Exact three-wave solutions for high nonlinear form of Benjamin-Bona-Mahony-Burgers equations

Mohammad Taghi Darvishi, Maliheh Najafi and Mohammad Najafi

Abstract—By means of the idea of three-wave method, we obtain some analytic solutions for high nonlinear form of Benjamin-Bona-Mahony-Burgers (shortly BBMB) equations in its bilinear form.

Keywords—Benjamin-Bona-Mahony-Burgers equations, Hirota's bilinear form, Three-wave method.

I. INTRODUCTION

THE generalized Benjamin-Bona-Mahony-Burgers equations has the following form

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + (g(u))_x = 0, \qquad (1)$$

where $u(x,t)$ represents the fluid velocity in the horizontal direction x, α is a positive constant, $\beta \in \mathbb{R}$ and $g(u)$ is a C^2-smooth nonlinear function. In the physical sense, equation (1) with the dissipative term $-\alpha u_{xx}$ is proposed if the good predictive power is desired, such problem arises in the phenomena for both the bore propagation and the water waves. Benjamin-Bona-Mahony-Burgers equations have a special kind of solutions: the solitary waves and the property of existence of a finite number of quantities (with physical interpretation) conserved by the solutions. The temporal decay rates of the solution of (1) and its special cases are studied in a number of works. [1] studied the large-time behavior of the global solutions to the Cauchy problem for (1). [2] used the decomposition method for the numerical solution subject to appropriate initial condition of (1) to consider solitary-wave solutions of generalized BBMB equations. [3] applied variational iteration method and homotopy perturbation methods to derive approximate explicit solutions for a special form of (1). [4] used hompotopy analysis method to BBMB equations. [5] concerned (1) for special case $\alpha = 1$ and $\beta = 0$, in this special case we have the long-wave equation

$$u_t - u_{xxt} - u_{xx} + (g(u))_x = 0. \qquad (2)$$

[6] implemented the Exp-Function method, to solve equation (1). [7] considered a high nonlinear form of BBMB equations in the following form

$$u_t + u_{xxx} - 3(uv)_x = 0,$$
$$u_x = v_y, \qquad (3)$$

and presented some exact solutions for it by the Exp-function method. [8] investigated the exponential time decay rate of

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solutions toward traveling waves for the Cauchy problem of equation (1). In this paper we consider equations (3) and solve it by three-wave method. By this method we obtain some exact solution for equations (3).

This paper is organized as follows: In the following section we have a brief review on the three-wave solution method. In Section III, we apply the three-wave method to obtain some exact solutions of (3). Finally, the paper is concluded in Section IV.

II. THREE-WAVE METHOD

Dai et al. [9], suggested the three-wave method for nonlinear evolution equations. The basic idea of this method applies the Painlevé analysis to make a transformation as

$$u = T(f) \qquad (4)$$

for some new and unknown function f. Then we use this transformation in a high dimensional nonlinear equation of the general form

$$F(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, \cdots) = 0, \qquad (5)$$

where $u = u(x,y,t)$ and F is a polynomial of u and its derivatives. By substituting (4) in (5), the first one converts into the Hirota's bilinear form, which it will solve by taking a special form for f and assuming that the obtained Hirota's bilinear form has three-wave solutions, then we can specify the unknown function f; cf. [9], [10].

III. SOLUTIONS OF HIGH NONLINEAR FORM OF BENJAMIN-BONA-MAHONY-BURGERS EQUATIONS

In this paper, we investigate explicit formula of solutions of the following high nonlinear form of Benjamin-Bona-Mahony-Burgers (BBMB) equations given in [7]

$$u_t + u_{xxx} - 3(uv)_x = 0,$$
$$u_x = v_y, \qquad (6)$$

where $u = u(x,y,t) : \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_t^+ \to \mathbb{R}$.
To solve (6), we introduce a new dependent variables u, v by

