

# Ruin probability for a Markovian risk model with two-type claims

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**Abstract**—In this paper, a Markovian risk model with two-type claims is considered. In such a risk model, the occurrences of the two type claims are described by two point processes  $\{N_i(t), t \geq 0\}, i = 1, 2$ , where  $\{N_i(t), t \geq 0\}$  is the number of jumps during the interval  $(0, t]$  for the Markov jump process  $\{X_i(t), t \geq 0\}$ . The ruin probability  $\Psi(u)$  of a company facing such a risk model is mainly discussed. An integral equation satisfied by the ruin probability  $\Psi(u)$  is obtained and the bounds for the convergence rate of the ruin probability  $\Psi(u)$  are given by using key-renewal theorem.

**Keywords**—Risk model; Ruin probability; Markov jump process; Integral equation

## I. INTRODUCTION

The classical risk model has been extensively studied since the work of Cramer[3], and has been generalized to various Markovian risk model which have been studied extensively [1, 2, 4, 7]. Recently, many authors have studied continuous-time risk models involving two classes of claims. Yuen et al. [9] consider the non-ruin probability for a correlated risk process involving two dependent classes of insurance risks, with exponential claims, which can be transformed into a surplus process with two independent classes of insurance risks, for which one claim number process is Poisson and the other is a renewal process with Erlang(2) claim inter-arrival times. Li and Garrido [5] consider a risk process with two classes of independent risks, namely the compound Poisson process and the renewal process with generalized Erlang(2) inter-arrivals times. A further extension was given by Li and Lu [6]. They derive a system of integro-differential equations for the Gerber-Shiu expected discounted penalty functions, when the ruin is caused by a claim belonging either to the first or to the second class and obtained explicit results when the claim sizes are exponentially distributed. Zhang et al. [8] extended the model of Li and Lu [6], by considering the claim number process of the second class to be a renewal process with generalized Erlang(n) inter-arrival times.

In this paper, we mainly consider a Markovian risk model with two-type claims. Integral equation for the ruin probability is found and the bounds for the convergence rate of the ruin probability are given.

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space containing all objects defined in the following,  $(\mathcal{S}_i, \mathcal{B}_i)(i = 1, 2)$  be two measurable spaces where  $\mathcal{S}_i$  is a subset of real line  $R$  and  $\mathcal{B}_i$

is a Borel  $\sigma$ -algebra on  $\mathcal{S}_i$ . Consider the risk model

$$U(t) = u + ct - \sum_{k=1}^{N_1(t)} Y_k - \sum_{k=1}^{N_2(t)} Z_k, \quad (1)$$

where  $u = U(0) \geq 0$  is the initial surplus,  $c > 0$  is the premium income rate,  $\{Y_k, k \geq 1\}$  are i.i.d nonnegative random sequence with common distribution function  $F_1$  and mean value  $\mu_1$ ;  $\{Z_k, k \geq 1\}$  are also i.i.d nonnegative random sequence but with common distribution function  $F_2$  and mean value  $\mu_2$ ,  $Y = \{Y_k, k \geq 1\}$  and  $Z = \{Z_k, k \geq 1\}$  denote the two-type claim processes;  $N_i(t)$  is the number of jumps during the interval  $(0, t]$  for the Markov jump process  $X_i = \{X_i(t), t \geq 0\}$  on space  $\mathcal{S}_i$  with bounded intensity function  $\lambda_i(x)$  and jumping measure  $Q_i(x, B)$ . Throughout this paper, we always assume that  $X_i$  is stationary ergodic with initial stationary distribution  $q_i(\cdot)$ , i.e.,  $\int_B \lambda_i(x) q_i(dx) = \int_{\mathcal{S}_i} \lambda_i(x) Q_i(x, B) q_i(dx)$  and  $X_1, X_2, Y, Z$  are mutually independent.

Let

$$\begin{aligned} T &= \inf\{t \geq 0 : U(t) < 0\}, (\inf \Phi = \infty) \\ \Psi(u) &= P(T < \infty | U(0) = u), \\ R(u) &= 1 - \Psi(u), \\ \Psi_x(u) &= P(T < \infty | U(0) = u, X_1(0) = x), \\ R_x(u) &= 1 - \Psi_x(u), x \in \mathcal{S}_1, \\ \tilde{\Psi}_y(u) &= P(T < \infty | U(0) = u, X_2(0) = y), \\ \tilde{R}_y(u) &= 1 - \tilde{\Psi}_y(u), y \in \mathcal{S}_2, \\ \Psi_{xy}(u) &= P(T < \infty | U(0) = u, X_1(0) = x, X_2(0) = y), \\ R_{xy}(u) &= 1 - \Psi_{xy}(u), x \in \mathcal{S}_1, y \in \mathcal{S}_2. \end{aligned}$$

