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Note to the global GMRES for solving the matrix equation $AXB = F$

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Abstract—In the present work, we propose a new projection method for solving the matrix equation $AXB = F$. For implementing our new method, generalized forms of block Krylov subspace and global Arnoldi process are presented. The new method can be considered as an extended form of the well-known global generalized minimum residual (Gl-GMRES) method for solving multiple linear systems and it will be called as the extended Gl-GMRES (EGl-GMRES). Some new theoretical results have been established for proposed method by employing Schur complement. Finally, some numerical results are given to illustrate the efficiency of our new method.

Keywords—Matrix equation, Iterative method, linear systems, block Krylov subspace method, global generalized minimum residual (Gl-GMRES).

I. Introduction

CONSIDER the multiple linear system

$$AX = C,$$

where $A \in \mathbb{R}^{n\times n}$ is a large and spars nonsingular matrix, C and X are $n \times s$ rectangular real matrices.

For nonsymmetric problems, recently, some *block Krylov subspace methods* have been developed; see [2, 4, 6, 8-10, 13, 14] and the references therein. The generalized minimum residual method and its weighted version, for solving the multiple linear system $AX = C$, are projection methods on the *block Krylov subspace*

$$\mathcal{K}_m(A, V) = span\{V, AV, A^2V, ..., A^{m-1}V\},$$

where $V \in \mathbb{R}^{n\times s}$ is given.

In this paper, we are interested to solve the following matrix equation

$$AXB = F, \qquad (1)$$

where $A \in \mathbb{R}^{p\times n}$, is a full column-rank matrix and $B \in \mathbb{R}^{s\times q}$ is a full row-rank matrix.

Recently, there has been an increased interest in solving matrix equations; for more details see [3,5] and references therein . In [3], Ding et al. proposed an iterative method for solving the matrix equation (1) by extending the well-known *Jacobi* and *Gauss-Seidel* methods. The proof of the following lemma and theorem were given in [3].

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Lemma I.1. *If A is a full column-rank matrix and B is a full row-rank matrix* $(p \geq n, s \leq q)$*, then in the sense of least-squares,* (1) *has the unique solution*

$$X = (A^TA)^{-1}A^TFB^T(BB^T)^{-1}.$$

Theorem I.2. *If the conditions of Lemma I.1 hold, the gradient based iterative algorithm of* (1)*,*

$$X(k) = X(k-1) + \mu A^T[F - AX(k-1)B]B^T,$$

$$0 < \mu < \frac{2}{\lambda_{\max}[AA^T]\lambda_{\max}[BB^T]} \quad \text{or} \quad \mu \leq \frac{2}{\|A\|^2\,\|B\|^2},$$

yields $X(k) \to X$.

It is obvious that finding a proper μ by the conditions described in Theorem I.2, is too expensive. It can be easily investigated by numerical examples that the value of μ approximated by Theorem I.2 may become too small which in application algorithm may become divergent.

It is known that the global generalized minimum residual (Gl-GMRES) method is suitable for solving multiple linear systems with large coefficient matrix. Hence, we are interested to present a new projection method, by extending (Gl-GMRES) method, for solving the matrix equation (1). To this end, we need to generalize the definition of the block Krylov subspace. On the other hand, it is obvious that each system of the form (1) can be reformed as $(A^TA)X(BB^T) = A^TFB^T$. Hence, without loss of generality, we will consider the following matrix equation

$$AXB = F, \qquad (2)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{s\times s}$ are *nonsingular* matrices and $X, F \in \mathbb{R}^{n\times s}$.

Notation: The vector $vex(X)$ denotes the vector of $\mathbb{R}^{ns}$ obtained by stacking the columns of the $n \times s$ matrix X, $det(Z)$ is the determinant of the square matrix Z and $tr(Z)$ denotes the trace of Z.

For any matrices X and Y of dimensions $n \times p$ and $q \times l$ respectively, the Kronecker product $X \otimes Y$ is the $nq \times pl$ matrix defined by $X \otimes Y = [X_{i,j}Y]$. The inner product $< .,. >_F$ for the matrices X and Y is defined as $< X, Y >= tr(X^TY)$ and the corresponding matrix norm is the well-known Frobenius norm.