$$u = -2(\ln f)_{xy} \quad , \quad v = -2(\ln f)_{xx}, \qquad (7)$$

where f is an unknown real function which will be determined. Substituting equation (7) into equation (6), we have

$$(D_y D_t + D_y D_x^3)f \cdot f = 0, \qquad (8)$$

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where the D-operator is defined by

$$D_x^m D_y^k D_t^n f(x,y,z,t) \cdot g(x,y,z,t) =$$

$$\left(\tfrac{\partial}{\partial x_1} - \tfrac{\partial}{\partial x_2}\right)^m \left(\tfrac{\partial}{\partial y_1} - \tfrac{\partial}{\partial y_2}\right)^k \left(\tfrac{\partial}{\partial t_1} - \tfrac{\partial}{\partial t_2}\right)^n$$

$$[f(x_1,y_1,t_1)g(x_2,y_2,t_2)],$$

and the right hand side is computed in

$$x_1 = x_2 = x,\ y_1 = y_2 = y,\ t_1 = t_2 = t.$$

Now we suppose the solution of (6) as

$$f(x,\xi,t) = \mathrm{e}^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \, \mathrm{e}^{\xi_1} \quad (9)$$

where

$$\xi_i = a_i x + b_i y + c_i t, \quad i = 1,2,3 \tag{10}$$

and a_i, b_i, c_i, δ_i are some constants to be determined later. Substituting equation (9) into equation (8) and equating all coefficients of $\sin(\xi_2)$, $\cos(\xi_2)$, $\sinh(\xi_3)$ and $\cosh(\xi_3)$ to zero, we get the following set of algebraic equations for $a_i, b_i, c_i,$ $\delta_i,\ (i=1,2,3)$

$$3 a_3{}^2 a_1 b_1 + b_3 c_3 + 3 a_1{}^2 b_3 a_3 + c_1 b_1 + a_1{}^3 b_1 + b_3 a_3{}^3 = 0,$$

$$a_1{}^3 b_3 + 3 b_3 a_3{}^2 a_1 + a_3{}^3 b_1 + 3 a_1{}^2 b_1 a_3 + b_3 c_1 + b_1 c_3 = 0,$$

$$b_2 a_2{}^3 + c_1 b_1 + a_1{}^3 b_1 - 3 a_1{}^2 b_2 a_2 - b_2 c_2 - 3 a_2{}^2 a_1 b_1 = 0,$$

$$b_1 c_2 + 3 a_1{}^2 b_1 a_2 + b_2 c_1 - a_2{}^3 b_1 + a_1{}^3 b_2 - 3 b_2 a_2{}^2 a_1 = 0,$$

$$b_3 c_3 - 3 a_3{}^2 b_2 a_2 + b_3 a_3{}^3 - b_2 c_2 - 3 a_2{}^2 b_3 a_3 + b_2 a_2{}^3 = 0,$$

$$3 b_3 a_3{}^2 a_2 - 3 b_2 a_2{}^2 a_3 - a_2{}^3 b_3 + a_3{}^3 b_2 + b_2 c_3 + b_3 c_2 = 0,$$

$$3 \delta_2{}^2 b_3 c_3 + 16 a_1{}^3 b_1 \delta_3 + 4 \delta_1{}^2 a_2{}^3 b_2 - \delta_1{}^2 b_2 c_2 + \\ 4 \delta_2{}^2 a_3{}^3 b_3 + 4 c_1 b_1 \delta_3 = 0.$$

$$\tag{11}$$

Solving the system of equations (11) with the aid of Maple, yields the following cases:

