of covering-based rough sets are defined respectively as follows:

The first is [15]: $X^* = X_* \bigcup(\bigcup\{\bigcup Md(x)|x \in X - X_*\})$;

The second is [19]: $\overline{X}$=$\bigcup\{K|K \in C \wedge K \bigcap X \neq \emptyset\}$;

The third is [20]: $X^\# = \bigcup\{\bigcup Md(x)|x \in X\}$;

The fourth is [27]: $X^@ = X_@ \bigcup\{K|K \in C \wedge K \bigcap(X - X_@) \neq \emptyset\}$;

The fifth is [26]: $X^+ = X_+ \bigcup\{N(x)|x \in X - X_+\}$;

The sixth is [21], [23], [25]: $X^\$ = \{x|N(x) \bigcap X \neq \emptyset\}$;

The seventh is [22]: $X^\% = X_\% \bigcup(\sim \bigcup\{Friends(y)|x \in X - X_\%, y \in e.f(x)\})$;

The eighth is: $X^\& = X_\& \bigcup(\bigcup\{\bigcap_{K \in Md(x)} K|x \in X - X_\&\})$;

(remark: symbol "∼" means obtaining complementary set.)

According to the above definitionwe find that the eight models's lower approximation are the same except the sixth. While the upper approximation of the eight models are different. To these models, we can't estimate which one is better or worse than others because different models may be applicable to different places. Zhu [25], [27], [35] study the upper approximation of them from the point of view of containable relation.

B. The comparison of covering-based rough sets

In this section, we will propose position 16, 17 and 18, and give the proofs of them in detail. Some concepts of this section such as $Md(x)$, $N(x)$, $Friend(x)$ are defined on $reduct(C)$.

Proposition 4.1: Let U be a universe, and C is a covering of U. $X \subseteq U$ is an arbitrary subset of U. In the eight models of covering-based rough sets defined in definition 4.2, the lower approximation of X produced in $reduct(C)$ is not greater than it produced in $RC(reduct(C))$.

Proof: Let $redct(C) = \{K_1, K_2, \ldots, K_m\}$. I is an index set and $i, j, p, q, r, s, h \in I$. For convenience, let X_* is the lower approximation of other seven models of covering-based rough sets that except the 6th type of model and it bases on $reduct(C)$. Correspondingly, RX_* is the lower approximation of the seven models of covering-based rough and it bases on $RC(reduct(C))$.

For any $K_i \in reduct(C)$, $RCE(K_i)$ is the refinement of K_i and we suppose that $RCE(K) = \{T_{i1}, T_{i2}, \ldots, T_{ir}\}$. If $K_i \subseteq X$, according to definition 4.2, we get that $K_i \subseteq X_*$. Since $\cup RCE(K_i) = K_i$, then $T_{ij} \subseteq K_i$ where $1 \le j \le r$. Accordingly, for each $T_{ij} \in RCE(K_i), T_{ij} \subseteq RX_*$. Similarly, if $X_* = K_1 \cup K_2 \cup \ldots \cup K_p$, then the union of $RCE(K_1), RCE(K_2), \ldots, RCE(K_p)$ that correspond to $K_1, K_2, \ldots, K_p$ equals to RX_*, that is, $RX_* = RCE(K_1) \cup RCE(K_2) \cup \ldots \cup RCE(K_p)$. Here, if there at least exists $K_q \in reduct(C) - \{K_1, K_2, \ldots, K_p\}$ and $T_{qh} \in RCE(K_q)$ such that $T_{qh} \subseteq X$, then $RX_* \subset X_*$. Of course, if K_q or T_{qj} are not exist, then $RX_* = X_*$. Accordingly, the result holds.

In the following, we prove that the 6th type of model of covering-based rough sets also satisfies proposition.

Let $reduct(C) = \{K_1, K_2, \ldots, K_m\}$. I is an index set and $i, j, p, q, r, h, s, e, f \in I$. $RN(x)$ is the neighbor of x produced in $RC(reduct(C))$. $RX_\$$ is the 6th type of model of the lower approximation of X produced in $RC(reduct(C))$.