We call  $T$  the time of ruin,  $\Psi(u)$  the ruin probability,  $R(u)$  the survival probability. Obviously, we have

$$\begin{aligned} \Psi(u) &= \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \Psi_{xy}(u) q_2(dy) q_1(dx) \\ &= \int_{\mathcal{S}_1} \Psi_x(u) q_1(dx) \\ &= \int_{\mathcal{S}_2} \tilde{\Psi}_y(u) q_2(dy). \end{aligned}$$

Let

$$\rho = \frac{c - \mu_1 \int_{\mathcal{S}_1} \lambda_1(x) q_1(dx) - \mu_2 \int_{\mathcal{S}_2} \lambda_2(x) q_2(dx)}{\mu_1 \int_{\mathcal{S}_1} \lambda_1(x) q_1(dx) + \mu_2 \int_{\mathcal{S}_2} \lambda_2(x) q_2(dx)}$$

be the relative security loading. Throughout the paper, we always assume that  $\rho > 0$ .

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II. INTEGRAL EQUATION OF RUIN PROBABILITY

**Lemma 2.1** Under the assumption that  $\rho > 0$ , we have  $\lim_{u \rightarrow \infty} \Psi(u) = 0$ .

**Proof** Put  $Y(t) = U(t) - u$ , since  $\lim_{t \rightarrow \infty} \frac{N_i(t)}{t} = \int_{S_i} \lambda_i(x)q_i(dx)$ ,  $i = 1, 2$ , then

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{Y_t}{t} &= \lim_{t \rightarrow \infty} \left( c - \frac{1}{t} \sum_{k=1}^{N_1(t)} Y_k - \frac{1}{t} \sum_{k=1}^{N_2(t)} Z_k \right) \\ &= c - \lim_{t \rightarrow \infty} \frac{1}{N_1(t)} \sum_{k=1}^{N_1(t)} Y_k \cdot \frac{N_1(t)}{t} \\ &\quad - \lim_{t \rightarrow \infty} \frac{1}{N_2(t)} \sum_{k=1}^{N_2(t)} Z_k \cdot \frac{N_2(t)}{t} \\ &= c - \mu_1 \int_{S_1} \lambda_1(x)q_1(dx) - \mu_2 \int_{S_2} \lambda_2(x)q_2(dx). \end{aligned}$$

By the theory of Markov process and the assumption that  $\lambda_i(x)$  is bounded, it is clear that  $X_i$  has only finite jumps during the interval  $(0, \tau]$ , thus  $\inf_{t \geq 0} Y_t$  is finite with probability one and thus

$$\lim_{u \rightarrow \infty} \Psi(u) = \lim_{u \rightarrow \infty} P(\inf_{t \geq 0} (u + Y(t)) < 0) = 0,$$

then Lemma 2.1 is proved.

**Corollary 2.1** For  $\Psi_x(u)$ ,  $x \in S_1$ ,  $\tilde{\Psi}_y(u)$ ,  $y \in S_2$  we have

$$\begin{aligned} \lim_{u \rightarrow \infty} \Psi_x(u) &= 0, & q_1(\cdot) - a.e. & \quad x \in S_1; \\ \lim_{u \rightarrow \infty} \tilde{\Psi}_y(u) &= 0, & q_2(\cdot) - a.e. & \quad y \in S_2. \end{aligned}$$

**Proof** Since  $\Psi(u) = \int_{S_1} \Psi_x(u)q_1(dx) + \int_{S_2} \tilde{\Psi}_y(u)q_2(dy)$ , by the dominated convergence theorem, we can get

$$\begin{aligned} 0 &= \lim_{u \rightarrow \infty} \Psi(u) = \int_{S_1} \lim_{u \rightarrow \infty} \Psi_x(u)q_1(dx) \\ &= \int_{S_2} \lim_{u \rightarrow \infty} \tilde{\Psi}_y(u)q_2(dy). \end{aligned}$$

Obviously, it is that  $\lim_{u \rightarrow \infty} \tilde{\Psi}_x(u) > 0$ ,  $q_1(\cdot) - a.e.$   $x \in S_1$  and  $\lim_{u \rightarrow \infty} \tilde{\Psi}_y(u) > 0$ ,  $q_2(\cdot) - a.e.$   $y \in S_2$ , thus

$$\begin{aligned} \lim_{u \rightarrow \infty} \Psi_x(u) &= 0, & q_1(\cdot) - a.e. & \quad x \in S_1; \\ \lim_{u \rightarrow \infty} \tilde{\Psi}_y(u) &= 0, & q_2(\cdot) - a.e. & \quad y \in S_2, \end{aligned}$$

the proof of Corollary 2.1 is completed.