A. Case 1:

$$a_1 = a_3 = b_2 = c_1 = c_3 = 0, c_2 = a_2^3 \tag{12}$$

for some arbitrary real constants a_2, δ_1, δ_2 and δ_3. Substitute (12) into (7) with equation (9), we obtain the solution as

$$f(x,y,t) = \mathrm{e}^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}$$

and

$$u(x,y,t) =$$

$$-2 \frac{\delta_1 \sin(\xi_2) a_2 \left(-b_1 \mathrm{e}^{-\xi_1} + \delta_2 \sinh(\xi_3) b_3 + \delta_3 b_1 \mathrm{e}^{\xi_1}\right)}{\left(\mathrm{e}^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}\right)^2} \tag{13}$$

and

$$v(x,y,t) = \frac{2\delta_1 \cos(\xi_2) a_2{}^2}{\mathrm{e}^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}}$$

$$+ \frac{2\delta_1{}^2 (\sin(\xi_2))^2 a_2{}^2}{\left(\mathrm{e}^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}\right)^2} \tag{14}$$

for

$$\xi_1 = b_1 y, \quad \xi_3 = b_3 y, \quad \xi_2 = a_2 x + a_2{}^3 t.$$

If $\delta_3 > 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) =$$

$$-2 \frac{\delta_1 \sin(\xi_2) a_2 \left(-2 b_1 \sqrt{\delta_3} \sinh(\xi_1 - \theta) + \delta_2 \sinh(\xi_3) b_3\right)}{\left(2 \sqrt{\delta_3} \cosh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

and

$$v(x,y,t) = \frac{2\delta_1 \cos(\xi_2) a_2{}^2}{2 \sqrt{\delta_3} \cosh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)}$$

$$+ \frac{2\delta_1{}^2 (\sin(\xi_2))^2 a_2{}^2}{\left(2 \sqrt{\delta_3} \cosh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

for

$$\beta = \frac{1}{2} \ln(\delta_3).$$

If $\delta_3 < 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) =$$

$$-2 \frac{\delta_1 \sin(\xi_2) a_2 \left(-2 b_1 \sqrt{-\delta_3} \cosh(\xi_1 - \theta) + \delta_2 \sinh(\xi_3) b_3\right)}{\left(2 \sqrt{-\delta_3} \sinh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

and

$$v(x,y,t) = \frac{2\delta_1 \cos(\xi_2) a_2{}^2}{2 \sqrt{-\delta_3} \sinh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)}$$

$$+ \frac{2\delta_1{}^2 (\sin(\xi_2))^2 a_2{}^2}{\left(2 \sqrt{-\delta_3} \sinh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

for

$$\theta = \frac{1}{2} \ln(-\delta_3).$$

B. Case 2:

$$a_1 = -i a_2, a_3 = c_3 = 0, b_1 = i b_2,$$

$$c_1 = -i a_2^3, c_2 = a_2^3, \delta_3 = \frac{\delta_1^2}{4}, \tag{15}$$

for some arbitrary real constants a_2, b_2, δ_1 and δ_2. We make the dependent variable transformation as follows

$$a_2 = i A_2 \quad , \quad b_2 = i B_2 \tag{16}$$

where A_2 and B_2 are real. Substitute (15) into (7) with (9), we obtain the solution as follows

$$f(x,y,t) = \mathrm{e}^{\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{-\xi_1}$$

and

$$u(x,y,t) =$$

$$-2 \frac{-A_2 B_2 \mathrm{e}^{-\xi_1} + \delta_1 \cosh(\xi_2) B_2 A_2 - \delta_3 A_2 B_2 \mathrm{e}^{\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}} \tag{17}$$

$$+2 \frac{S_1 D_1}{\left(\mathrm{e}^{-\xi_1} + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}\right)^2}$$

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where

$$S_1 = \left(-A_2 e^{-\xi_1} - \delta_1 \sinh\left(\xi_2\right) A_2 + \delta_3 A_2 e^{\xi_1}\right)$$

and

$$D_1 = \left(B_2 e^{-\xi_1} - \delta_1 \sinh\left(\xi_2\right) B_2 + \delta_2 \sinh\left(\xi_3\right) b_3 - \delta_3 B_2 e^{\xi_1}\right)$$

