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Completion Number of a Graph

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***Abstract*—In this paper a new concept of partial complement of a graph G is introduced and using the same a new graph parameter, called completion number of a graph G, denoted by $\varsigma(G)$ is defined. Some basic properties of graph parameter, completion number, are studied and upperbounds for completion number of classes of graphs are obtained , the paper includes the characterization also.**

***Index Terms*—Completion Number,Maximum Independent subset, Partial complements, Partial self complementary.**

I. Introduction

The graph operation of complementation in a graph G keeps the vertex set $V(G)$ unchanged and acts on the edge set $E(G)$. This operation can be viewed in many different directions and each such attempt results in introducing new complements. It is always quite interesting to investigate when the original graphs appears back after applying the operation of complementation. Such graphs are called Self complementary graphs. A lot of literature is available on self complementary graphs [1],[3],[4]and the problem of finding algorithm to test whether a given graph is self complementary is as hard as the graph isomorphism problem[2].

With these problems in mind, this paper introduce a new concept of partial complement of a graph and using it a new graph parameter called completion number of a graph is defined in section II. An upper bound for completion number of some standard classes of graphs are discussed in section III. Some results on completion number of graphs are shown in section V. Finally, conclusion are drawn in section V.

II. Definitions

Let $G = (V, E)$ be a graph and $V(G)$ for two vertices u and v in V_1 remove the edge between them if it exists and add the edge between them if it does not exist. The graph thus obtained, denoted by $CV_1(G)$, is called a partial complement of G with respect to subset V_1 of V. G is said to be partial self complementary (P.S.C) if there exists a subsets of V_1 of V such that $CV_1(G) = G$

We consider a sequence of partial complements,denoted by $\{CV_i(G_{i-1})\}$ where V_i is a maximum independent set of $V(G_{i-1})$ with $G_0 = G$ and $CV_i(G_{i-1} = G_i)$, for $i \geq 1$. The above sequence terminates at the k^{th} step when $G_k = K_p$. This k is called the completion number of G. The formal defination is given below.

Definition .1. *Let G be a (p, q) graph with vertex set $V(G)$ and edge set $E(G)$. Let $V_i(G_{i-1}) \subseteq V(G_{i-1})$ where $G_0 = G$ and $G_i = Cv_i(G_{i-1})$, the partial complement of G_{i-1} with respect to maximum independent subset $v_i(G_{i-1})$ of $V(G_{i-1})$ for $i \geq 1$. The least positive integer k for which the sequence $\{Cv_i(G_{i-1})\}_{i\geq 1}$ terminates with a complete graph is called the completion number of G, denoted by $C(G)$. In the first of the following two examples the sequence $\{CV_i(G_{i-1})\}$ with $V_i(G_{i-1})$ for the graph C_5 are shown and hence $C(C_5)$ is concluded to be equal to 5. In the second example it is shown that completion number of petersen graph is five by exhibiting $V_i(G_{i-1})$ at each step.*

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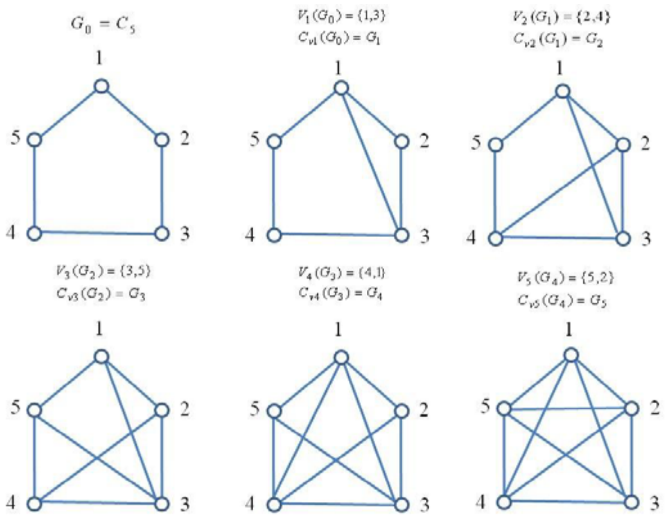
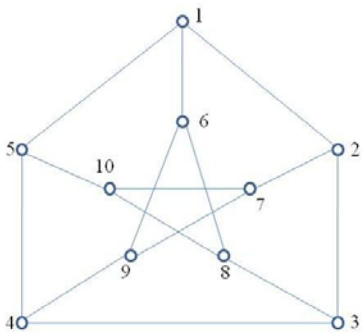