Definition I.3. (R. Bouyouli et al.[1]). Let $A = [A_1, A_2, ..., A_p]$ and $B = [B_1, B_2, ..., B_\ell]$ be matrices of dimensions $n \times ps$ and $n \times \ell s$, respectively, where A_i and B_j are $n \times s$ matrices. Then the $p \times \ell$ matrix $A^T \diamondsuit B = [(A^T \diamondsuit B)_{ij}]$ is defined by $(A^T \diamondsuit B)_{ij} =< A_i, B_j >_F$.

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Definition I.4. Let M be a matrix partitioned into four blocks

$$M = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right],$$

where the submatrix D is assumed to be square and nonsingular. The *Schur* complement of D in M , denoted by (M/D), is defined by

$$(M/D) = A - BD^{-1}C.$$

Generalization and properties of the Schur complements are found in [1].

Proposition I.5. *Assuming that the matrix D is nonsingular and E is a matrix such that the product EA is well defined, then*

$$\left(\left[\begin{array}{cc} EA & EB \\ C & D \end{array} \right] \Big/ D \right) = E \left(\left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \Big/ D \right).$$

Proof: See[2]. ■

Proposition I.6. *Let* $A \in \mathbb{R}^{n\times s}, B \in \mathbb{R}^{n\times ks}, C \in \mathbb{R}^{k\times p}, G \in \mathbb{R}^{k\times k}$*and* $E \in \mathbb{R}^{n\times s}$. *If the matrix G is nonsingular matrix then*

$$E^T \diamondsuit \left(\left(\begin{array}{cc} A & B \\ C\otimes I_s & G\otimes I_s \end{array} \right) \Big/ G\otimes I_s \right) = \left(\begin{array}{cc} E^T\diamondsuit A & E^T\diamondsuit B \\ C & G \end{array} \right) \Big/ G.$$

Proof: See[2]. ■

Proposition I.7. *If the matrices M and D are square and nonsingular, then*

$$M^{-1} = \left[\begin{array}{cc} (M/D)^{-1} & -(M/D)^{-1}BD^{-1} \\ -D^{-1}C(M/D)^{-1} & D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1} \end{array} \right].$$

Proof: See[15]. ■

The outline of this paper is organized as follows. In Section 2, the generalized block Krylov subspace and global Arnoldi process are presented which are needed for implementing our new method. A new method, called by the extended Gl-GMRES (EGl-GMRES) method, is proposed in Section 3. Furthermore, we establish some convergent results for EGl-GMRES. In Section 4, we give some numerical experiments to demonstrate the efficiency of our new method. Finally, the paper is ended with a brief conclusion in Section 5.

II. Generalized global Arnoldi process

We can easily see that the matrix equation (2) is equivalent to the following linear system of equations

$$(B^T \otimes A)vec(X) = vec(F).$$

However, the size of the linear equations $(B^T \otimes A)vec(X) = vec(F)$ is too large and the block Krylov subspace methods consume more computer time and memory once the size of the system is large. To overcome these complications and drawbacks, by extending the global generalized minimum residual (Gl-GMRES) method, we propose an extended global (EGl-GMRES) for solving the matrix equation (2). To this end, we need to generalize the definition of the block Krylov subspace in the following.

Definition II.1. Suppose that $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{s\times s}$, and $V \in \mathbb{R}^{n\times s}$, we define the *generalized block Krylov subspace* as follows

$$\mathcal{G}K_m \equiv \mathcal{G}K_m(A, V, B) \equiv span\{V, AVB, A^2VB^2, ..., A^{m-1}VB^{m-1}\}. \quad (3)$$

Now, we present a generalized form of global Arnoldi process which constructs a $F-$orthonormal basis for the $\mathcal{G}K_m$.

Algorithm II.2. *(Generalized global Arnoldi process)*

1. Choose an $n \times s$ matrix *V*. Set $\beta = \|V\|_F$,$V_1 = V/\beta$
2. For $j = 1, 2, ..., m$ Do:
3. $W = AV_j$
4. $W = WB$
5. For $i = 1, 2, ..., j$ Do:
6. $h_{ij} = < W, V_i >_F$
7. $W = W - h_{ij}V_i$
8. EndDo
9. $h_{j+1,j} = \|W\|_F$. If $h_{j+1,j} = 0$ Stop
10. $V_{j+1} = W/h_{j+1,j}$
11. EndDo.

Denote by $\mathcal{V}_m$, the $n \times ms$ matrix with columns $V_1, V_2, ..., V_m$, $\overline{H}_m$, the $(m+1)\times m$ Hessenberg matrix whose nonzero entries $h_{ij}, i = 1, 2, ..., m + 1,\ j = 1, ..., m$, are defined by Algorithm II.2, and by H_m, the matrix obtained from $\overline{H}_m$ by deleting its last row.

