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Almost periodic solution for a food-limited population model with delay and feedback control

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Abstract—In this paper, we consider a food-limited population model with delay and feedback control. By applying the comparison theorem of the differential equation and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the permanence and existence of a unique globally attractive positive almost periodic solution of the system are obtained.

Keywords—Almost periodic solution; Food-limited population; Feedback control; Permanence.

I. Introduction

When growth limitations are based on the proportion of available resources not utilized, the food-limited model was proposed in [1] as follows:

$$\dot{x}(t) = \frac{rx(t)(K - x(t))}{K + \gamma x(t)}, \tag{1}$$

here the population density is denoted by $x(t)$ and the positive constants r and K represent the growth rate of the population and the carrying capacity of the habitat, respectively. Assuming that a growing population requires more food (growth and maintenance) than a saturated one (maintenance only), a further modification is to assume that the average growth rate is a function of some specified delayed argument $t - \tau$ (see, e.g., [2], [3]). The model (1) becomes

$$\dot{x}(t) = \frac{rx(t)(K - x(t - \tau))}{K + \gamma x(t - \tau)}, \quad \tau > 0, \tag{2}$$

which is called as delayed food-limited model. Eq.(2) has been extensively studied in the literature. A majority of results on Eq.(2) deal with global attractivity of the positive equilibrium and oscillatory behavior of solutions (see [2], [4], [5]). These studies were also carried out on Eq.(2) with time periodic coefficients(see [6], [7], [8]).

To the best of our knowledge, no work has been done for the existence of almost periodic solutions of system (2) yet. It is well know that the assumption of almost periodicity of the coefficients in (2) is a way of incorporating the time-dependent variability of the environment, especially, when the various components of the environment are periodic with not necessary commensurate periods (e.g. seasonal effects of weather, food supplies, mating habits and harvesting). Also, as we know, the method used to investigate the positive T-periodic solution of the non-linear ecosystem (for example, by using coincidence degree theory (see [9]) or Brower's fixed point theorem (see [10])) could not be used to investigate the almost periodic solution of the system (2).

On the other hand, we note that ecosystems in the real world are continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. In ecology, a question of practical interest is whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control theory, we call the disturbance functions as control variables. In 1993, Gopalsamy and Weng [11] introduce a models with feedback controls, in which the control variables satisfy certain differential equation. In the last decades, much work has been done on the ecosystem with feedback controls (see [12], [13], [14], [15], [16], [17] and the references therein). In particular, Li and Liu [12], Lalli et al. [13], Liu and Xu [14] and Li [15] have studied delay equations with feedback controls.

Stimulated by above reasons, in this paper we will consider an almost periodic food-limited population model with delay and feedback control as follows:

$$\begin{cases} \dot{x}(t) = \dfrac{r(t)x(t)(k(t) - x(t - \tau))}{k(t) + \eta(t)x(t - \tau)} - d(t)x(t)u(t - \tau), \\ \dot{u}(t) = -\beta(t)u(t) + \alpha(t)x(t - \tau), \end{cases} \tag{3}$$

where $x(t)$ is the population density, $u(t)$ is the control variable at time $t, r(t)$ and $k(t)$ represent the growth rate of the population and the carrying capacity of the habitat at time t, respectively, $\tau > 0$ is time delay. And all the coefficients $r(t), k(t), d(t), \eta(t), \beta(t)$ and $\alpha(t)$ are continuous, bounded, positive almost periodic functions on $R = (-\infty, +\infty)$.

Let f be a continuous bounded function on R and we set

$$f^u = \sup_{t \in R} f(t), \quad f^l = \inf_{t \in R} f(t).$$

Throughout this paper, we assume the coefficients of the almost periodic system (3) satisfy

$$\min\{r^l, k^l, d^l, \eta^l, \beta^l, \alpha^l\} > 0,$$
$$\max\{r^u, k^u, d^u, \eta^u, \beta^u, \alpha^u\} < +\infty.$$

The aim of this paper is, by constructing a suitable Lyapunov functional and applying the analysis technique of Feng and Liu [18] and Shi and Chen [19], to obtain sufficient conditions for the existence of a unique globally attractive positive almost periodic solution of system (3).

The remaining part of this paper is organized as follows: In Section 2, by applying the theory of differential inequality, we

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present the permanence results for system (3). In Section 3, by constructing a suitable Lyapunov function, a set of sufficient conditions which ensure the existence and uniqueness of almost periodic solution of system (3) are obtained. In Section 4, we end this paper with a suitable example which is given to illustrate the feasibility of the main results.

