

Dynamical behaviors in a discrete predator–prey model with a prey refuge

Kejun Zhuang, Zhaohui Wen

Abstract—By incorporating a prey refuge, this paper proposes new discrete Leslie–Gower predator–prey systems with and without Allee effect. The existence of fixed points are established and the stability of fixed points are discussed by analyzing the modulus of characteristic roots.

Keywords—Leslie–Gower; predator–prey model; prey refuge; Allee effect.

I. INTRODUCTION

THE dynamical relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. There have been plenty of papers about the dynamics on the predator–prey system with and without different kinds of functional responses. It is worth mentioning that the consequences of hiding behavior of prey on the dynamics of predator–prey interactions can be recognized significant.

In fact, the effects of prey refuges on the population dynamics are very complex in nature, but for modelling purposes, it can be considered as constituted by two components [2]: the first effects, which affect positively the growth of prey and negatively that of predators, comprise the reduction of prey mortality due to decrease in predation success. The second one may be the trad–offs and by–products of the hiding behavior of prey which could be advantageous or detrimental for all the interacting populations.

As far as we know, most of the works on predator–prey system with a prey refuge are only the continuous models governed by differential equations without time delay [3], [4]. However, many authors have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations [5], [6]. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. For example, the stability of a discrete–time predator–prey system with and without Allee effect was studied [7]. And no such work has been done for the discrete Leslie–Gower predator–prey model with a prey refuge.

As a result, the main purpose of this paper is to construct the discrete predator–prey system incorporating a prey refuge

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with the help of forward Euler scheme. The paper is organized as follows: in Section 2, the discrete predator–prey model without Allee effect is formulated. In Section 3, the existence and stability of fixed points are investigated. In Section 4, the discrete model with Allee effect is formulated and analyzed. Finally, Section 5 draws the conclusion.

II. THE MATHEMATICAL MODEL

In [8], [9], Leslie introduced the following predator–prey model where the "carrying capacity" of the predator's environment is proportional to the number of prey:

$$\begin{cases} \frac{dH}{dt} = (r_1 - a_1P - b_1H)H, \\ \frac{dP}{dt} = (r_2 - a_2\frac{P}{H})P, \end{cases} \quad (1)$$

where H , P represent the prey and predator density, respectively. r_1 , a_1 , b_1 , r_2 , a_2 are positive constants. The parameters r_1 and r_2 are the intrinsic growth rates of the prey and the predator, respectively. The value r_1/b_1 denotes the carrying capacity of the prey and r_2H/a_2 takes on the role of a prey-dependent carrying capacity for the predator. There have been many important and interesting results about system (1), such as the global stability, permanence, periodic solutions, almost periodic solutions and so on.

Stimulated by the works of [10], [11], [12], Chen extended system (1) by incorporating a refuge protecting mH of the prey as follows,

$$\begin{cases} \frac{dH}{dt} = (r_1 - b_1H)H - a_1(1-m)HP, \\ \frac{dP}{dt} = \left(r_2 - a_2\frac{P}{(1-m)H}\right)P, \end{cases} \quad (2)$$

where $m \in [0, 1)$. This leaves $(1-m)H$ of the prey available to the predator. For system (2), the stability property of the positive equilibrium was studied and the influence of the refuge was explicitly discussed.

In the present work, applying the forward Euler scheme to system (2), we obtain the discrete–time predator–prey system with a prey refuge as follows:

$$\begin{cases} H_{n+1} = H_n + \delta H_n [r_1 - b_1H_n - a_1(1-m)P_n], \\ P_{n+1} = P_n + \delta P_n \left(r_2 - a_2\frac{P_n}{(1-m)H_n}\right), \end{cases} \quad (3)$$

where δ is the step size and all the coefficients are positive constants. Notice that if the predator density disappears in this model, then the prey density satisfies the discrete logistic–type model.

III. EXISTENCE AND STABILITY OF FIXED POINTS

In this section, we first determine the existence of fixed points of (3), then investigate their stability by calculating the eigenvalues for the variational matrix of (3) at each fixed point.

Solving the following nonlinear equations

$$\begin{cases} H(r_1 - b_1H - a_1(1 - m)P) = 0, \\ P\left(r_2 - a_2\frac{P}{(1-m)H}\right) = 0, \end{cases}$$

we can get the two fixed points: $E_1(r_1/b_1, 0)$ and $E_2(H^*, P^*)$, where $H^* = \frac{r_1 a_2}{a_1 r_2 (1-m)^2 + a_2 b_2}$, $P^* = \frac{r_1 r_2 (1-m)}{a_1 r_2 (1-m)^2 + a_2 b_1}$. Obviously, E_2 is the only positive fixed point for all parameter values.

To study the stability of the fixed points of the model, we first give the useful lemma, which can be easily proved by the relations between roots and coefficients of a quadratic equation [13].