It is obvious that, the generalized global Arnoldi process constructs an F-orthonormal basis $V_1, V_2, ..., V_m$ of the matrix block Krylov $\mathcal{G}K_m(A, V, B)$, i.e.,

the matrices $V_1, V_2, ..., V_m$ satisfy in the following conditions

$$tr(V_i^TV_j) = 0,\ \ tr(V_i^TV_i) = 1, \text{ for } i \neq j, \quad i, j = 1, 2, ..., m. \quad (4)$$

Theorem II.3. *Let* $\mathcal{V}_m$, H_m, *and* $\overline{H}_m$ *defined as before. Then the following relations hold*

$$A\mathcal{V}_m(I_m \otimes B) = \mathcal{V}_m(H_m \otimes I_s) + [0_{n\times s}, ..., 0_{n\times s}, h_{m+1,m}V_{m+1}], \quad (5)$$

$$A\mathcal{V}_m(I_m \otimes B) = \mathcal{V}_m(H_m \otimes I_s) + h_{m+1,m}V_{m+1}(e_m^T \otimes I_s)$$
$$= \mathcal{V}_{m+1}(\overline{H}_m \otimes I_s), \quad (6)$$

where $e_m^T = \left[\begin{array}{cccc} 0, & ..., & 0, & 1 \end{array} \right]_{1\times m}$.

Proof: The relation (5) follows from fact that

$$A\mathcal{V}_m(I_m \otimes B) = [AV_1B, AV_2B,..., AV_mB],$$

and lines 3, 4 and 7 of Algorithm II.2, The relation (6) is reformulation of (5). ■

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III. Extended global GMRES EGl-GMRES method

In this section, we present our new method, EGl-GMRES for solving the matrix equation (2).

Given an initial guess X_0, with the corresponding residual $R_0 = F - AX_0B$, the EGl-GMRES constructs the new approximate solution X_m to the solution of (2) such that

$$X_m \in X_0 + \mathcal{G}K_m(A, R_0, B), \tag{7}$$

and

$$R_m = F - AX_mB \perp_F A\ \mathcal{G}K_m(A, R_0, B)\ (I_m \otimes B). \tag{8}$$

Consider the *F*-orthonormal basis $\mathcal{V}_m$, constructed with generalized global Arnoldi process. From the relation (7), we deduce that

$$X_m = X_0 + \mathcal{V}_m(y_m \otimes I_s), \tag{9}$$

where the vector $y_m \in \mathbb{R}^m$ is obtained by imposing the orthogonality condition (8). By substituting (9) in R_m, we get

$$\begin{aligned} R_m = F - AX_mB &= F - A(X_0 + \mathcal{V}_m(y_m \otimes I_m))B \\ &= R_0 - A\mathcal{V}_m(I_m \otimes B)(y_m \otimes I_s). \end{aligned}$$

where the vector $y_m \in \mathbb{R}^m$ is obtained by imposing the condition (8). On the other hand, it is easy to see that y_m is the solution of following least-square problem too

$$\min_{y\in\mathbb{R}^m} \|R_0 - A\mathcal{V}_m(y \otimes I_s)B\|_F = \min_{y\in\mathbb{R}^m} \|R_0 - A\mathcal{V}_m(I_m \otimes B)y\|_F. \tag{10}$$

Straightforward computations show that

$$R_m = \mathcal{V}_m[(\beta e_1 - \overline{H}_m y_m) \otimes I_s].$$

Hence, by rewriting the equation (10), we conclude that y_m is the solution of the following least-square problem

$$\min_{y\in\mathbb{R}^m} \left\|\beta e_1 - \overline{H}_m y\right\|_2 \tag{11}$$

Now, we propose the EGl-GMRES algorithm for solving the matrix equation (2) as follows.

Algorithm III.1. *(EGl-GMRES)*
1. Choose X_0, a tolerance ε, compute $R_0 = F - AX_0B$ and Set $V = R_0$
2. For $m = 1,\ 2,\ 3, ...$
3. Construct the *F*-orthonormal basis $V_1, V_2, ..., V_m$ by Algorithm II.2
4. Find y_m as the solution of

$$\min_{y\in\mathbb{R}^m} \left\|\beta e_1 - \overline{H}_m y\right\|_2$$

5. Compute the approximate solution $X_m = X_0 + \mathcal{V}_m(y_m \otimes I_s)$ and $R_m = F - AX_mB$.
6. If $\|R_m\|_F < \varepsilon$ Stop.
7. Set $X_0 = X_m$, $R_0 = R_m$, $V = R_0$, and go to 2.
8. EndDo.