also

$$v\left(x,y,t\right) = -2\,\frac{A_2{}^2 e^{-\xi_1} + \delta_1 \cosh(\xi_2) A_2{}^2 + \delta_3 A_2{}^2 e^{\xi_1}}{e^{-\xi_1} + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 e^{\xi_1}}$$
$$+2\,\frac{\left(-A_2 e^{-\xi_1} - \delta_1 \sinh(\xi_2) A_2 + \delta_3 A_2 e^{\xi_1}\right)^2}{\left(e^{-\xi_1} + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 e^{\xi_1}\right)^2} \qquad (18)$$

for

$$\xi_1 = A_2 x - B_2 y - A_2{}^3 t, \quad \delta_3 = \frac{\delta_1^2}{4}$$

$$\xi_2 = A_2 x + B_2 y - A_2{}^3 t, \quad \xi_3 = b_3 y.$$

If $\delta_3 > 0$ then we obtain the exact breather cross-kink solution

$$u\left(x,y,t\right) =$$
$$-2\,\frac{-2 A_2 B_2 \sqrt{\delta 3}\,\cosh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) B_2 A_2}{2\sqrt{\delta 3}\,\cosh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3)}$$
$$+2\,\frac{S_2 D_2}{\left(2\sqrt{\delta 3}\,\cosh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

where

$$S_2 = \left(-2 A_2 \sqrt{\delta_3}\,\sinh(\xi_1 - \theta) - \delta_1 \sinh\left(\xi_2\right) A_2\right)$$

and

$$D_2 =$$
$$\left(2 B_2 \sqrt{\delta_3}\,\sinh(\xi_1 - \theta) - \delta_1 \sinh\left(\xi_2\right) B_2 + \delta_2 \sinh\left(\xi_3\right) b_3\right)$$

also

$$v\left(x,y,t\right) = -2\,\frac{2 A_2{}^2 \sqrt{\delta_3}\,\cosh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) A_2{}^2}{\sqrt{\delta_3}\,\cosh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3)}$$
$$+2\,\frac{\left(-2 A_2 \sqrt{\delta_3}\,\sinh(\xi_1 - \theta) - \delta_1 \sinh(\xi_2) A_2\right)^2}{\left(2\sqrt{\delta_3}\,\cosh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

for

$$\theta = \frac{1}{2}\,\ln(\delta_3).$$

If $\delta_3 < 0$ then we obtain the exact breather cross-kink solution

$$u\left(x,y,t\right) = -2\,\frac{-2 A_2 B_2 \sqrt{-\delta 3}\,\sinh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) B_2 A_2}{2\sqrt{-\delta 3}\,\sinh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3)}$$
$$+2\,\frac{S_3 D_3}{\left(2\sqrt{-\delta 3}\,\sinh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

where

$$S_3 = \left(-2 A_2 \sqrt{-\delta_3}\,\cosh(\xi_1 - \theta) - \delta_1 \sinh\left(\xi_2\right) A_2\right)$$

and

$$D_3 =$$
$$\left(2 B_2 \sqrt{-\delta_3}\,\cosh(\xi_1 - \theta) - \delta_1 \sinh\left(\xi_2\right) B_2 + \delta_2 \sinh\left(\xi_3\right) b_3\right)$$

also

$$v\left(x,y,t\right) = -2\,\frac{2 A_2{}^2 \sqrt{-\delta_3}\,\sinh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) A_2{}^2}{\sqrt{-\delta_3}\,\sinh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3)}$$
$$+2\,\frac{\left(-2 A_2 \sqrt{-\delta_3}\,\cosh(\xi_1 - \theta) - \delta_1 \sinh(\xi_2) A_2\right)^2}{\left(2\sqrt{-\delta_3}\,\sinh(\xi_1 - \theta) + \delta_1 \cosh(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_3).$$