Lemma 3.1. Let $F(\lambda) = \lambda^2 - B\lambda + C$. Suppose that $F(1) > 0$, λ_1 and λ_2 are the two roots of $F(\lambda) = 0$. Then

- (i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;
- (ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;
- (iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;
- (iv) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $B \neq 0, 2$;
- (v) λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $B^2 - 4AC < 0$ and $C = 1$.

Let λ_1 and λ_2 are two eigenvalues of the fixed point. We recall some definitions of topological types for a fixed point (x, y) . (x, y) is called a *sink* if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. A sink is locally asymptotically stable. (x, y) is called a *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. A source is locally unstable. (x, y) is called a *saddle* if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$). And (x, y) is called *non-hyperbolic* if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

A. Stability of fixed point E_1

The Jacobian matrix of (3) at E_1 is given by

$$J_1 = \begin{bmatrix} 1 - \delta r_1 & -\frac{a_1 r_1 \delta (1-m)}{b_1} \\ 0 & 1 + \delta r_2 \end{bmatrix}.$$

The corresponding characteristic equation can be written as

$$\lambda^2 - (\text{tr}J_1)\lambda + \det J_1 = 0, \quad (4)$$

where $\text{tr}J_1$ is the trace and $\det J_1$ is the determinant of the Jacobian matrix J_1 . Hence the two eigenvalues of Jacobian matrix J_1 are $\lambda_1 = 1 - r_1\delta$ and $\lambda_2 = 1 + r_2\delta > 1$. According to Lemma 3.1, we can obtain that there are only two different topological types of E_1 for all permissible parameter values.

Proposition 3.2. The fixed point E_1 is a saddle if $r_1\delta > 1$; E_1 is a source if $r_1\delta < 1$.

B. Stability of interior fixed point E_2

Now, we shall discuss the stability of positive fixed point E_2 . The Jacobian matrix of (3) at E_2 is in the form of

$$J_2 = \begin{bmatrix} 1 - \delta b_1 H^* & -a_1 \delta (1 - m) H^* \\ r_2^2 \delta (1 - m) / a_2 & 1 - r_2 \delta \end{bmatrix}.$$

Then the characteristic equation is

$$\lambda^2 + p\lambda + q = 0, \quad (5)$$

where $p = 2 - \delta(r_2 + b_1 H^*)$, $q = (1 - \delta b_1 H^*)(1 - \delta r_2) + a_1 \delta^2 r_2^2 (1 - m)^2 H^* / a_2$.

Proposition 3.3. If $r_2\delta + b_1\delta H^* < 4$, then there exist at least four different topological types of $E_2(H^*, P^*)$ for all parameter values.

(i) E_2 is a sink if and only if both the following conditions hold:

- (i.1) $a_1 > \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (2r_2\delta + 2b_1\delta H^* - r_2 b_1 \delta^2 H^* - 4)$;
- (i.2) $a_1 < \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (b_1\delta H^* + r_2\delta - b_1 r_2 \delta^2 H^*)$.

(ii) E_2 is a saddle if and only if

$$a_1 < \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (2r_2\delta + 2b_1\delta H^* - r_2 b_1 \delta^2 H^* - 4).$$

(iii) E_2 is a source if and only if

$$a_1 > \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (b_1\delta H^* + r_2\delta - b_1 r_2 \delta^2 H^*).$$

(iv) E_2 is not hyperbolic if one of the following conditions holds:

- (iv.1) $a_1 = \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (2r_2\delta + 2b_1\delta H^* - r_2 b_1 \delta^2 H^* - 4)$;
- (iv.2) $a_1 = \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (b_1\delta H^* + r_2\delta - b_1 r_2 \delta^2 H^*)$.

Proof: Let $F(\lambda) = \lambda^2 + p\lambda + q$. Then

$$F(1) = r_2 b_1 \delta^2 H^* + \frac{a_1 r_2^2 \delta^2 (1 - m)^2 H^*}{a_2}$$

is always positive. By direct computation, we have $F(-1) > 0$ if and only if

$$a_1 > L = \frac{a_2}{\delta^2 r_2^2 (1 - m)^2 H^*} (2r_2\delta + 2b_1\delta H^* - r_2 b_1 \delta^2 H^* - 4),$$

and $q < 1$ if and only if

$$a_1 < M = \frac{a_2}{\delta^2 r_2^2 (1 - m)^2 H^*} (b_1\delta H^* + r_2\delta - b_1 r_2 \delta^2 H^*).$$

According to the assumption $r_2\delta + b_1\delta H^* < 4$, it can be concluded that $L < M$.