The EGl-GMRES algorithm requires the storage of $\mathcal{V}_m$. That is, in order to save the vector $\mathcal{V}_m$ we need an m dimensional vectors space whose entries are $n \times s$ matrices. To cure the storage problem, encountered also in Gl-GMRES, the value of m is limited by storage constraint and by avoiding rounding errors. Hence, Algorithm III.1 can be restarted after m iterations. The corresponding algorithm is called the restarted EGl-GMRES(m), see [12].

Let $\mathcal{W}_m = A\mathcal{V}_m(I_m \otimes B)$, by imposing the condition (8), we have

$$\begin{aligned} 0 = \mathcal{W}_m^T \Diamond R_m &= \mathcal{W}_m^T \Diamond R_0 - \mathcal{W}_m^T \Diamond (\mathcal{W}_m(y_m \otimes I_s) \\ &= \mathcal{W}_m^T \Diamond R_0 - (\mathcal{W}_m^T \Diamond \mathcal{W}_m)y_m. \end{aligned}$$

Hence y_m is the solution of the following linear system

$$(\mathcal{W}_m^T \Diamond \mathcal{W}_m)y_m = \mathcal{W}_m^T \Diamond R_0. \tag{12}$$

Now, we give some new expressions for the residual matrix R_m by means of the *Schur* complement.

Straightforward computations show that

$$R_m = R_0 - \mathcal{W}_m[(\mathcal{W}_m^T \Diamond \mathcal{W}_m)^{-1}(\mathcal{W}_m^T \Diamond R_0) \otimes I_p]$$

$$= R_0 - \mathcal{W}_m[(\mathcal{W}_m^T \Diamond \mathcal{W}_m)^{-1} \otimes I_p][(\mathcal{W}_m^T \Diamond R_0) \otimes I_p].$$

Hence, from the definition of the Schur complement, we derive

$$R_m = \left(\begin{bmatrix} R_0 & \mathcal{W}_m \\ (\mathcal{W}_m^T \Diamond R_0) \otimes I_p & (\mathcal{W}_m^T \Diamond \mathcal{W}_m) \otimes I_p \end{bmatrix} \Big/ (\mathcal{W}_m^T \Diamond \mathcal{W}_m) \otimes I_p \right). \tag{13}$$

Theorem III.2. *Assume that $\mathcal{W}_m^T \Diamond \mathcal{W}_m$ is nonsingular. The residual matrix R_m, obtained by EGl-GMRES at step m, satisfies in the following relation*

$$\|R_m\|_F^2 = \frac{det[\overline{\mathcal{V}}_{m+1}^T \Diamond \overline{\mathcal{V}}_{m+1}]}{det[\mathcal{W}_m^T \Diamond \mathcal{W}_m]} \tag{14}$$

where $\overline{\mathcal{V}}_{m+1} = [R_0, \mathcal{W}_m]$.

Proof: From the orthogonality condition (8), we have

$$R_m^T \Diamond R_m = R_0^T \Diamond R_m.$$

By using Proposition I.6 and Eq. (13), we conclude that

$$\begin{aligned} R_0^T \Diamond R_m &= \left(\begin{bmatrix} R_0^T \Diamond R_0 & R_0^T \Diamond \mathcal{W}_m \\ \mathcal{W}_m^T \Diamond R_0 & \mathcal{W}_m^T \Diamond \mathcal{W}_m \end{bmatrix} \Big/ \mathcal{W}_m^T \Diamond \mathcal{W}_m \right) \\ &= (\overline{\mathcal{V}}_{m+1}^T \Diamond \overline{\mathcal{V}}_{m+1} / \mathcal{W}_m^T \Diamond \mathcal{W}_m). \end{aligned}$$

Or equivalently

$$R_m^T \Diamond R_m = (\overline{\mathcal{V}}_{m+1}^T \Diamond \overline{\mathcal{V}}_{m+1} / \mathcal{W}_m^T \Diamond \mathcal{W}_m), \tag{15}$$

note that $\|R_m\|_F^2 = R_m^T \Diamond R_m$ is a scalar, therefore we can conclude the result. ■

Theorem III.3. *At step m, assume that R_m denotes the residual produced by EGl-GMRES methods. Then we have*

$$\|R_m\|_F^2 = \frac{\det \begin{pmatrix} \beta^2 & \beta e_1^T H_m \\ \beta H_m^T e_1 & \overline{H}_m^T \overline{H}_m \end{pmatrix}}{\det[\overline{H}_m^T \overline{H}_m]}, \tag{16}$$

where $\beta = \|R_0\|_F$.