II. Permanence

Now let us state several definitions and lemmas which will be useful in the proof of our main result of this section.

Definition 1. *System* (3) *is said to be permanent if there exist two positive constants m, M and T_0 such that each positive solution $(x(t),u(t))^T$ of system* (3) *satisfies $m \le x(t) \le M, m \le u(t) \le M$, for all $t > T_0$.*

Lemma 1. *$R^2_+ = \{(x,u)|x>0,u>0\}$ is positive invariant with respect to system* (3).

Lemma 2. *[11] If $a>0, b>0$, and $\dot{x} \ge (\le) x(b-ax^{\alpha})$, where α is positive constant, then*

$$\liminf_{t\to+\infty} x(t) \ge (\frac{b}{a})^{\frac{1}{\alpha}},\ \big(\limsup_{t\to+\infty} x(t) \le (\frac{b}{a})^{\frac{1}{\alpha}}\big).$$

Lemma 3. *[20] If $a>0, b>0$ and $\dot{x} \ge (\le) b-ax$, when $t\ge 0$ and $x(0)>0$, we have*

$$\liminf_{t\to+\infty} x(t) \ge \frac{b}{a} \quad \big(\limsup_{t\to+\infty} x(t) \le \frac{b}{a}\big).$$

Set

$$\begin{aligned} M_1 &:= k^u\exp(\tau r^u),\\ M_2 &:= \frac{\alpha^u M_1}{\beta^l},\\ m_1 &:= k^l \exp\left[\left(\frac{r^l k^l}{k^u+\eta^u M_1} - \frac{r^l M_1}{k^u+\eta^u M_1}\right)\tau\right],\\ m_2 &:= \frac{\alpha^l m_1}{\beta^u}. \end{aligned}$$

We also introduce two assumptions:

(H_1) $\dfrac{r^l k^l}{k^u+\eta^u M_1} - d^u M_2 > 0.$

(H_2) $\dfrac{r^l(1+\eta^l)}{k^u+\eta^u M_1} - \dfrac{\alpha^u}{m_2} > 0.$

Theorem 1. *Suppose that (H_1) holds, then system* (3) *is permanence, i.e. there exists positive constants m_i and $M_i (i=1,2)$ such that for any positive solution $(x(t),u(t))^T$ of system* (3) *satisfies*

$$\begin{aligned} 0 < m_1 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M_1,\\ 0 < m_2 \le \liminf_{t\to+\infty} u(t) \le \limsup_{t\to+\infty} u(t) \le M_2. \end{aligned}$$

Proof: Let $(x(t),u(t))^T$ be a positive solution of (3), from the first equation of system (3) it follows that

$$\begin{aligned} \dot{x}(t) &\le \frac{r(t)x(t)(k(t)-x(t-\tau))}{k(t)}\\ &\le r(t)x(t) \quad \text{for all } t\in R. \end{aligned} \tag{4}$$

Hence, for any $\theta<0$, integrating inequality (4) from $t+\theta$ to t, we obtain

$$x(t+\theta) \ge x(t)\exp\bigg(\int_t^{t+\theta} r(s)\,\mathrm{d}s\bigg). \tag{5}$$

So for any $t\in R$, from (5) and the first equation of system (3) we further obtain

$$\begin{aligned} \dot{x}(t) &\le \frac{r(t)x(t)(k(t)-x(t-\tau))}{k(t)}\\ &\le \frac{r^u x(t)(k^u - x(t-\tau))}{k^l}\\ &\le \frac{r^u x(t)(k^u - x(t)\exp(\int_t^{t-\tau} r(s)\,\mathrm{d}s))}{k^l}. \end{aligned}$$

Since for any $t\in R$ and $s\in[-\tau,0]$,

$$\int_t^{t+s} r(\theta)\,\mathrm{d}\theta \ge -\tau r^u,$$

we have

$$\begin{aligned} \dot{x}(t) &\le \frac{r^u x(t)(k^u - x(t)\exp(-\tau r^u))}{k^l}\\ &= x(t)\bigg(\frac{r^u k^u}{k^l} - \frac{r^u\exp(-\tau r^u)}{k^l}x(t)\bigg). \end{aligned} \tag{6}$$

Applying Lemma 2 to (6) leads to

$$\limsup_{t\to+\infty} x(t) \le k^u\exp(\tau r^u) := M_1. \tag{7}$$

From (7), for small enough positive constant $\varepsilon>0$, there exists a $T_1>0$ large enough such that

$$x(t) \le M_1+\varepsilon \qquad \text{for all } t\ge T_1. \tag{8}$$

Then, from the second equation of system (3) and (8), we obtain that for $t\ge T_1$,

$$\begin{aligned} \dot{u}(t) &\le -\beta(t)u(t) + \alpha(t)(M_1+\varepsilon)\\ &\le -\beta^l u(t) + \alpha^u(M_1+\varepsilon). \end{aligned}$$