In $reduct(C)$, if $x \in K_i, K_j$, then $N(x) = K_i \cap K_j$. Let $RCE(K_i) = \{T_{i1}, T_{i2}, \ldots, T_{ip}\}$, $RCE(K_j) = \{T_{j1}, T_{j2}, \ldots, T_{jq}\}$. If there exists $s(1 \le s \le p)$ elements in $RCE(K_i)$ and $h(1 \le h \le q)$ elements in $RCE(K_j)$ contain x. Let $A = T_{i1} \cap T_{i2} \cap \ldots \cap T_{is}, B = T_{j1} \cap T_{j2} \cap \ldots \cap T_{jh}$. Then $A \subseteq K_i$, $B \subseteq K_j$. Accordingly, $RN(x) = A \cap B \subseteq K_i \cap K_j = N(x)$, that is, $RN(x) \subseteq N(x)$. Similarly, in $RC(reduct(C))$, if $x \in K_1, K_2, \ldots, K_r$, then $RN(x)N(x)$ still holds. So, if $N(x) \subseteq X$, then $RN(x) \subseteq X$. That is, for any $x \in U$, if $x \in X_\$$, then $x \in RX_\$$. Here, if there at least exists $y \in U - X_\$$ such that $RN(y) \subseteq X$, then $X_\$ \subset RX_\$$. If there is not exist such y or $RN(y)$, then $X_\$ = RX_\$$. Thus, in the 6th type of model of covering-based rough sets, the lower approximation of X produced in $reduct(C)$ is not greater than it produce in $RC(reduct(C))$.

In conclusion, we get that, in the eight models of covering-based rough sets defined in definition 4.2, the lower approximation of X produced in $reduct(C)$ is not greater than it produce in $RC(reduct(C))$. ■

Proposition 4.2: Let U be a universe, and C is a covering of U. $X \subseteq U$ is an arbitrary subset of U. In the eight models of covering-based rough sets defined in definition 4.2, beside the 1th and the 3th types, the upper approximation of X to reduct(X) is not less than it is to $RC(reduct(C))$.

Proof: Let $reduct(C) = \{K_1, K_2, \ldots. K_m\}$. I is an index set and $i, j, p, qr, h, s \in I$.

(1)The upper approximation $\overline{X} = \cup\{K|K \in C \wedge K \cap X \neq \emptyset\}$ of the 2th type of covering-based rough sets.

Let $\overline{RX} = \cup\{K|K \in RC(reduct(C)) \wedge K \cap X \neq \emptyset\}$ is the upper approximation of X in $RC(reduct(X))$.

For any $K_i \in reduct(C)$, assume $RCE(K_i) = \{T_{i1}, T_{i2}, \ldots, T_{ip}\}$, if $K_i \subseteq \overline{X}$ and there exists $T_{i1}, T_{i2}, \ldots, T_{is}(1 \le s \le p)$ such that $T_{i1} \cap X \neq \emptyset, T_{i2} \cap X \neq \emptyset, \ldots, T_{is} \cap X \neq \emptyset$. Let $A_i = \{T_{i1}, T_{i2}, \ldots, T_{is}\}$, then $\cup A_i = K_i$. If $\overline{X} = K_1 \cup K_2 \cup \ldots \cup K_j$, correspondingly, $\overline{RX} = \{A_q|q = 1, 2, \ldots, j\}$ and $\overline{RX} \subseteq \overline{X}$. That is, the upper approximation of X in the 2th type of model of covering-based rough sets to $reduct(C)$ is not less than it is to $RC(reduct(C))$.

(2) The upper approximation $X^@ = X_@ \cup \{K|K \in C \wedge K \cap (X - X_@) \neq \emptyset\}$ of the 4th type of model of covering-based rough sets.

Let $RX_@$ is the lower approximation of X and $RX^@ = RX_@ \cup \{K|K \in RC(reduct(C)) \wedge K \cap (X - RX_@) \neq \emptyset\}$ is the upper of X in $RC(reduct(X))$, respectively.