Fig. 1. Thus $e(C_5) \leq 5$ and as all independent sets induce K_2,$e(C_5) = 5$



$V_1(G_0) = \{1,3,9,10\}$
$V_2(G_1) = \{2,4,6,10\}$
$V_3(G_2) = \{3,5,6,7\}$
$V_4(G_3) = \{4,1,7,8\}$
$V_5(G_4) = \{5,2,8,9\}$

Fig. 2. Thus $e(PetersenGraph) \leq 5$ and as choice of every independent set is optimal e(Petersen Graph)=5

Note 1. *From the above examples it follows that two non isomorphic graphs can have the same completion number.*

This paper has two parts. In the first part upperbound for completion number of some standard classes of graphs like cycles, paths, wheels, bipartite graphs, and unicyclic graphs of the form $C_n \cdot K_2$ are obtained and in the second part we derive some results pertaining to completion number of a graph are obtained. In the same section a characterization theorem for graphs with completion number k for every $k \geq 0$ is derived. For the notation and terminology undefined, we refer to [5].

III. Upper Bound for completion number of some standard classes of Graphs

A. Bipartite Graphs

Let G be a (p, q) bipartite graph with bipartition $\{V_1, V_2\}$ with $|v_1| = P_1$ and $|v_2| = P_2$. If G is connected then completion number of G,$C(G)$, satisfies the relation $C(G) \leq 2 + P_1P_2 - q$. Thus it follows that,

Completion number of the even cycle $C_{2n} \leq 2 + n^2 - 2n = n^2 - 2n + 2$.

Completion number of the path $P_{2n} \leq 2 + n^2 - (2n - 1) = n^2 - 2n + 3$.

Completion number of $P_{2n+1} \leq 2+n(n+1)-2n = n^2-n+2$.

Completion number of $C_{2n} \cdot K_2 \leq 2 + n(n+1) - (2n+1) = n^2 - n + 1$.

The above upper bounds are not sharp. The sharper bounds in the case of paths $P_{4n+2}, P_{4n}, P_{4n+1}$ and P_{4n+3} are given below.

Consider P_{4n+2}. It is a bipartite graph in which each partite set has $2n + 1$ elements. Let $\{v_1, v_2, \ldots, v_{4n+1}\}$ be the vertex set of P_{4n+2} and the partite sets V_1 and V_2 be given by $V_1 = \{v_1, v_3, \ldots, v_{4n+1}\}$,
$V_2 = \{v_2, v_4, \ldots, v_{4n+2}\}$
$G_0 = P_{4n+2}$
$V_1(G_0) = \{V_1, V_3, V_{2n+1}, V_{2n+4}, V_{2n+6}, \ldots, V_{4n+2}\}$, $Cv_1(G_0) = G_1$
$V_2(G_1) = \{V_1, V_3, V_{2n+2}, V_{2n+3}, V_{2n+5}, \ldots, V_{4n+1}\}$, $Cv_2(G_1) = G_2$.
In V_1 n independent sets of size three and in V_2,$(n - 1)$ independent sets of size three are possible. Thus totally there are $(2n - 1)$ independent sets of size three are possible. Thus $Cv_i(G_{i-1}) = G_i$,$3 \leq i \leq 2n + 1$.
At the end G_{2n+1} has two induced complete graphs,K_{2n+1} and $(2n - 1)$ induced triangles. $V_i(G_{i-1})$ for $2n + 2 \leq i \leq 4n^2 - 2n + 4$ are independent sets with two vertices and $Cv_i(G_{i-1}) = G_i$, $2n + 2 \leq i \leq 4n^2 - 2n + 4$.
$G_{4n^2-2n+4} = K_{4n+2}$. Hence $C(P_{4n+2}) \leq 4n^2 - 2n + 4$
similarly, $C(P_{4n}) \leq 4n^2 - 4n + 1$
$C(P_{4n+1}) \leq 4n^2 - 4n + 5$
$C(P_{4n+3}) \leq 4n^2 + 4$