In the following, by using the backward differential technique, we give an integral equation satisfied by the ruin probability  $\Psi(u)$ .

**Theorem 2.1** If the relative security loading  $\rho > 0$ , then

$$\begin{aligned} \Psi(0) &= \frac{1}{c} \left( \mu_1 \int_{S_1} \lambda_1(x)q_1(dx) + \mu_2 \int_{S_2} \lambda_2(x)q_2(dx) \right), \\ \Psi(u) &= \frac{1}{c} \int_{S_1} \lambda_1(x)q_1(dx) \int_u^\infty \bar{F}_1(z)dz \\ &\quad + \frac{1}{c} \int_0^u \left[ \int_{S_1} \lambda_1(x)\Psi_x(u-z)q_1(dx) \right] \bar{F}_1(z)dz \\ &\quad + \frac{1}{c} \int_{S_2} \lambda_2(x)q_2(dx) \int_u^\infty \bar{F}_2(z)dz \\ &\quad + \frac{1}{c} \int_0^u \left[ \int_{S_2} \lambda_2(y)\tilde{\Psi}_y(u-z)q_2(dy) \right] \bar{F}_2(z)dz, \end{aligned}$$

where  $\bar{F}_i(z) = 1 - F_i(z)$ ,  $i = 1, 2$ .

**Proof** Using the backward differential technique, we have

$$\begin{aligned} R_{xy}(u) &= (1 - \lambda_1(x)\Delta)(1 - \lambda_2(y)\Delta)R_{xy}(u + c\Delta) \\ &\quad + \lambda_1(x)\Delta(1 - \lambda_2(y)\Delta) \int_{S_1} Q_1(x, dx_1) \times \\ &\quad \int_0^{u+c\Delta} R_{x_1y}(u + c\Delta - z)dF_1(z) \\ &\quad + \lambda_2(y)\Delta(1 - \lambda_1(x)\Delta) \int_{S_2} Q_2(y, dy_1) \times \\ &\quad \int_0^{u+c\Delta} R_{xy_1}(u + c\Delta - z)dF_2(z) + o(\Delta). \quad (2) \end{aligned}$$

Thus

$$\begin{aligned} cR'_{xy}(u) &= (\lambda_1(x) + \lambda_2(y))R_{xy}(u) \\ &\quad - \lambda_1(x) \int_{S_1} Q_1(x, dx_1) \int_0^u R_{x_1y}(u-z)dF_1(z) \\ &\quad - \lambda_2(y) \int_{S_2} Q_2(y, dy_1) \int_0^u R_{xy_1}(u-z)dF_2(z). \end{aligned}$$

Replacing  $u$  by  $t$  and integrating from  $t = 0$  to  $t = u$ , we obtain

$$\begin{aligned} c(R_{xy}(u) - R_{xy}(0)) &= \lambda_1(x) \int_0^u R_{xy}(t)dt \\ &\quad - \lambda_1(x) \int_0^u \int_{S_1} Q_1(x, dx_1) \int_0^t R_{x_1y}(t-z)dF_1(z)dt \\ &\quad + \lambda_2(y) \int_0^u R_{xy}(t)dt \\ &\quad - \lambda_2(y) \int_0^u \int_{S_2} Q_2(y, dy_1) \int_0^t R_{xy_1}(t-z)dF_2(z)dt \\ &= \lambda_1(x) \int_0^u R_{xy}(t)dt - \lambda_1(x) \int_{S_1} Q_1(x, dx_1) \int_0^u R_{x_1y}(t)dt \\ &\quad + \lambda_1(x) \int_{S_1} Q_1(x, dx_1) \int_0^u \bar{F}_1(z)R_{x_1y}(u-z)dz \\ &\quad + \lambda_2(y) \int_0^u R_{xy}(t)dt - \lambda_2(y) \int_{S_2} Q_2(y, dy_1) \int_0^u R_{xy_1}(t)dt \\ &\quad + \lambda_2(y) \int_{S_2} Q_2(y, dy_1) \int_0^u \bar{F}_2(z)R_{xy_1}(u-z)dz. \quad (3) \end{aligned}$$