Setting $\varepsilon\to 0$ in above inequality leads to

$$\dot{u}(t) \le -\beta^l u(t) + \alpha^u M_1.$$

Since $u(t)>0$ for all $t\in R$ holds, then $u(0)>0$, so applying Lemma 3 to above inequality we obtain

$$\limsup_{t\to+\infty} u(t) \le \frac{\alpha^u M_1}{\beta^l} := M_2. \tag{9}$$

From (9), for above $\varepsilon>0$, there exists a $T_2\ge T_1>0$ large enough such that

$$u(t) \le M_2+\varepsilon \qquad \text{for all } t\ge T_2. \tag{10}$$

From the first equation of system (3) and (8) and (10), we obtain that for $t\ge T_2$,

$$\begin{aligned} \dot{x}(t) &\ge \frac{r^l x(t)(k^l - x(t-\tau))}{k^u+\eta^u(M_1+\varepsilon)} - d^u(M_2+\varepsilon)x(t)\\ &= x(t)\bigg[\frac{r^l k^l}{k^u+\eta^u(M_1+\varepsilon)} - d^u(M_2+\varepsilon)\\ &\quad - \frac{r^l}{k^u+\eta^u(M_1+\varepsilon)}x(t-\tau)\bigg]. \end{aligned}$$

Setting $\varepsilon \to 0$ in above inequality leads to

$$\dot{x}(t) \geq x(t)\left[\frac{r^l k^l}{k^u+\eta^u M_1} - d^u M_2 - \frac{r^l}{k^u+\eta^u M_1}x(t-\tau)\right].$$

Then by (7) and applying Lemma 3 given in [21], there exists a constant m_1 such that

$$\liminf_{t\to+\infty} x(t) \geq m_1 = k^l \exp\left[\left(\frac{r^l k^l}{k^u+\eta^u M_1} - d^u M_2 - \frac{r^l M_1}{k^u+\eta^u M_1}\right)\tau\right]. \qquad (11)$$

From (11), for above $\varepsilon > 0$, there exists a $T_3 \geq T_2 > 0$ large enough such that

$$x(t) \geq m_1 - \varepsilon \qquad \text{for all } t \geq T_3. \qquad (12)$$

Then, from the second equation of system (3) and (12), we obtain that for $t \geq T_3$,

$$\begin{aligned}\dot{u}(t) &\geq -\beta(t)u(t) + \alpha(t)(m_1-\varepsilon)\\ &\geq -\beta^u u(t) + \alpha^l(m_1-\varepsilon).\end{aligned}$$

Setting $\varepsilon \to 0$ in above inequality leads to

$$\dot{u}(t) \geq -\beta^u u(t) + \alpha^l m_1.$$

Then applying Lemma 3 to above inequality, we have

$$\liminf_{t\to+\infty} u(t) \geq \frac{\alpha^l m_1}{\beta^u} := m_2. \qquad (13)$$

(7), (9), (11), and (13) show that under the assumption of Theorem 1, system (3) is permanence. This completes the proof of Theorem 1. ■

Next we will prove for $t \in R$, the above conclusions hold.

We denote by (S) the set of all solutions $z(t) = (x(t), u(t))^T$ of system (3) on R satisfying $m_1 \leq x(t) \leq M_1, m_2 \leq u(t) \leq M_2$ for $t \in R$.

Theorem 2. $(S) \neq \emptyset$.

Proof: From properties of almost periodic functions, there exists a sequence $t_n, t_n \to \infty$ as $n \to \infty$, such that

$$\begin{aligned}&r(t+t_n) \to r(t), \quad k(t+t_n) \to k(t), \quad \eta(t+t_n) \to \eta(t),\\ &\beta(t+t_n) \to \beta(t), \quad \alpha(t+t_n) \to \alpha(t), \quad d(t+t_n) \to d(t)\end{aligned}$$

as $n \to \infty$ uniformly on R. Let $z(t) = (x(t), u(t))^T$ be a solution of Eq.(3) satisfying $m_1 \leq x(t) \leq M_1, m_2 \leq u(t) \leq M_2$ for $t \in R$. Clearly, the sequence $z(t+t_n)$ is uniformly bounded and equicontinuous on each bounded subset of R. Therefore by Ascoli's theorem, we know that there exists a subsequence $z(t+t_k)$ which converges to a continuous function $p(t) = (p_1(t), p_2(t))^T$ as $k \to \infty$ uniformly on each bounded subset of R. Let $\bar{T} \in R$ be given, then for $t \in R$, we have