B. Bound for Completion number of C_{2n+1}

The first and second independent sets have n vertices in each of them. Afterwards there is a sequence of maximal independent sets with three vertices. Depending upon the nature of these sets the problem is divided into six parts.

C. Bound for Completion number of C_{12n+1}

Consider the cycle C_{12n+1} with vertex set$\{v_1, v_2, \ldots, v_{12n+1}$, $G_0 = C_{12n+1}$,
$V_1(G_0) = \{V_1, V_3, \ldots, V_{12n-1}\}$,$Cv_1(G_0) = G_1$
$V_2(G_1) = \{V_2, V_4, \ldots, V_{12n}\}$,$Cv_2(G_1) = G_2$.
The graphs G_2 has two disjoint induced complete subgraphs on $6n$ vertices and a vertex not on either of them. Then $V_i(G_{i-1})$ for $3 \leq i \leq 6n$ are independent sets with three vertices one of which is the vertex which is not in any of the two induced complete subgraphs K_{6n} and $Cv_i(G_{i-1}) = G_i$, $3 \leq i \leq 6n$. At the end G_{6n} has two induced complete subgraphs K_{6n} and $(6n - 2)$ induced triangle K_3. Then $V_i(G_{i-1})$ for $6n + 1$ is $36n^2 - 12n + 5$ are independent sets with two vertices and $Cv_i(G_{i-1}) = G_i$, $6n + 1$ is $36n^2 - 12n = 5$ Thus $G_{36n^2-12n+5} = K_{12n+1}$ and $C(C_{12n+1}) \leq 36n^2 - 12n + 5$ similarly
$C(C_{12n+3}) \leq 36n^2 + 2$, $C(C_{12n+5}) \leq 36n^2 + 12n + 5$
$C(C_{12n+7}) \leq 36n^2 + 24n + 8$
$C(C_{12n+9}) \leq 36n^2 + 36n + 11$
$C(C_{12n+11}) \leq 36n^2 + 48n + 20$

D. Bound for Completion number of UniCyclic Graphs of the form $c_{2n+1} \cdot K_2$

Consider $C_{2n+1} \cdot K_2$ with vertex set $\{v_1, v_2, v_3, \ldots, v_{2n+2}\}$. Let free end of K_2 be V_1 other end be V_2 and $v_2, v_3, \ldots, v_{12n+2}$ be the vertices of C_{2n+1}. $G_0 = C_{12n+1} \cdot K_2$
$v_1(G_0) = \{v_1, v_3, v_5, \ldots, v_{2n+1}$, $Cv_1(G_0) = G_1$
$v_2(G_1) = \{v_2, v_4, v_6, \ldots, v_{2n+2}$, $Cv_2(G_1) = G_2$
G_2 has two disjoint induced complete subgraphs. Then $V_i(G_{i-1})$ are independent sets with two vertices and $Cv_i(G_{i-1}) = G_i$, $3 \leq i \leq n^2 + 1$.
Thus $G_{n^2+1} = K_{2n+1}$ and thus $C(C_{2n+1} \cdot K_2) \leq n^2 + 1$.