According to proposition 4.1, we get $X_@ \subseteq RX_@$. Because $X - X_@ = (X - RX_@) + (RX_@ - X_@)$, then $\{K|K \in C \wedge K \cap (X - X@) \neq \emptyset\}$ =$\{K|K \in C \wedge K \cap (X - RX_@) \neq \emptyset\}$+$\{K|K \in C \wedge K \cap (RX_@ - X_@) \neq \emptyset\}$. For any $K_i \in reduct(C)$, assume $RCE(K_i) = \{T_{i1}, T_{i2}, \ldots, T_{ip}\}$. If $K_i \cap (X - RX_@) \neq \emptyset$, then there exists $T_{i1}, T_{i2}, \ldots, T_{is}(1 \le s \le p)$ such that $T_{i1} \cap (X_@ - RX_@) \neq \emptyset$, $T_{i2} \cap (X_@ - RX_@) \neq \emptyset$, $\ldots$, $T_{is} \cap (X_@ - RX_@) \neq \emptyset$. If $A_i = \{T_{i1}, T_{i2}, \ldots, T_{is}\}$, then $\cup A_i \subseteq K_i$. Similarly, if $K_1, K_2, \ldots, K_j \in C$ and $K_1 \cap (X_@ - RX_@) \neq \emptyset$, $K_2 \cap (X_@ - RX_@) \neq \emptyset$, $\ldots$, $K_j \cap (X_@ - RX_@) \neq \emptyset$, then there exists $A_1, A_2, \ldots, K_j$ such that $\cup\{\cup A_q|q = 1, 2, \ldots, j\} \subseteq \cup\{K_q|q = 1, 2, \ldots, j\}$. Accordingly, we can get that: $\{K|K \in RC(reduct(C)) \wedge K \cap (X - RX_@) \neq \emptyset\} \subseteq \{K|K \in reduct(C) \wedge K \cap (X - RX_@) \neq \emptyset\}$ On the other hand, $RX_@ - X_@ \subseteq \{K|K \in reduct(C) \wedge K \cap (RX_@ - X_@) \neq \emptyset\}$. So, $(RX_@ - X_@)$+$\{K|K \in RC(reduct(C)) \wedge K \cap (X - RX_@) \neq \emptyset\}$$\subseteq$$\{K|K \in reduct(C) \wedge K \cap (X - RX_@) \neq \emptyset\}$ +$\{K|K \in reduct(C) \wedge K \cap (RX_@ - X_@) \neq \emptyset\}$. Consequently, $RX_@$+$\{K|K \in RC(reduct(C)) \wedge K \cap (X - RX_@) \neq \emptyset\}$$\subseteq$ $X_@ + \{K|K \in reduct(C) \wedge K \cap (X - RX_@) \neq \emptyset\}$, namely, $RX^@ \subseteq X^@$. So, the result holds.

(3)The upper approximation $X^+ = X_+ \cup \{N(x)|x \in X - X_+\}$ of the 5th type of model of covering;

In $RC(reduct(C))$, we suppose that RX_+ is the lower approximation of X, $RN(x)$ is the neighborhood of x, and $RX^+ = RX_+ \cup \{RN(x)|x \in X - RX_+\}$ is the upper approximation.

According to proposition 4.1, we get that $X_+ \subseteq RX_+$. For any $x \in U$, we have that $RN(x) \subseteq N(x)$. Since $X - X_+ = (X - RX_+)$+$(RX_+ - X_+)$, then $\{N(x)|x \in X - X_+\}$=$\{N(x)|x \in X - RX_+\}$+$\{N(x)|x \in RX_+ - X\}$. Because $X_+ \subseteq RX_+$, then $\{RN(x)|x \in X - RX_+\}$$\subseteq$$\{N(x)|x \in X - RX_+\}$. As $RX_+ - X_+ \subseteq \{N(x)|x \in RX_+ - X_+\}$, then $(RX_+ - X_+)$+$\{RN(x)|x \in X - RX_+\}$$\subseteq$$\{N(x)|x \in X - RX_+\}$+$\{N(x)|x \in RX_+ - X_+\}$. Consequently, RX_++$\{RN(x)|x \in X - RX_+\}$$\subseteq$$X_+$ +

$\{N(x)|x \in X - X_+\}$, that is , $RX_+ \subseteq X_+$. Therefore, we get the result.

(4) The upper approximation $X^{\$} = \{x|N(x) \cap X \neq \emptyset\}$ of the 6th type of model of covering;

Let $RN(x) = \cap\{K|x \in K \wedge K \in RC(reduct(C))\}$ is the neighbor of x in $RC(reduct(C))$. $RX^{\$} = \{x|RN(x) \cap X \neq \emptyset\}$ is the upper approximation of X in $RC(reduct(C))$. For any $x \in U$, we get that $RN(x) \subseteq N(x)$. If $RN(x) \cap X \neq \emptyset$, then $N(x) \cap X \neq \emptyset$. So, $\{x|RN(x) \cap X \neq \emptyset\} \subseteq \{x|N(x) \cap X \neq \emptyset\}$, that is, $RX^{\$} \subseteq X^{\$}$. Thereby, the result holds.