$$\begin{aligned}&x(t+t_k+\bar{T}) - x(t_k+\bar{T})\\ &= \int_{\bar{T}}^{t+\bar{T}} \left[\frac{r(s+t_k)x(s+t_k)(k(s+t_k)-x(s+t_k-\tau))}{k(s+t_k)+\eta(s+t_k)x(s+t_k-\tau)}\right.\\ &\left. -d(s+t_k)x(s+t_k)u(s+t_k-\tau)\right]\mathrm{d}s,\\ &u(t+t_k+\bar{T}) - u(t_k+\bar{T})\\ &= \int_{\bar{T}}^{t+\bar{T}} (-\beta(s+t_k)u(s+t_k) +\\ &\alpha(s+t_k)x(s+t_k-\tau))\,\mathrm{d}s.\end{aligned}$$

Applying Lebesgue dominated convergence theorem, and letting $k \to \infty$ in above equalities, we obtain

$$\begin{aligned}p_1(t+\bar{T}) - p_1(\bar{T}) &= \int_{\bar{T}}^{t+\bar{T}} \left[\frac{r(s)x(s)(k(s)-x(s-\tau))}{k(s)+\eta(s)x(s-\tau)}\right.\\ &\left. -d(s)x(s)u(s-\tau)\right]\mathrm{d}s,\\ p_2(t+\bar{T}) - p_2(\bar{T}) &= \int_{\bar{T}}^{t+\bar{T}} (-\beta(s)u(s) + \alpha(s)x(s-\tau))\,\mathrm{d}s\end{aligned}$$

for all $t \in R$. Since $\bar{T} \in R$ is arbitrarily given, then $p(t) = (p_1(t), p_2(t))^T$ is a solution of system (3) on R. It is clear that $m_1 \leq p_1(t) \leq M_1, m_2 \leq p_2(t) \leq M_2$ for $t > 0$. Thus $p(t) \in (S)$.

This completes the proof. ■

III. EXISTENCE OF A UNIQUE ALMOST PERIODIC SOLUTION

Now, we give the definition of the almost periodic function.

Definition 2. [22] *A function $f(t, x)$, where f is an m-vector, t is a real scalar and x is an n-vector, is said to be almost periodic in t uniformly with respect to $x \in X \subset R^n$, if $f(t, x)$ is continuous in $t \in R$ and $x \in X$, and if for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$, there exists a τ such that the inequality*

$$||f(t+\tau, x) - f(t, x)|| = \sum_{i=1}^{m} |f_i(t+\tau, x) - f_i(t, x)| < \varepsilon$$

is satisfied for all $t \in R, x \in X$. The number τ is called an ε-translation number of $f(t, x)$.

Definition 3. *[23] A function $f : R \to R$ is said to be asymptotically almost periodic function, if there exists an almost periodic function $q(t)$ and a continuous function $r(t)$ such that*

$$f(t) = q(t) + r(t), \quad t \in R \quad \textit{and} \quad r(t) \to 0 \quad \textit{as} \quad t \to \infty.$$

We refer to [24], [25] for the relevant definitions and the properties of almost periodic functions. In the followings, by constructing an suitable Lyapunov functional, we get the sufficient conditions for the existence of the globally attractive solution for system (3).

Theorem 3. *Assume that (H_1) and (H_2) hold, then for any two positive solutions $z_1(t) = (x_1(t), u_1(t))^T$ and $z_2(t) = (x_2(t), u_2(t))^T$ of system (3), we have*

$$\lim_{t\to\infty} |z_1(t) - z_2(t)| = 0.$$

Proof: Let $z_1(t) = (x_1(t), u_1(t))^T$ and $z_2(t) = (x_2(t), u_2(t))^T$ be any two positive solutions of system (3).