C. Case 3:

$$a_3 = -a_1, b_3 = b_1, c_3 = -c_1, c_1 = -a_1{}^3 + 3\,a_2{}^2 a_1,$$
$$c_2 = -a_2 \left(3\,a_1{}^2 - a_2{}^2\right), \delta_3 = \frac{a_1 \delta_2{}^2 b_1 - a_2 b_2 \delta_1{}^2}{4\,a_1 b_1} \qquad (19)$$

for some arbitrary real constants $a_1, a_2, b_1, b_2, \delta_1$ and δ_2. Substitute equation (19) into (7) with equation (9), we obtain the solution as

$$f\left(x,y,t\right) = e^{-\xi_1} + \delta_1 \cos\left(\xi_2\right) + \delta_2 \cosh\left(\xi_3\right) + \delta_3 e^{\xi_1}$$

and

$$u\left(x,y,t\right) =$$
$$-2\,\frac{a_1 b_1 e^{-\xi_1} - \delta_1 \cos(\xi_2) b_2 a_2 - \delta_2 \cosh(\xi_3) b_1 a_1 + \delta_3 a_1 b_1 e^{\xi_1}}{e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 e^{\xi_1}} \qquad (20)$$
$$+\frac{2 S_4 D_4}{\left(e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 e^{\xi_1}\right)^2}$$

where

$$S_4 = \left(-a_1 e^{-\xi_1} + \delta_1 \sin\left(\xi_2\right) a_2 - \delta_2 \sinh\left(\xi_3\right) a_1 + \delta_3 a_1 e^{\xi_1}\right)$$

and

$$D_4 = \left(-b_1 e^{-\xi_1} + \delta_1 \sin\left(\xi_2\right) b_2 + \delta_2 \sinh\left(\xi_3\right) b_1 + \delta_3 b_1 e^{\xi_1}\right)$$

also

$$v\left(x,y,t\right) =$$
$$-2\,\frac{a_1{}^2 e^{-\xi_1} - \delta_1 \cos(\xi_2) a_2{}^2 + \delta_2 \cosh(\xi_3) a_1{}^2 + \delta_3 a_1{}^2 e^{\xi_1}}{e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 e^{\xi_1}} \qquad (21)$$
$$+2\,\frac{\left(-a_1 e^{-\xi_1} + \delta_1 \sin(\xi_2) a_2 - \delta_2 \sinh(\xi_3) a_1 + \delta_3 a_1 e^{\xi_1}\right)^2}{\left(e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 e^{\xi_1}\right)^2}$$

for

$$\xi_1 = a_1 x + b_1 y + \left(-a_1{}^3 + 3\,a_2{}^2 a_1\right) t$$
$$\xi_2 = -a_2 x - b_2 y + a_2 \left(3\,a_1{}^2 - a_2{}^2\right) t$$
$$\xi_3 = -a_1 x + b_1 y + \left(a_1{}^3 - 3\,a_2{}^2 a_1\right) t$$

and

$$\delta_3 = \frac{a_1 \delta_2{}^2 b_1 - a_2 b_2 \delta_1{}^2}{4\,a_1 b_1}. \qquad (22)$$

If $\delta_3 > 0$ then we obtain the exact breather cross-kink solution

$$u\left(x,y,t\right) =$$
$$-2\,\frac{2 a_1 b_1 \sqrt{\delta_3}\,\cosh(\xi_1 - \theta) - \delta_1 \cos(\xi_2) b_2 a_2 - \delta_2 \cosh(\xi_3) b_1 a_1}{2\sqrt{\delta 3}\,\cosh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)} \qquad (23)$$
$$+\frac{2 S_5 D_5}{\left(2\sqrt{\delta_3}\,\cosh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)\right)^2}$$

where

$$S_5 = \left(-2 a_1 \sqrt{\delta_3}\,\sinh(\xi_1 - \theta) + \delta_1 \sin\left(\xi_2\right) a_2 - \delta_2 \sinh\left(\xi_3\right) a_1\right)$$