E. The completion Number of a Wheel

Completion number of $W_{1,n} = K_1 + C_n$ is same as completion number of C_n. The completion number of generalized wheel $W_{m,n} = \overline{K_m} + C_n$ is given by $C(W_{m,n}) = C(C_n) + 1$.

IV. Some Results on Completion number of a Graph

The completion number of a (p, q) graph $C(G)$ satisfies, $0 \leq C(G) \leq C_2^p - q$. $C(G) = 0$ if and only if $G = K_p$ and $C(G) = C_2^p - q$ if and only if G is the graph obtained by removing q edges from K_p such that no three edges form a triangle. In the following theorem graphs with completion number one are characterized .

Theorem 1. *Completion number of a (p, q) graph G is one if and only if either $G = \overline{K_m}$ for some integer $m \geq 1$ or $G = K_n + K_m$ for some integers m and n with $n \geq 1$ and $m \geq 1$.*

Proof: The necessary part is trivial. To prove the sufficient part,consider a graph with completion number one. Consider the maximal independent set V_1 of $V(G)$. If $V_1 = V(G)$ then the graph is $\overline{k_n}$ for some integer $n \geq 1$. If V_1 is a proper subset of $V(G)$, as the completion number is one, every vertex of the maximal independent set is adjacent to all the vertices of $V(G) - V_1$ and every two vertices of $V(G) - V_1$ are adjacent. Hence $G = K_n + K_m$ where $n = |V_1(G)|$ and $m = p - n$. ■

Let G be a graph with $C(G) = k$. Then $C(G + K_n) = k$ and $C(G + K_n) = k + 1$ for any integer $n > 1$
Given any integer $K \geq 0$, there exists graph G with $C(G) = k$
Let G_1 be a graph with completion number K_1 and G_2 be a graph with completion number K_2. Then completion number of $G_1 + G_2$ is $k_1 + k_2$.

Note 2. *In General if G_i is a graph with completion number k_i, $1 \leq i \leq n$ $C \sum_{i-1}^{n} G_i = \sum_{i-1}^{n} C(G_i)$*

Note 3. *The converse of the above is not true.i.e, if G is a graph with completion number k then G need not be sum of i graphs $G_1, G_2, \ldots, G_i$ with $C(G_j) = K_j$, $1 \leq j \leq i$ and $k = k_1 + k_2, \ldots, k_i$ for any $i > 1$ as illustrated in the following example.*

Example: Consider graph $G = C_4$ $C(G) = 4$ But $G = \sum_{j=1}^{i} G_j$ with $C(G_j) = K_j$ and $\sum_{j=1}^{i} k_j = 4$ for any i, $2 \leq i \leq 4$. The following results characterizes a graph with completion number k that can be written as sum of k graphs each with completion number one.

Theorem 2. *Let G be a graph with completion number k. Then $G = G_1 + G_2, \ldots, G_k$ with $C(G_i) = 1$, $1 \leq i \leq k$ if and only if either G is the complete k partite graph. $G = K_{p_1,p_2,\ldots,p_k}$ where $P_i = |V(G_i)|$ or $G = K_{m_1,m_2,\ldots,m_k} + K_{n_1,n_2,\ldots,n_k}$ where $G_i = K_{m_i} + K_{n_i}$ with $m_i > 1$ and $n_i \geq$ 1,$1 \leq i \leq k$.*

Proof: If $G = K_{p_1,p_2,\ldots,p_k}$ or $G = K_{m_1,m_2,\ldots,m_k} + K_{n_1,n_2,\ldots,n_k}$ then it is easy to see that G is of completion number k. Conversely if G is of completion number k and $G = G_1 + G_2 +, \ldots, G_k$ then each G_j is of completion number one,$i \leq j \leq k$. Then by theorem (4.3.1)either G_j is K_m for some $m > 1$ or $G_j = K_m + K_n$ for some m and n with $m > 1$ and $n \geq 1$.Thus $G = \sum_{j=1}^{k} G_j$ is either $K_{p_1,p_2,\ldots,p_k}$ or $K_{m_1,m_2,\ldots,m_k} + K_{n_1,n_2,\ldots,n_k}$. Thus if $C(G) = K$ and $G = \sum_{i=1}^{k} G_j$ then either G is a complete K-partite graph or sum of a complete K-partite graph and a complete graph. ■

In the following theorem graphs with completion number k are characterized.