Proof: Invoking Eq. (6), we derive

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$$\mathcal{W}_m^T \diamondsuit \mathcal{W}_m = (\mathcal{V}_{m+1}(\overline{H}_m \otimes I_p))^T \diamondsuit (\mathcal{V}_{m+1}(\overline{H}_m \otimes I_p)).$$

As $\mathcal{V}_{m+1}$ is an orthonormal basis for $\mathcal{EK}_m$, we deduce that $\mathcal{V}_{m+1}^T \diamondsuit \mathcal{V}_{m+1} = I$. Therefore,

$$\mathcal{W}_m^T \diamondsuit \mathcal{W}_m = \overline{H}_m^T (\mathcal{V}_{m+1}^T \diamondsuit \mathcal{V}_{m+1}) \overline{H}_m = \overline{H}_m^T \overline{H}_m \quad (17)$$

Using Eq. (5), we get

$$\begin{aligned} R_0^T \diamondsuit \mathcal{W}_m &= R_0^T \diamondsuit [\mathcal{V}_m (H_m \otimes I_p) + h_{m+1,m} V_{m+1} (e_m^T \otimes I_p)] \\ &= (R_0^T \diamondsuit \mathcal{V}_m) H_m + h_{m+1,m} (R_0^T \diamondsuit V_{m+1}) e_m^T. \quad (18) \end{aligned}$$

It is known that $R_0^T = \beta V_1$, and $V_1^T \diamondsuit V_i = 0$ for $i \neq 1$. Hence, we can rewrite (18) as follows

$$R_0^T \diamondsuit \mathcal{W}_m = (R_0^T \diamondsuit \mathcal{V}_m) H_m = \beta e_1^T H_m. \quad (19)$$

On the other hand,

$$\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1} = \begin{bmatrix} R_0^T \diamondsuit R_0 & R_0^T \diamondsuit \mathcal{W}_m \\ \mathcal{W}_m^T \diamondsuit R_0 & \mathcal{W}_m^T \diamondsuit \mathcal{W}_m \end{bmatrix}. \quad (20)$$

By substituting Eqs. (17) and (19) in the above relation, the result follows from Theorem III.2 immediately. ■

Theorem III.4. *Let $\overline{\mathcal{V}}_{m+1} = [R_0, \mathcal{W}_m]$. Assume that $\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1}$ and $\mathcal{W}_m^T \diamondsuit \mathcal{W}_m$ are nonsingular matrices, then residual R_m satisfies the following relation:*

$$\|R_m\|_F^2 = 1/(e_1^T (\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1})^{-1} e_1)$$

Proof: Since the matrices $\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1}$ and $\mathcal{W}_m^T \diamondsuit \mathcal{W}_m$ are nonsingular, the Schur complement $(\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1} / \mathcal{W}_m^T \diamondsuit \mathcal{W}_m)$ is nonzero. Therefore, by Proposition 1.10, we get

$$e_1^T (\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1})^{-1} e_1 = ((\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1} / \mathcal{W}_m^T \diamondsuit \mathcal{W}_m))^{-1}.$$

Now, the result can be concluded from Theorem 3.3. ■

Theorem III.5. *The residual R_m satisfies the following relation*

$$\frac{4\chi(\overline{\mathcal{V}}_{m+1})}{(1+\chi(\overline{\mathcal{V}}_{m+1}))^2} \leq \frac{\|R_m\|_F^2}{\|R_0\|_F^2} \leq 1,$$

where $\chi(\overline{\mathcal{V}}_{m+1})$ is the condition number of the matrix $\overline{\mathcal{V}}_{m+1}$.