From (H_2), $\frac{\alpha^l m_1}{M_2^2} > 0, d^l m_1 > 0$ and $d^l m_2 > 0$, it follows that there exists an enough small $\varepsilon > 0$ such that

$$\begin{cases} A_1(\varepsilon) = \frac{r^l(1+\eta^l)}{k^u+\eta^u(M_1+\varepsilon)} - \frac{\alpha^u}{m_2-\varepsilon} > \varepsilon, \\ A_2(\varepsilon) = \frac{\alpha^l(m_1-\varepsilon)}{(M_2+\varepsilon)^2} > \varepsilon, \\ A_3(\varepsilon) = d^l(m_1-\varepsilon) > \varepsilon, \\ A_4(\varepsilon) = d^l(m_2-\varepsilon) > \varepsilon. \end{cases} \tag{14}$$

It follows from (8), (10), (12) and (13) that for above $\varepsilon > 0$, there exists a $T \geq T_3 > 0$ such that for $t \geq T$,

$$m_1 - \varepsilon \leq x(t) \leq M_1 + \varepsilon, \quad m_2 - \varepsilon \leq u(t) \leq M_2 + \varepsilon.$$

Set

$$V_1(t) = |\ln x_1(t) - \ln x_2(t)|.$$

Calculating the upper right derivatives of $V_1(t)$ along the solution of (3), it follows that

$$\begin{aligned} & D^+V_1(t) \\ = \ & \operatorname{sgn}(x_1(t) - x_2(t))[(\ln x_1(t))' - (\ln x_2(t))'] \\ = \ & \operatorname{sgn}(x_1(t) - x_2(t))\left[\frac{r(t)(k(t) - x_1(t-\tau))}{k(t) + \eta(t)x_1(t-\tau)}\right. \\ & -d(t)x_1(t)u_1(t-\tau) \\ & \left.-\frac{r(t)(k(t) - x_2(t-\tau))}{k(t) + \eta(t)x_2(t-\tau)} + d(t)x_2(t)u_2(t-\tau)\right] \\ = \ & \operatorname{sgn}(x_1(t) - x_2(t)) \\ & \times\left[\frac{r(t)k(t)(1+\eta(t))(x_2(t-\tau) - x_1(t-\tau))}{(k(t)+\eta(t)x_1(t-\tau))(k(t)+\eta(t)x_2(t-\tau))}\right. \\ & +d(t)x_2(t)(u_2(t-\tau) - u_1(t-\tau)) \\ & \left.+d(t)u_1(t-\tau)(x_2(t) - x_1(t))\right] \\ \leq \ & \operatorname{sgn}(x_1(t) - x_2(t)) \\ & \times\left[\frac{r(t)k(t)(1+\eta(t))(x_2(t-\tau) - x_1(t-\tau))}{k(t)(k(t)+\eta(t)x_2(t-\tau))}\right] \\ & -d(t)x_2(t)|u_1(t-\tau) - u_2(t-\tau)| \\ & -d(t)u_1(t-\tau)|x_1(t) - x_2(t)| \\ = \ & -\frac{r(t)(1+\eta(t))}{k(t)+\eta(t)x_2(t-\tau)}|x_1(t-\tau) - x_2(t-\tau)| \\ & -d(t)x_2(t)|u_1(t-\tau) - u_2(t-\tau)| \\ & -d(t)u_1(t-\tau)|x_1(t) - x_2(t)|. \end{aligned}$$

Let

$$V_2(t) = |\ln u_1(t) - \ln u_2(t)|.$$

Calculating the upper right derivatives of $V_2(t)$ along the solution of (3), it follows that

$$\begin{aligned} D^+V_2(t) &= \operatorname{sgn}(u_1(t) - u_2(t))[(\ln u_1(t))' - (\ln u_2(t))'] \\ &= \operatorname{sgn}(u_1(t) - u_2(t))\alpha(t)\left[\frac{x_1(t-\tau)}{u_1(t)} - \frac{x_2(t-\tau)}{u_2(t)}\right] \\ &= -\frac{\alpha(t)x_1(t-\tau)}{u_1(t)u_2(t)}|u_1(t) - u_2(t)| \\ &\quad +\frac{\alpha(t)}{u_2(t)}|x_1(t-\tau) - x_2(t-\tau)|. \end{aligned}$$

Now let us define

$$V(t) = V_1(t) + V_2(t).$$

Therefore, for $t \geq T$, it follows from above analysis that

$$\begin{aligned} D^+V(t) \leq & -\left[\frac{r(t)(1+\eta(t))}{k(t)+\eta(t)x_2(t-\tau)} - \frac{\alpha(t)}{u_2(t)}\right] \\ & \times|x_1(t-\tau) - x_2(t-\tau)| \\ & -\frac{\alpha(t)x_1(t-\tau)}{u_1(t)u_2(t)}|u_1(t) - u_2(t)| \\ & -d(t)x_2(t)|u_1(t-\tau) - u_2(t-\tau)| \\ & -d(t)u_1(t-\tau)|x_1(t) - x_2(t)| \\ \leq & -\left[\frac{r^l(1+\eta^l)}{k^u+\eta^u(M_1+\varepsilon)} - \frac{\alpha^u}{m_2-\varepsilon}\right] \\ & \times|x_1(t-\tau) - x_2(t-\tau)| \\ & -\frac{\alpha^l(m_1-\varepsilon)}{(M_2+\varepsilon)^2}|u_1(t) - u_2(t)| \\ & -d^l(m_1-\varepsilon)|u_1(t-\tau) - u_2(t-\tau)| \\ & -d^l(m_2-\varepsilon)|x_1(t) - x_2(t)|. \end{aligned}$$