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and

$$D_5 = \left(-2b_1\sqrt{\delta_3}\sinh(\xi_1-\theta) + \delta_1\sin(\xi_2)b_2 + \delta_2\sinh(\xi_3)b_1\right)$$

also

$$v(x,y,t) =$$
$$-2\frac{2a_1{}^2\sqrt{\delta_3}\cosh(\xi_1-\theta)-\delta_1\cos(\xi_2)a_2{}^2+\delta_2\cosh(\xi_3)a_1{}^2}{2\sqrt{\delta_3}\cosh(\xi_3-\theta)+\delta_1\cos(\xi_2)+\delta_2\cosh(\xi_3)}$$

$$+2\frac{\left(-2a_1\sqrt{\delta_3}\sinh(\xi_1-\theta)+\delta_1\sin(\xi_2)a_2-\delta_2\sinh(\xi_3)a_1\right)^2}{\left(2\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_1\cos(\xi_2)+\delta_2\cosh(\xi_3)\right)^2}$$

for

$$\theta = \frac{1}{2}\ln(\delta_3).$$

If $\delta_3 < 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) =$$
$$-2\frac{2a_1b_1\sqrt{-\delta_3}\sinh(\xi_1-\theta)-\delta_1\cos(\xi_2)b_2a_2-\delta_2\cosh(\xi_3)b_1a_1}{2\sqrt{-\delta 3}\sinh(\xi_1-\theta)+\delta_1\cos(\xi_2)+\delta_2\cosh(\xi_3)}$$

$$+\frac{2S_6D_6}{\left(2\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_1\cos(\xi_2)+\delta_2\cosh(\xi_3)\right)^2} \tag{24}$$

where

$$S_6 = \left(-2a_1\sqrt{-\delta_3}\cosh(\xi_1-\theta) + \delta_1\sin(\xi_2)a_2 - \delta_2\sinh(\xi_3)a_1\right)$$

and

$$D_6 = \left(-2b_1\sqrt{-\delta_3}\cosh(\xi_1-\theta) + \delta_1\sin(\xi_2)b_2 + \delta_2\sinh(\xi_3)b_1\right)$$

also

$$v(x,y,t) =$$

$$-2\frac{2a_1{}^2\sqrt{-\delta_3}\sinh(\xi_1-\theta)-\delta_1\cos(\xi_2)a_2{}^2+\delta_2\cosh(\xi_3)a_1{}^2}{2\sqrt{-\delta_3}\sinh(\xi_3-\theta)+\delta_1\cos(\xi_2)+\delta_2\cosh(\xi_3)}$$

$$+2\frac{\left(-2a_1\sqrt{-\delta_3}\cosh(\xi_1-\theta)+\delta_1\sin(\xi_2)a_2-\delta_2\sinh(\xi_3)a_1\right)^2}{\left(2\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_1\cos(\xi_2)+\delta_2\cosh(\xi_3)\right)^2}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_3).$$

D. Case 4:

$$a_1 = a_3, a_2 = ia_3, b_2 = 0, c_1 = -4a_3{}^3,$$
$$c_2 = -4ia_3{}^3, c_3 = -4a_3{}^3 \tag{25}$$

for some arbitrary real constants $b_1, a_3, \delta_1, \delta_2$ and δ_3. Substitute (25) into (7) with (9), we obtain the solution as follows

$$f(x,y,t) = e^{-\xi_1} + \delta_1\cosh(\xi_2) + \delta_2\cosh(\xi_3) + \delta_3e^{\xi_1}$$

and

$$u(x,y,t) =$$
$$-2\frac{a_3b_1e^{-\xi_1}+\delta_2\cosh(\xi_3)b_3a_3+\delta_3a_3b_1e^{\xi_1}}{e^{-\xi_1}+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)+\delta_3e^{\xi_1}} \tag{26}$$

$$+\frac{2S_7D_7}{e^{-\xi_1}+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)+\delta_3e^{\xi_1}}$$

where

$$S_7 = \left(-a_3e^{-\xi_1} - \delta_1\sinh(\xi_2)a_3 - \delta_2\sinh(\xi_3)a_3 + \delta_3a_3e^{\xi_1}\right)$$