Proof: It is not difficult to see that $(\mathcal{W}_m^T \diamondsuit \mathcal{W}_m)^{-1}$ is a positive definite matrix. Evidently,

$$\|R_m\|_F^2 = (\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1} / \mathcal{W}_m^T \diamondsuit \mathcal{W}_m)$$
$$= R_0^T \diamondsuit R_0 - [R_0^T \diamondsuit \mathcal{W}_m](\mathcal{W}_m^T \diamondsuit \mathcal{W}_m)^{-1}[R_0^T \diamondsuit \mathcal{W}_m]^T \leq R_0^T \diamondsuit R_0.$$

Using Theorem III.4, Kantorovich inequality and the fact that

$$R_0^T \diamondsuit R_0 = e_1^T (\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1}) e_1,$$

we have

$$\begin{aligned} R_0^T \diamondsuit R_0 &\geq \frac{1}{e_1^T (\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1})^{-1} e_1} \\ &\geq \frac{4\chi(\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1})}{(1+\chi(\overline{\mathcal{V}}_{m+1}^T \diamondsuit \overline{\mathcal{V}}_{m+1}))^2} R_0^T \diamondsuit R_0. \end{aligned}$$

Hence the result is fulfilled. ■

Remark III.6. Theorem III.5 shows that EGl-GMRES is not convergent as long as the matrix $\overline{\mathcal{V}}_m$ is well conditioned.

IV. Numerical Experiments

In this section, we give some numerical experiments to illustrate the efficiency of our new method. All numerical procedures were computed in Mathematica 6 and run on an Intel Pentium IV processor. Also, we will compare our new method with the method given in [3]. For simplicity we called the method, proposed in [3], as Ding's Method.

In all of the numerical results, the matrix F in (1) is generated such that X is the solution of the matrix equation $AXB = F$, where nonzero elements of X are $X_{ii} = 1$ for $i = 1, 2, ..., min(n, s)$. The initial guess X_0 was chosen such that $X_0 = 0$ and the tests were stopped as soon as

$$\|R_m\|_F = \|F - AX_mB\|_F \leq 0.5 \times 10^{-6}.$$

Example IV.1. *Consider the matrix equation $AXB = F$ where the matrices A and B both taken from Harwell-Boeing collocation. In fact, we have chosen NOS6 (675×675) and NOS5 (468×468) from the set Lanpro* [1].

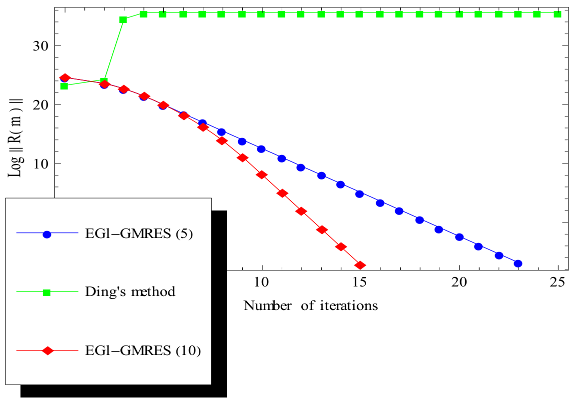

Fig. 1. The matrices A and B are NOS6 and NOS5, respectively.

The results of performing of EGl-GMRES and Ding's method are illustrated in Figures 1 and 2. As seen, the EGl-GMRES outperforms Ding's method.

V. Conclusion

We introduced a generalized forms for the block Krylov subspace and global Arnoldi process. Then, an extended global GMRES (EGl-GMRES) method was presented for solving the matrix equation $AXB = F$. In a similar way discussed in [10], we can propose weighted version of EGl-GMRES method which can converge faster than EGl-GMRES method. Extended Gl-FOM for solving the matrix equation $AXB = F$ has been presented in [11]. In order to accelerate the speed of convergence, the weighted versions of both EGl-FOM and

[1] http://math.nist.gov/MatrixMarket/data/Harwell-Boeing/lanpro/lanpro.html

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EGl-GMRES can be utilized for solving the matrix equation $AXB = F$.

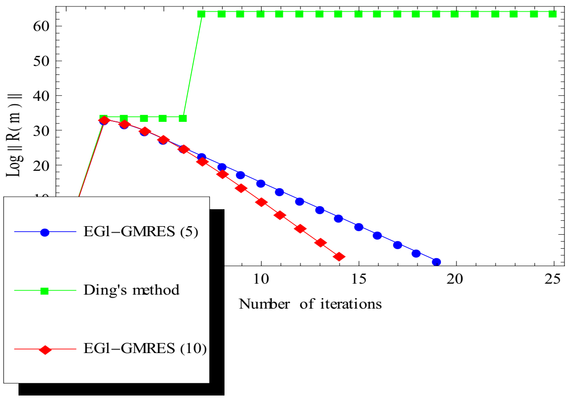

Fig. 2. The matrices A and B are NOS5 and NOS5, respectively.

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