From (14), we know that there must be an positive constant ε such that

$$\begin{aligned} D^+V(t) \leq & -\varepsilon|x_1(t-\tau) - x_2(t-\tau)| - \varepsilon|u_1(t) - u_2(t)| \\ & -\varepsilon|u_1(t-\tau) - u_2(t-\tau)| - \varepsilon|x_1(t) - x_2(t)|. \end{aligned}$$

Integrating the above inequality on internal $[T, t]$, it follows that for $t \geq T$

$$\begin{aligned} & V(t) + \varepsilon\int_T^t |x_1(s-\tau) - x_2(s-\tau)|\,\mathrm{d}s \\ & +\varepsilon\int_T^t |u_1(s) - u_2(s)|\,\mathrm{d}s \\ & +\varepsilon\int_T^t |u_1(s-\tau) - u_2(s-\tau)|\,\mathrm{d}s \\ & +\varepsilon\int_T^t |x_1(s) - x_2(s)|\,\mathrm{d}s \leq V(T) < +\infty. \end{aligned}$$

Therefore,

$$\limsup_{t\to\infty}\int_T^t |u_1(s) - u_2(s)|\,\mathrm{d}s \leq \frac{V(T)}{\varepsilon} < +\infty,$$
$$\limsup_{t\to\infty}\int_T^t |x_1(s) - x_2(s)|\,\mathrm{d}s \leq \frac{V(T)}{\varepsilon} < +\infty.$$

From the above inequalities, one could easily deduce that

$$\lim_{t\to+\infty}|x_1(t) - x_2(t)| = 0, \quad \lim_{t\to+\infty}|u_1(t) - u_2(t)| = 0.$$

This completes the proof. ■

Theorem 4. *Suppose that all conditions of Theorem 3 hold, then there exists a unique almost periodic solution of system* (3).

Proof: From Theorem 2, there exists a bounded positive solution

$$z(t) = (u_1(t), u_2(t))^T, \quad t \geq 0.$$

Suppose that $z(t) = (u_1(t), u_2(t))^T$ is a solution of (3), then there exists a sequence $\{t'_k\}, t'_k \to \infty$ as $k \to \infty$, such that

$(u_1(t+t'_k), u_2(t+t'_k))^T$ is a solution of the following system:

$$\begin{cases} x'(t) = \dfrac{r(t+t'_k)x(t)(k(t+t'_k)-x(t-\tau))}{k(t+t'_k)+\eta(t+t'_k)x(t-\tau)} \\ \qquad -d(t+t'_k)x(t)u(t-\tau), \\ u'(t) = -\beta(t+t'_k)u(t)+\alpha(t+t'_k)x(t-\tau). \end{cases}$$

From above discussion and Theorem 1, we have that not only $u_i(t+t'_k)$ $(i=1,2)$ but also $\dot{u}_i(t+t'_k)$ $(i=1,2)$ are uniformly bounded, thus $u_i(t+t'_k)$ $(i=1,2)$ is uniformly bounded and equi-continuous. By Ascoli's theorem there exists a uniformly convergent subsequence $\{u_i(t+t_k)\} \subseteq \{u_i(t+t'_k)\}$ such that for any $\varepsilon > 0$, there exists a $k(\varepsilon) > 0$ with the property that if $m, k > K(\varepsilon)$ then

$$|u_i(t+t_m) - u_i(t+t_k)| < \varepsilon \ (i=1,2).$$

It shows that $u_i(t)$ $(i=1,2)$ are asymptotically almost periodic functions, then, $u_i(t+t_k)(i=1,2)$ are the sum of an almost periodic function $q_i(t+t_k)(i=1,2)$ and a continuous function $p_i(t+t_k)(i=1,2)$ defined on R, such that

$$u_i(t+t_k) = p_i(t+t_k) + q_i(t+t_k) \quad \text{for all } t \in R,$$

where

$$\lim_{k\to+\infty} p_i(t+t_k) = 0, \quad \lim_{k\to+\infty} q_i(t+t_k) = q_i(t),$$

$q_i(t)$ is an almost periodic function, which implies that $\lim_{k\to+\infty} u_i(t+t_k) = q_i(t)$ $(i=1,2)$.