(5) The upper approximation $X^{\%} = X_{\%} \cup \{\sim \cup\{Friends(y)|x \in X - X_{\%}, y \in Re.f(x)\}\}$ of the 7th type of model of covering; $RX_{\%}$ is the lower approximation of X in $RC(reduct(C))$. $RFriend(x)$ and $Re.f(x)$ are the friend and enemy of x in $RC(reduct(C))$. $RX^{\%} = RX_{\%} \cup \{\sim \cup\{Friends(y)|x \in X - RX_{\%}, y \in Re.f(x)\}\}$ is the upper approximation of X in $RC(reduct(C))$.

From proposition 4.1, we get that $X_{\%} \subseteq RX_{\%}$. Then, $X - RX_{\%} \subseteq X - X_{\%}$, that is, $\forall x \in X - RX_{\%}$. There surely be that $x \in X - X_{\%}$. On the contrary, we could not hold that. We suppose that $RCE(K_i) = \{T_{i1}, T_{i2}, \ldots, T_{ip}\}$. For any $x \in X - RX_{\%}$ and $K_i \subseteq Friends(x)$, if $x \in T_{i1}, T_{i2}, \ldots, T_{is} (1 \leq s \leq p)$, then $T_{i1}, T_{i2}, \ldots, T_{is} \subseteq RFriends(x)$. Assume $A_i = \{T_{i1}, T_{i2}, \ldots, T_{is}\}$, then $\cup A_i \subseteq K_i$. Consequently, if $Friends(x) = \cup\{K_1, K_2, \ldots, K_j\}$, correspondingly, there is $RFriends(x) = \cup\{A_1 \cup A_2 \cup \ldots \cup A_j\}$ and $RFriends(x) \subseteq Friends(x)$. According to definition 4.1, we get that $e.f(x) \subseteq Re.f(x)$. So, $\cup\{Friends(y)|x \in X - RX_{\%}, y \in e.f(x)\} \subseteq \cup\{RFriends(y)|x \in X - RX_{\%}, y \in Re.f(x)\}$. In the following, we get that $\sim \cup\{RFriends(y)|x \in X - RX_{\%}, y \in Re.f(x)\} \subseteq \sim \cup\{Friends(y)|x \in X - RX_{\%}, y \in e.f(x)\}$. For any $x \in RX_{\%} - X_{\%}$, there surely be that $x \notin e.f(x)$. Thereby, $x \notin \cup\{Friends(y)|x \in RX_{\%} - X_{\%}, y \in e.f(x)\}$. Then $x \in (\sim \cup\{Friends(y)|x \in RX_{\%} - X_{\%}, y \in e.f(x)\})$. Further more, $RX_{\%} - X_{\%} \subseteq \{\sim \cup\{Friends(y)|x \in RX_{\%} - X_{\%}, y \in e.f(x)\}\}$. Therefore, $(RX_{\%} - X_{\%})$ + $(\sim \cup\{RFriends(x)|x \in X - RX_{\%}, y \in Re.f(x)\}) \subseteq (\sim \cup\{Friends(y)|x \in X - RX_{\%}, y \in e.f(x)\})$+ $(\sim \cup\{Friends(y)|x \in RX_{\%} - X_{\%}, y \in e.f(x)\})$. Accordingly, $RX_{\%}$ + $(\sim \cup\{RFriends(x)|x \in X - RX_{\%}, y \in Re.f(x)\}) \subseteq X_{\%}$ + $(\sim \cup\{Friends(y)|x \in X - X_{\%}, y \in e.f(x)\})$, that is, $RX^{\%} \subseteq X^{\%}$. So, the result holds.

(6) The upper approximation $X^{\&} = X_{\&} \cup (\cup\{\bigcap_{K \in Md(x)} K|x \in X - X_{\&}\})$ of the 8th type of model of covering.