and

$$D_7 = \left(-b_1e^{-\xi_1} - \delta_2\sinh(\xi_3)b_3 + \delta_3b_1e^{\xi_1}\right)$$

also

$$v(x,y,t) =$$

$$-2\frac{a_3{}^2e^{-\xi_1}+\delta_1\cosh(\xi_2)a_3{}^2+\delta_2\cosh(\xi_3)a_3{}^2+\delta_3a_3{}^2e^{\xi_1}}{e^{-\xi_1}+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)+\delta_3e^{\xi_1}} \tag{27}$$

$$+2\frac{\left(-a_3e^{-\xi_1}-\delta_1\sinh(\xi_2)a_3-\delta_2\sinh(\xi_3)a_3+\delta_3a_3e^{\xi_1}\right)^2}{e^{-\xi_1}+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)+\delta_3e^{\xi_1}}$$

for

$$\xi_1 = a_3x + b_1y - 4a_3{}^3t$$

$$\xi_2 = a_3x - 4a_3{}^3t$$

$$\xi_3 = -a_3x - b_3y + 4a_3{}^3t.$$

If $\delta_3 > 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) =$$

$$-2\frac{2a_3b_1\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_2\cosh(\xi_3)b_3a_3}{2\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)}$$

$$+\frac{2S_8D_8}{2\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)}$$

where

$$S_8 = \left(-2a_3\sqrt{\delta_3}\sinh(\xi_1-\theta) - \delta_1\sinh(\xi_2)a_3 - \delta_2\sinh(\xi_3)a_3\right)$$

and

$$D_8 = \left(-2b_1\sqrt{\delta_3}\sinh(\xi_1-\theta) - \delta_2\sinh(\xi_3)b_3\right)$$

also

$$v(x,y,t) =$$

$$-2\frac{2a_3{}^2\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_1\cosh(\xi_2)a_3{}^2+\delta_2\cosh(\xi_3)a_3{}^2}{2\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)}$$

$$+2\frac{\left(-2a_3\sqrt{\delta_3}\sinh(\xi_1-\theta)-\delta_1\sinh(\xi_2)a_3-\delta_2\sinh(\xi_3)a_3\right)^2}{2\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)}$$

for

$$\theta = \frac{1}{2}\ln(\delta_3).$$

If $\delta_3 < 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) =$$

$$-2\frac{2a_3b_1\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_2\cosh(\xi_3)b_3a_3}{2\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)}$$

$$+\frac{2S_9D_9}{2\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)}$$

where

$$S_9 = \left(-2a_3\sqrt{-\delta_3}\cosh(\xi_1-\theta) - \delta_1\sinh(\xi_2)a_3 - \delta_2\sinh(\xi_3)a_3\right)$$

and

$$D_9 = \left(-2b_1\sqrt{-\delta_3}\cosh(\xi_1-\theta) - \delta_2\sinh(\xi_3)b_3\right)$$

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also

$$v\left(x,y,t\right)=$$

$$-2\,\frac{2a_3{}^2\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_1\cosh(\xi_2)a_3{}^2+\delta_2\cosh(\xi_3)a_3{}^2}{2\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)}$$

$$+2\,\frac{\left(-2a_3\sqrt{-\delta_3}\cosh(\xi_1-\theta)-\delta_1\sinh(\xi_2)a_3-\delta_2\sinh(\xi_3)a_3\right)^2}{2\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_1\cosh(\xi_2)+\delta_2\cosh(\xi_3)}$$

for

$$\theta=\frac{1}{2}\,\ln(-\delta_3).$$

IV. Conclusions

In this paper, using the three-wave solution method we obtained some explicit formulas of solutions for the Benjamin-Bona-Mahony-Burgers equations. Three-wave solution method with the aid of a symbolic computation software like Maple or Mathematica is an easy and straightforward method which can be apply to other nonlinear partial differential equations. It must be noted that, all obtained solutions have checked in the Benjamin-Bona-Mahony-Burgers equations. All solutions satisfy in the equations.

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