Due to Lemma 3.1, it is easy to see that, E_2 is a sink if $a_1 > \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (2r_2\delta + 2b_1\delta H^* - r_2 b_1 \delta^2 H^* - 4)$ and $a_1 < \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (b_1\delta H^* + r_2\delta - b_1 r_2 \delta^2 H^*)$; E_2 is a saddle if $a_1 < \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (2r_2\delta + 2b_1\delta H^* - r_2 b_1 \delta^2 H^* - 4)$; E_2 is a source if $a_1 > \frac{a_2}{\delta^2 r_2^2 (1-m)^2 H^*} (b_1\delta H^* + r_2\delta - b_1 r_2 \delta^2 H^*)$ and E_2 is non-hyperbolic for the other parameter values. ■

From Proposition 3.3, we can find that periodic oscillations may occur at some critical values.

IV. ALLEE EFFECT ON PREY POPULATION

In this section, we consider the predator prey system (3) as subject to an Allee effect on prey population and have the following system:

$$\begin{cases} H_{n+1} = H_n + \delta H_n \left[(r_1 - b_1 H_n) \frac{H_n}{u + H_n} - a_1 (1 - m) P_n \right], \\ P_{n+1} = P_n + \delta P_n \left(r_2 - a_2 \frac{P_n}{(1 - m) H_n} \right), \end{cases} \quad (6)$$

where we take $H_n/(u + H_n)$ as the Allee effect function and u as the Allee constant satisfying the assumption

$$\frac{a_2 r_1 - a_1 r_2 u + 2a_1 m r_2 u - a_1 m^2 r_2 u}{a_2 b_1 + a_1 r_2 - 2a_1 m r_2 + a_1 m^2 r^2} > 0. \quad (7)$$

If we choose the Allee constant $u = 0$ (i.e., if there is no Allee effect on the prey population), then system (6) reduces to (3) immediately. System (6) has two fixed points: $E_3(H_1^*, P_1^*)$ and $E_4(r_1/b_1, 0)$, where

$$H_1^* = \frac{a_2 r_1 - a_1 r_2 u + 2a_1 m r_2 u - a_1 m^2 r_2 u}{a_2 b_1 + a_1 r_2 - 2a_1 m r_2 + a_1 m^2 r^2},$$

$$P_1^* = \frac{r_2(1-m)H_1^*}{a_2}.$$

Next, we shall discuss the stability of positive fixed point $E_3(H_1^*, P_1^*)$. After some simple calculations, the Jacobian matrix of (6) at E_3 is

$$J_3 = \begin{bmatrix} \alpha & -a_1 \delta H_1^*(1-m) \\ r_2^2 \delta(1-m)/a_2 & 1 - r_2 \delta \end{bmatrix},$$

where $\alpha = 1 + \frac{\delta H_1^*(r_1 - 2b_1 H_1^*)}{u + H_1^*} + \frac{\delta H_1^{*3}(b_1 H_1^* - r_1)}{(u + H_1^*)^2}$. Then the characteristic equation is

$$F_1(\lambda) = \lambda^2 - (\text{tr} J_3)\lambda + \det J_3 = 0, \quad (8)$$

where

$$\text{tr} J_3 = \alpha + 1 - r_2 \delta$$

and

$$\det J_3 = \alpha(1 - r_2 \delta) + \frac{a_1 r_2^2 \delta^2 (1-m)^2 H_1^*}{a_2}.$$

Again by using Lemma 3.1, we obtain that the modulus of two roots of (8) is less than 1 (i.e., the positive fixed point J_3 is asymptotically stable) if and only if $F_1(1) > 0$, $F_1(-1) > 0$ and $\det J_3 < 1$.

We observe that $F_1(1) > 0$ holds if and only if $r_2 \delta(1-\alpha) + a_1 r_2^2 \delta^2 (1-m)^2 H_1^*/a_2 > 0$ holds. If $\alpha < 1$, then $F_1(1) > 0$.

Then we investigate the condition $F_1(-1) > 0$ when $\alpha < 1$. It implies that $F_1(-1) > 0$ holds if and only if $2 + \alpha - r_2 \delta(1 + \alpha) + \frac{a_1 r_2^2 \delta^2 (1-m)^2 H_1^*}{a_2} > 0$ holds. For simplicity, if $r_2 \delta < 1$, then $F_1(-1) > 0$.

Now, we can get the conclusions on the stability of fixed point E_3 .

Proposition 4.1. By assumption (7), the positive fixed point E_3 of system (6) is asymptotically stable if the follows conditions are satisfied:

- (i) $\alpha < 1$;
- (ii) $r_2 \delta < 1$;
- (iii) $\alpha(1 - r_2 \delta) + \frac{a_1 r_2^2 \delta^2 (1-m)^2 H_1^*}{a_2} < 1$.

V. CONCLUSION

In this paper, the new discrete Leslie–Gower predator–prey model with a prey refuge was proposed. Existence and stability of fixed points were investigated. Afterwards, the predator–prey model with Allee effect was considered. By mathematical analysis, we have shown the stability of the positive fixed point. However, it may be very complicated structure when our system is delayed and the predator population is subject to an Allee effect. Thus it would be very interesting to improve such structure in the future.

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