Let RX_+ and $RMd(x)$ in $RC(reduct(C))$ are respectively the lower approximation and minimal description. $X^{\&} = X_{\&} \cup (\cup\{\bigcap_{(K \in Md(x))} K|x \in X - RX_{\&}\})$ is the upper approximation of X in $RC(reduct(C))$.

According to proposition 4.1, we get that $X_{\&} \subseteq RX_{\&}$. During the proof of (1), we get that $\cup A_i \subseteq K_i$. If $Md(x) = \{K_1, K_2, \ldots, K_j\}$, correspondingly, there is $RMd(x) = \{A_1 \cup A_2 \cup \ldots \cup A_j\}$ such that $\bigcap_{q=1}^{j}(\cup A_q) \subseteq \bigcap_{q=1}^{j} K_q$. So, $\cup\{\bigcap_{K \in RMd(x)} K|x \in X - RX_{\&}\} \subseteq \cup\{\bigcap_{K \in Md(x)} K|x \in X - RX_{\&}\}$. Moreover, $X - X_{\&}$=$(X - RX_{\&})$+$(RX_{\&} - X_{\&})$. Then, $\cup\{\bigcap_{K \in Md(x)} K|x \in X - X_{\&}\}$ =$\cup\{\bigcap_{K \in Md(x)} K|x \in X - RX_{\&}\}$+$\cup\{\bigcap_{K \in Md(x)} K|x \in RX_{\&} - X_{\&}\}$. Because of $RX_{\&} - X_{\&} \subseteq \cup\{\bigcap_{K \in Md(x)} K|x \in RX_{\&} - X_{\&}\}$, then $(RX_{\&} - X_{\&})$ +$\cup\{\bigcap_{K \in RMd(x)} K|x \in X - RX_{\&}\} \subseteq \cup\{\bigcap_{K \in Md(X)} K|x \in X - RX_{\&}\}$+$\cup\{\bigcap_{K \in Md(x)} K|x \in RX_{\&} - X_{\&}\}$. So, $RX_{\&}$ +$\cup\{\bigcap_{RMd(x)} K|x \in X - RX_{\&}\} \subseteq X_{\&}$ + $\cup\{\bigcap_{K \in Md(x)} K|x \in X - X_{\&}\}$, that is, $RX^{\&} \subseteq X^{\&}$. Thereby, the above result holds. Accordingly, we prove the proposition 4.2. ■

For the first and third models of covering-based rough sets, proposition does not hold. We can get it from the following example.

Example 4.1: Let $U = \{a, b, c, d\}$ be a universe, $C = \{\{a\}, \{a, b, c\}, \{a, b, c, d\}\}$ is a covering of U, $X = \{a, b, d\}$ is a subset of U. In the first and the third model of covering-based rough sets, Please computer the upper approximation of X on $reduct(C)$ and $RC(reduct(C))$ respectively.

According to known conditions, $reduct(C)$=C$RC(reduct(C)) = \{\{a\}, \{a, b, c\} \{a, d\}, \{b, d\}, \{c, d\}, \{d\}\}$, for any $x \in U$, $Md(x)$ and $RMd(x)$ represent the minimal description of x on $reduct(C)$ and $RC(reduct(C))$ respectively. Then, $Md(a) = \{\{a\}\}$, $Md(b) = \{\{a, b, c\}\}$, $Md(c) = \{\{a, b, c\}\}$, $Md(d) = \{\{a, b, c, d\}\}$, $RMd(a) = \{\{a\}\}$, $RMd(b) = \{\{a, b, c\}, \{b, d\}\}$, $RMd(c) = \{\{a, b, c\}, \{c, d\}\}$, $RMd(d) = \{\{d\}\}$.

(1) In the first model:

The upper approximation of X on $reduct(C)$ is:

$X^* = X_* \bigcup(\bigcup\{\bigcup Md(x)|x \in X - X_*\}) = \{a, b\} \bigcup\{a, b, c\} = \{a, b, c\}$;

The upper approximation of X on $RC(reduct(C))$ is:

$X^* = X_* \bigcup(\bigcup\{\bigcup RMd(x)|x \in X - X_*\}) = \{a, b\} \bigcup (\bigcup \{\{a, b, c\} \bigcup\{c, d\}\}) = \{a,\ b, c, d\}$;

(2) In the third model:

The upper approximation of X on $reduct(C)$ is: $X^{\#} = \bigcup\{\bigcup Md(x)|x \in X\} = \bigcup\{\{a\}, \{a, b, c\}\} = \{a, b, c\}$

The upper approximation of X on $RC(reduct(C))$is: $X^{\#} = \bigcup\{\bigcup RMd(x)|x \in X\}$ $= \{a, b\} \bigcup(\bigcup\{\{a, b, c\} \bigcup\{c, d\}\}) = \{a, b, c, d\}$

From the above example we find that the upper approximation of X on $reduct(C)$ is larger than it on $RC(reduct(C))$, that is, proposition 4.2 is not hold in the first and the third model.