On the other hand,

$$\begin{aligned} \lim_{k\to+\infty} \dot{u}_i(t+t_k) &= \lim_{k\to+\infty}\lim_{h\to 0} \frac{u_i(t+t_k+h)-u_i(t+t_k)}{h} \\ &= \lim_{h\to 0}\lim_{k\to+\infty} \frac{u_i(t+t_k+h)-u_i(t+t_k)}{h} \\ &= \lim_{h\to 0} \frac{q_i(t+h)-q_i(t)}{h} \\ &= \dot{q}_i(t). \end{aligned}$$

So the limit $\dot{q}_i(t)(i=1,2)$ exists.

Now we will prove that $(q_1(t), q_i(t))^T$ is an almost solution of system (3).

From the properties of almost periodic functions, there exists an sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that

$$\begin{aligned} &r(t+t_n) \to r(t), \quad k(t+t_n) \to k(t), \quad \eta(t+t_n) \to \eta(t), \\ &\beta(t+t_n) \to \beta(t), \quad \alpha(t+t_n) \to \alpha(t), \quad d(t+t_n) \to d(t) \end{aligned}$$

as $n \to +\infty$ uniformly on R. It is easy to know that $u_i(t+t_n) \to q_i(t)$ as $n \to +\infty (i=1,2)$. Then we have

$$\begin{aligned} &\dot{q}_1(t) \\ &= \lim_{n\to+\infty} \dot{u}_1(t+t_n) \\ &= \lim_{n\to+\infty} \Big[\frac{r(t+t_n)u_1(t+t_n)(k(t+t_n)-u_1(t+t_n-\tau))}{k(t+t_n)+\eta(t+t_n)u_1(t+t_n-\tau)} \\ &\quad -d(t+t_n)x(t+t_n)u(t+t_n-\tau)\Big] \\ &= \frac{r(t)q_1(t)(k(t)-q_1(t-\tau))}{k(t)+\eta(t)q_1(t-\tau)} - d(t)x(t)u(t-\tau), \\ &\dot{q}_2(t) \\ &= \lim_{n\to+\infty} \dot{u}_2(t+t_n) \\ &= \lim_{n\to+\infty} \big[-\beta(t+t_n)u_2(t+t_n) \\ &\quad +\alpha(t+t_n)u_1(t+t_n-\tau)\big] \\ &= -\beta(t)q_2(t) + \alpha(t)q_1(t-\tau). \end{aligned}$$

This proves that $(q_1(t), q_2(t))^T$ satisfies system (3) and $(q_1(t), q_2(t))^T$ is a positive almost periodic solution, by Theorem 3, it follows that there exists a unique positive almost solution of system (3). The proof is completed. ∎

IV. Example

Consider the system

$$\begin{cases} \dot{x}(t) = \dfrac{(40+\sin^2(t))x(t)(1+\sin^2(t)-x(t-\frac{1}{41}))}{1+\sin^2(t)+(1+\cos^2(t))x(t-\frac{1}{41})} \\ \qquad -\frac{\cos^2(t)+1}{1000}x(t)u(t-\frac{1}{41}), \\ \dot{u}(t) = -(1+\cos^2 t)u(t) + (1+\sin^2(t))x(t-\frac{1}{41}). \end{cases} \tag{15}$$

In this case, we have $r^l = 40, r^u = 41, k^l = 1, k^u = 2, \eta^l = 1, \eta^u = 2, \beta^l = 1, \beta^u = 2, \alpha^l = 1, \alpha^u = 2, d^u = \frac{1}{500}, d^l = \frac{1}{1000}$ and $\tau = \frac{1}{41}$. And so

$$\begin{aligned} M_1 &= k^u e^{\tau r^u} = 2e > 0, \\ M_2 &= \frac{\alpha^u M_1}{\beta^l} = 4e > 0, \\ m_1 &= k^l \exp[(\frac{r^l k^l}{k^u+\eta^u M_1} - d^u M_2 \\ &\quad - \frac{r^l M_1}{k^u+\eta^u M_1})\tau] = e^{-0.13} > 0, \\ m_2 &= \frac{\alpha^l m_1}{\beta^u} = \frac{1}{2}e^{-0.13}, \\ (H_1) &\quad \frac{r^l k^l}{k^u+\eta^u M_1} - d^u M_2 = 3.086 > 0, \\ (H_2) &\quad \frac{r^l(1+\eta^l)}{k^u+\eta^u M_1} - \frac{\alpha^u}{m_2} \approx 1.66 > 0. \end{aligned}$$

Then by Theorem 3, we obtain that system (15) has a unique, globally attractive, positive, almost periodic solution.