Theorem 3. *Let G be a (p, q) graph. Then $C(G) = k$ if and only if G is one of the following types.*

1) *G is a complete k-partite graph $K_{p_1,p_2,\ldots,p_k}$*
2) *G is sum of a complete k-partite graph and a complete graph.i.e,$G = K_{p_1,p_2,\ldots,p_k} + K_{p-\sum_{i=1}^{k} p_i}$*
3) *G is the graph obtained from the complete graph K_p as follows.*

Let $V_1, V_2, \ldots, V_k$ be the k maximal independent sets of G in succession and $v \in V_j$. Let v also belong to $V_{i_1} \cap V_{i_2}, \ldots, _{i_t}$, $i_1, i_2, \ldots, i_t \leq k$ and $b_{j_1}, b_{j_2}, \ldots, b_{ja}$ where $a = P_{i_1} + P_{i_2} +, \ldots, P_{i_t} - t$ with $\{V_{ij}\} = P_{ij}$, $1 \leq j \leq t$ be the vertices of $V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_t}$ other than v. Then remove the star $K_{1,a}$ from K_p at v joining v to each of $b_{j_1}, b_{j_2}, \ldots, b_{j_a}$. continue the process for every vertex in $V_1 \cup V_2 \cup, \ldots, \cup V_k$

Proof: In the first two cases it is easy to observe that the assertion of the theorem is true. The proof of the third case is given below. Let G be a graph with completion number k. $V_1, V_2, \ldots, V_k$ be the maximal independent subsets of $V(G)$ in successsion. Let $v \in V_i$, $1 \leq i \leq k$ and v also belong to $V_{i_1}, \cap, V_{i_2} \cap, \ldots, \cap V_{i_t}$. Let $b_{j_1}, b_{j_2}, \ldots, b_{j_a}$ be the vertices in $V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_t}$ other than v. Then in the process of taking succesive partial complements with respect to $V_1, V_2, \ldots, V_k$, the edges added with the vertex v as the common vertex are the edges joining v to each of $b_{j_1}, b_{j_2}, \ldots, b_{j_a}$ i.e, the star with v has the root and $b_{j_1}, b_{j_2}, \ldots, b_{j_a}$ as pendent vertices. The above is true for every vertex in $V_1 \cup V_2 \cup, \ldots, V_k$. Thus G can be obtained by removing the above said stars at v from K_p. Conversely Let G be the graph obtained by removing stars at vertices v of K_p. Let the star removed at V_i,$1 \leq i \leq p$ be the one with V_i as the root and $v_{i_1}, v_{i_2}, \ldots, v_{i_{n_i}}$ as pendent vertices. Further $v_{i_1}, v_{i_2}, \ldots, v_{i_{t_i}}$ be the t_i vertices which are mutually non adjacent with $t_i \leq n_i$. Then form an independent set $V_i = \{v_i, v_{i_1}, v_{i_2}, \ldots, v_{i_{t_i}}\}$. Similarly if $\{V_{i_1}, V_{i_2}, \ldots, V_{i_{k_i}}\}$ independent sets of $V(G)$. If an independent set V_i has k_i vertices the same set will be repeated k_j times,once each as the independent set corresponding to each pendent vertex. If there are k distinct maximal independent then completion number is k, if the distinct independent sets are arranged in increasing order of size and partial complements are taken in succession then resulting graph is K_p ■

V. Conclusion

An upper bound for completion number of graphs from some standard classes are obtained. Some results pertaining to completion number are obtained. A characterization of graphs with completion number k is given.

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