Next, we will illustrate and validate proposition 4.2 by another example.

Example 4.2: Let $U = \{a, b, c, d, e, f, g\}$ be a universe, $C = \{\{a, b\}, \{b, c, e\}, \{c, d, f\}, \{b, c, e, f, g\}, \{a, b, c, e, f, g\}\}$ is a covering of U, $X = \{a, b, d, e\}$ is a subset of U. Please computer and compare the lower and upper approximations of X on $reduct(C)$ and $RC(reduct(C))$ in eight models.

According to definition 2.5, we know that $reduct(C) = \{\{a, b\}, \{b, c, e\}, \{c, d, f\}, \{b, c, e, f, g\}\}$. Then with definition 3.6 shows that $RC(reduct(C)) = \{\{a\}, \{a, b\}, \{b, c, e\}, \{c, d\}, \{d\}, \{d, f\}, \{b, g\}, \{c, g\}, \{e, g\}, \{f, g\}, \{g\}\}$.

(1) The lower approximation of X on $reduct(C)$ and $RC(reduct(C))$.

According to definition 4.2, we know that the upper approximation of X on $reduct(C)$ in eight models are:

$X_* = \underline{X} = X_\# = X_@ = X_+ = X_\% = X_\& = \bigcup\{K|K \in C \wedge K \subseteq X\} = \{a, b\}$;

$X_\$ = \{x|N(x) \subseteq X\} = \{a, b\}$;

The lower approximation of X on $RC(reduct(C))$ in eight models are:

$X_* = \underline{X} = X_\# = X_@ = X_+ = X_\% = X_\& = \bigcup\{K|K \in C \wedge K \subseteq X\} = \{a, b, d\}$;

$X_\$ = \{x|N(x) \subseteq X\} = \{a, b, d, e\}$.

Thus, the lower approximation of X on $RC(reduct(C))$ is larger than it on $reduct(C)$ in eight models.

(2) The upper approximation of X on $reduct(C)$ and $RC(reduct(C))$.

According to definition 4.2, we know that the upper approximation of X on $reduct(C)$ in eight models are:

$X^* = X_* \bigcup\{Md(x)|x \in X - X_*\} = \{a, b, c, d, e, f\}$;

$\overline{X} = \bigcup\{K|K \in C \wedge K \bigcap X \neq \emptyset\} = \{a, b, c, d, e, f, g\}$;

$X^\# = \bigcup\{Md(x)|x \in X\} = \{a, b, c, d, e, f\}$;

$X^@ = X_@ \bigcup\{K|K \in C \wedge K \bigcap(X - X_@) \neq \emptyset\} = \{a, b, c, d, e, f, g\}$;

$X^+ = X_+ \bigcup\{N(x)|x \in X - X_+\} = \{a, b, c, d, e, f\}$;

$X^\$ = \{x|N(x) \bigcap X \neq \emptyset\} = \{a, b, c, d, e, f, g\}$;

$X^\% = X_\% \bigcup(\sim \bigcup\{Friends(y)|x \in X - X_\%, y \in e.f(x)\}) = \{a, b, c, d, e, f, g\}$;

$X^\& = X_\& \bigcup(\bigcup\{\bigcap_{K \in Md(x)} K|x \in X - X_\&\}) = \{a, b, c, d, e, f\}$;

The upper approximation of X on $RC(reduct(C))$ in eight models are:

$X^* = X_* \bigcup\{RMd(x)|x \in X - X_*\} = \{a, b, c, d, e, g\}$;

$\overline{X} = \bigcup\{K|K \in C \wedge K \bigcap X \neq \emptyset\} = \{a, b, c, d, e, f, g\}$;

$X^\# = \bigcup\{RMd(x)|x \in X\} = \{a, b, c, d, e, g\}$;