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References

[1] F.E. Smith, Population dynamics in Daphnia magna, Ecology 44 (1963) 651-663.
[2] K. Gopalsamy, M.S.C. Kulenovic, G. Ladas, Time lags in a food-limited population model, Appl. Anal. 31 (1988) 225-237.
[3] A. Wan, J. Wei, Hopf bifurcation analysis of a food-limited population model with delay, Nonlinear Anal. Real World Appl. 11 (2010) 1087-1095.
[4] J.W.H. So, J.S. Yu, On the uniform stability for a food-limited population model with time delay, Proc. Roy. Soc. Edinburgh Sect. A 125 (1995) 991-1002.
[5] S.Y. Tang, L.S. Chen, Global attractivity in a food-limited population model with impulsive effects, J. Math. Anal. Appl. 292 (2004) 211-221.
[6] F.D. Chen, D.X. Sun, J.L. Shi, Periodicity in a food-limitied population model with toxicants and state dependent delays, J. Math. Anal. Appl. 288 (2003) 136-146.

[7] K. Gopalsamy, M.S.C. Kulenovic, G. Ladas, Environmental periodicity and time delay in a food-limited population model, J. Math. Anal. Appl. 147(1990) 545-555.
[8] J.J. Wang, J.R. Yan, On oscillation of a food-limited population model with impulse and delay, J. Math. Anal. Appl. 334 (2007) 349-357.
[9] Z.H. Zhou, Z.H. Yang, Periodic solutions in higher-dimensional Lotka-Volterra neutral competition systems with sate-dependent delays, Appl. Math. Comput. 189 (1) (2007) 986-995.
[10] X.Y. Song, Y.F. Li, Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect, Nonlinear Anal. Real World Appl. 9 (1) (2008) 64-79.
[11] C.X. Shen, Permanence and global attractivity of the food-chin system with Holling IV type functional response, Appl. Math. Comput. 194 (1) (2007) 179-185.
[12] Y.K. Li, P. Liu, Positive periodic solutons of a class of functional differential systems with feedback controls, Nonlinear Anal. 57 (2004) 655-666.
[13] B.S. Lalli, J. Yu, Feedback regulation of a logistic growth, Dynam. Systems Appl. 5 (1996) 117-124.
[14] Q. Liu, R. Xu, Persisrence and global stablity for a delayed nonautonomous single-species model with dispersal and feedback control, Differential Equations Dynam. Systems 11 (3-4) (2003) 353-367.
[15] Y.K. Li, Positive periodic solutions for neutral functional differential equations with distributed delays and feedback control, Nonlinear Anal. Real World Appl. 9 (2008) 2214-2221.
[16] F.D. Chen, Z. Li, Note on the permanence of a competitive system with infinite delay and feedback controls, Nonlinear Anal. Real World Appl. 8 (2007) 680-687.
[17] K. Gopalsamy, P. Wen, Feedback regulation of logistic growth, Internat.J. Math. Math. Sci. 16 (1993) 177-192.
[18] C.H, Feng, Y.J. Liu, Almost periodic solutions for delay Lotka-Volterra competitive systems, Acta Math. Appl. Sinica 28 (3) (2005) 459-465 (in chinese).
[19] C.L. Shi, Z. Li, F.D. Chen, The permanence and extinction of a nonlinear growth rate single-species non-autonomous dispersal models with time daleys, Nonlinear Anal. Real World Appl. 8 (5) (2007) 536-1550.
[20] F.D. Chen, Z. Li, Y.J. Huang, Note on the permanence of a competitive system with infinite delay and feedback controls, Nonlinear Anal. Real World Appl. 8 (2007) 680-687.
[21] Y. Nakata, Y. Muroya, Permanence of nonautonomous Lotka-Volterra cooperative systems with delays, Nolinear Anal. Real World Appl. 11 (2010) 528-534.
[22] F.D. Chen, Almost periodic solution of the non-autonomous two-species competitive model with stage structure, Appl. Math. Comput. 181 (2006) 685-693.
[23] Q. Wang, B.X. Dai, Almost periodic solution for n-species Lotka-Volterra competitive system with daley and feedback controls, Appl. Math. Comput. 200 (1) (2008) 133-146.
[24] C.Y. He, Almost Periodic Differential Equations, Higher Education Press, 1992. in Chinese.
[25] A.M. Fink, Almost periodic differential equations, Lecture Notes in Mathematics, Vol. 377, Springer-Verlag. Berlin, 1974.