Integrating both sides of Eq.(3) about  $q_1(\cdot)$  and  $q_2(\cdot)$ , we get

$$c[R(u) - R(0)] = \int_0^u \left[ \int_{S_1} \lambda_1(x) R_x(u-z) q_1(dx) \right] \bar{F}_1(z) dz + \int_0^u \left[ \int_{S_2} \lambda_2(y) \tilde{R}_y(u-z) q_2(dy) \right] \bar{F}_2(z) dz.$$

Let  $t \rightarrow \infty$  in the above equation, by the dominated convergence theorem, then

$$c[R(\infty) - R(0)] = \int_0^\infty \left[ \int_{S_1} \lambda_1(x) R_x(\infty) q_1(dx) \right] \bar{F}_1(z) dz + \int_0^\infty \left[ \int_{S_2} \lambda_2(y) \tilde{R}_y(\infty) q_2(dy) \right] \bar{F}_2(z) dz.$$

It follows from Corollary 2.1 that

$$c\Psi(0) = \mu_1 \int_{S_1} \lambda_1(x) q_1(dy) + \mu_2 \int_{S_2} \lambda_2(y) q_2(dy),$$

then

$$\Psi(0) = \frac{1}{c} \left( \mu_1 \int_{S_1} \lambda_1(x) q_1(dx) + \mu_2 \int_{S_2} \lambda_2(x) q_2(dx) \right),$$

and

$$\begin{aligned} \Psi(u) &= \Psi(0) - \frac{1}{c} \times \\ &\int_0^u \left[ \int_{S_1} \lambda_1(x) (1 - \Psi_x(u-z)) q_1(dx) \right] \bar{F}_1(z) dz \\ &- \frac{1}{c} \int_0^u \left[ \int_{S_2} \lambda_2(y) (1 - \tilde{\Psi}_y(u-z)) q_2(dy) \right] \bar{F}_2(z) dz \\ &= \frac{1}{c} \int_{S_1} \lambda_1(x) q_1(dx) \int_u^\infty \bar{F}_1(z) dz \\ &+ \frac{1}{c} \int_0^u \left[ \int_{S_1} \lambda_1(x) \Psi_x(u-z) q_1(dx) \right] \bar{F}_1(z) dz \\ &+ \frac{1}{c} \int_{S_2} \lambda_2(x) q_2(dx) \int_u^\infty \bar{F}_2(z) dz \\ &+ \frac{1}{c} \int_0^u \left[ \int_{S_2} \lambda_2(y) \tilde{\Psi}_y(u-z) q_2(dy) \right] \bar{F}_2(z) dz. \end{aligned}$$

Thus the theorem is completed.

### III. BOUNDS FOR CONVERGENCE RATE OF RUIN PROBABILITY

Let  $h_i(r) = \int_0^{+\infty} e^{rx} dF_i(x) - 1$ ,  $\tilde{\lambda}_i = \sup_{x \in S_i} \{\lambda_i(x)\}$ ,  $\hat{\lambda} = \inf_{x \in S_i} \{\lambda_i(x)\}$ ,  $i = 1, 2$ . In the following, we assume that

$$\tilde{\rho} = \frac{c}{\tilde{\lambda}_1 \mu_1 + \tilde{\lambda}_2 \mu_2} - 1 > 0, \hat{\rho} = \frac{c}{\hat{\lambda}_1 \mu_1 + \hat{\lambda}_2 \mu_2} - 1 > 0,$$

and assume that there exists a real number  $r_\infty > 0$  such that  $h_i(r) \rightarrow \infty$  when  $r \rightarrow \infty$  ( we allow for the possibility  $r_\infty = \infty$ ).

**Lemma 3.1** Under the above assumptions, there exist  $\tilde{R}, \hat{R}$  such that

$$\frac{\tilde{\lambda}_1}{c} h_1(\tilde{R}) + \frac{\tilde{\lambda}_2}{c} h_2(\tilde{R}) = \tilde{R}, \quad \frac{\hat{\lambda}_1}{c} h_1(\hat{R}) + \frac{\hat{\lambda}_2}{c} h_2(\hat{R}) = \hat{R}.$$

The proof of Lemma 3.1 is omitted.