$X^@ = X_@ \bigcup\{K|K \in C \wedge K \bigcap(X - X_@) \neq \emptyset\} = \{a, b, c, d, e, g\}$;

$X^+ = X_+ \bigcup\{N(x)|x \in X - X_+\} = \{a, b, c, d, e\}$;

$X^\$ = \{x|N(x) \bigcap X \neq \emptyset\} = \{a, b, d, e\}$;

$X^\% = X_\% \bigcup(\sim \bigcup\{Friends(y)|x \in X - X_\%, y \in e.f(x)\}) = \{a, b, d, e\}$;

Thus, in the eight models, all of the upper approximations of X on $RC(reduct(C))$ are not larger than it on $reduct(C)$ except the first and the third model.

Then, on what conditions proposition 4.2 is true to the first and the third model?

Proposition 4.3: Let U be a universe, and C is a covering of U. $X \subseteq U$, for any $x \in U$, if $\cup RMd(x) \subseteq \cup Md(x)$, then the upper approximation of X in $reduct(C)$ is not less than it is in $RC(reduct(C))$ to the first and third models of covering-based rough sets.

Proof: (1) To the 1th type of model of covering-based rough sets, the upper approximation $X^* = X_* \cup (\cup\{\cup Md(x)|x \in X - X_*\})$. Because $\cup RMd(x) \subseteq \cup Md(x)$, and $\cup\{\cup RMd(x)|x \in X - RX_*\} \subseteq \cup\{\cup Md(x)|x \in X - RX_*\}$. On the other hand, $RX_* - X_* \subseteq \cup\{\cup Md(x)|x \in RX_* - X_*\}$. So, $RX_* - X_* + \cup\{\cup RMd(x)|x \in X - RX_*\} \subseteq \cup\{\cup Md(x)|x \in RX_* - X_*\} + \cup\{\cup Md(x)|x \in X - RX_*\}$, that is , $RX_* \cup (\cup\{\cup RMd(x)|x \in X - RX_*\}) \subseteq X_* \cup (\cup\{\cup Md(x)|x \in X - X_*\})$. Thus, $RX^* \subseteq X^*$. Therefore, we get that the upper approximation of X in $reduct(C)$ is not less than it is in $RC(reduct(C))$ to the 1th type of model of covering-based rough sets.

(2) The upper approximation $X^\# = \cup\{\cup Md(x)|x \in X\}$ of X is the 3th type of model of covering-based rough sets. Because $\cup RMd(x) \subseteq \cup Md(x)$, then $\cup\{\cup RMd(x)| x \in X\} \subseteq \cup\{\cup Md(x)|x \in X\}$, that is , $RX^\# \subseteq X^\#$. Thus, we prove that the upper approximation of X in $reduct(C)$ is not less than it is in $RC(reduct(C))$ to the third model of covering-based rough sets. ■

Proposition 4.1, 4.2 and 4.3 based on the $reduct(C)$ of covering C are discussed the lower and upper approximations of the eight models of covering and are compared the upper and lower approximation before the refinement and after the refinement. The reason we do that is that: for one thing, by reducing the covering C, we can delete redundant information, and thereby get a better lower and upper approximations and increase the capacity of discernment. For another, the number of new covering-elements to refine the reduction of covering is more than it is directly refined this covering. The reason is that the determinate elements maybe increase after delete some reducible elements, which will bring to more covering elements. This means that the capacity of discernment to this model will enhance. If we directly discuss this problem in covering, we still get the same result.

V. Conclusions and Future Work

Covering-based rough sets is an important extension of rough sets and there are more and more applications and studies about it. In this paper, covering-based rough sets is studied from a new point of view of the refinement of covering-element. On the basis of refinement of covering-elements, the lower approximations of the eight models of covering-based rough sets are not greater than the original lower approximations. Correspondingly, all the upper approximations of the eight models are not less than the original upper approximations with exceptions of two models (the first and the third models) in some special situations. The refinement of covering-element enhances the capacity of discernment fundamentally to each of covering-based rough sets models. This is very meaningful to the study of rough sets theory and application. Meanwhile, the algorithms of Zhu [32], [33] is improved. And the algorithm of the refinement of covering-element is proposed. In the future work, we will continue study the properties of the refinement of covering-element. And we will use partially ordered set and lattice to study the refinement of covering-element.

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