**Theorem 3.1** For the probability  $\Psi(u)$ , we have

$$\limsup_{u \rightarrow \infty} e^{\tilde{R}u} \Psi(u) \leq \frac{1 + \tilde{\rho}}{(1 + \tilde{\rho}) \left( \frac{\tilde{\lambda}_1}{c} h'_1(\tilde{R}) + \frac{\tilde{\lambda}_2}{c} h'_2(\tilde{R}) - 1 \right)}, \quad (4)$$

$$\liminf_{u \rightarrow \infty} e^{\hat{R}u} \Psi(u) \geq \frac{1 + \hat{\rho}}{(1 + \hat{\rho}) \left( \frac{\hat{\lambda}_1}{c} h'_1(\hat{R}) + \frac{\hat{\lambda}_2}{c} h'_2(\hat{R}) - 1 \right)}. \quad (5)$$

**Proof** By theorem 2.1, we have

$$\begin{aligned} \Psi(u) &\leq \frac{\tilde{\lambda}_1}{c} \int_u^\infty \bar{F}_1(z) dz + \frac{\tilde{\lambda}_2}{c} \int_u^\infty \bar{F}_2(z) dz \\ &+ \frac{\tilde{\lambda}_1}{c} \int_0^u \Psi(u-z) \bar{F}_1(z) dz + \frac{\tilde{\lambda}_2}{c} \int_0^u \Psi(u-z) \bar{F}_2(z) dz \\ &= \frac{\tilde{\lambda}_1}{c} \int_u^\infty \bar{F}_1(z) dz + \frac{\tilde{\lambda}_2}{c} \int_u^\infty \bar{F}_2(z) dz \\ &+ \int_0^u \Psi(u-z) \left( \frac{\tilde{\lambda}_1}{c} \bar{F}_1(z) + \frac{\tilde{\lambda}_2}{c} \bar{F}_2(z) \right) dz. \end{aligned}$$

Multiplying the above inequality by  $e^{\tilde{R}u}$ , we have

$$\begin{aligned} e^{\tilde{R}u} \Psi(u) &\leq \frac{\tilde{\lambda}_1}{c} e^{\tilde{R}u} \int_u^\infty \bar{F}_1(z) dz + \frac{\tilde{\lambda}_2}{c} e^{\tilde{R}u} \int_u^\infty \bar{F}_2(z) dz \\ &+ \int_0^u e^{\tilde{R}(u-z)} \Psi(u-z) e^{\tilde{R}z} \left( \frac{\tilde{\lambda}_1}{c} \bar{F}_1(z) + \frac{\tilde{\lambda}_2}{c} \bar{F}_2(z) \right) dz. \end{aligned}$$

Thus, by lemma 3.1, we have that

$$\int_0^\infty e^{\tilde{R}z} \left( \frac{\tilde{\lambda}_1}{c} \bar{F}_1(z) + \frac{\tilde{\lambda}_2}{c} \bar{F}_2(z) \right) dz = 1,$$

and then

$$\begin{aligned} 0 &\leq \lim_{u \rightarrow \infty} \frac{\tilde{\lambda}_1}{c} e^{\tilde{R}u} \int_u^\infty \bar{F}_1(z) dz + \frac{\tilde{\lambda}_2}{c} e^{\tilde{R}u} \int_u^\infty \bar{F}_2(z) dz \\ &\leq \lim_{u \rightarrow \infty} \int_u^\infty e^{\tilde{R}z} \left( \frac{\tilde{\lambda}_1}{c} \bar{F}_1(z) + \frac{\tilde{\lambda}_2}{c} \bar{F}_2(z) \right) dz = 0, \end{aligned}$$

so by the key-renewal theorem, we obtain

$$\limsup_{u \rightarrow \infty} e^{\tilde{R}u} \Psi(u) \leq \frac{c_1}{c_2},$$

where

$$\begin{aligned} c_1 &= \int_0^\infty e^{\tilde{R}u} \int_u^\infty \left( \frac{\tilde{\lambda}_1}{c} \bar{F}_1(z) + \frac{\tilde{\lambda}_2}{c} \int_u^\infty \bar{F}_2(z) \right) dz du, \\ c_2 &= \int_0^\infty z e^{\tilde{R}z} \left( \frac{\tilde{\lambda}_1}{c} \bar{F}_1(z) + \frac{\tilde{\lambda}_2}{c} \int_u^\infty \bar{F}_2(z) \right) dz. \end{aligned}$$

So from the two above equations, we can get

$$c_1 = \frac{\tilde{\rho}}{\tilde{R}(1 + \tilde{\rho})}, c_2 = \frac{1}{\tilde{R}} \left( \frac{\tilde{\lambda}_1}{c} h'_1(\tilde{R}) + \frac{\tilde{\lambda}_2}{c} h'_2(\tilde{R}) - 1 \right).$$

Then the proof of (4) is completed.

We can get the proof of (5) by imitating the above proof of (4).□

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