

# The Gerber-Shiu Functions of a Risk Model with Two Classes of Claims and Random Income

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*Abstract*—In this paper, we consider a risk model involving two independent classes of insurance risks and random premium income. We assume that the premium income process is a Poisson Process, and the claim number processes are independent Poisson and generalized Erlang(n) processes, respectively. Both of the Gerber-Shiu functions with zero initial surplus and the probability generating functions (p.g.f.) of the Gerber-Shiu functions are obtained.

*Keywords*—Poisson process; generalized Erlang risk process; Gerber-Shiu function; generating function; generalized Lundberg equation

## I. INTRODUCTION

RECENTLY, many authors have studied continuous-time risk models involving two classes of claims. Yuen et al. [2] consider the non-ruin probability for a correlated risk process involving two dependent classes of insurance risks, with exponential claims, which can be transformed into a surplus process with two independent classes of insurance risks, for which one claim number process is Poisson and the other is a renewal process with Erlang(2) claim inter-arrival times. Li and Garrido [3] consider a risk process with two classes of independent risks, namely the compound Poisson process and the renewal process with generalized Erlang(2) inter-arrivals times. In this paper the authors derive a system of integro-differential equations for the non-ruin probabilities and obtain explicit results (via Laplace transforms) for claim amounts having distributions belonging to the rational family ( $K_n$  class). They also derive the probability of the supremum value of the surplus process before ruin. Note that the model of Yuen et al. [6], by using standard properties for the sum of independent compound Poisson processes, can be reduced to the one proposed by Li and Garrido [3]. A further extension was given by Li and Lu [4]. They derive a system of integro-differential equations for the Gerber-Shiu expected discounted penalty functions, when the ruin is caused by a claim belonging either to the first or to the second class and obtained explicit results when the claim sizes are exponentially distributed. Recently, Chadjiconstantinidis and Papaioannou [1] further treated the model of Li and Lu [4]. In the absence of a constant dividend barrier, they proved that the Gerber-Shiu function satisfies some defective renewal equations. Exact representations for the solutions of these equations are derived through an associated compound geometric distribution and an analytic expression for this quantity was given when the claim severities have rationally distributed Laplace transforms. Further, the same risk model is considered in the presence

of a constant dividend barrier. Additionally, Zhang et al. [7] extended the model of Li and Lu [4], by considering the claim number process of the second class to be a renewal process with generalized Erlang(n) inter-arrival times. The authors derived an integro-differential equation system for the Gerber-Shiu functions, and obtained their Laplace transforms when the corresponding Lundberg equation has distinct roots. Also, in the same paper, they derived analytic expressions for the Gerber-Shiu functions when the claim size distributions belong to the rational family.

It should be noted that a common assumption in the above literatures is that the premium is collected continuously with positive deterministic constant. However, it is evident that the deterministic premium income fails to capture the uncertainty of the customers' arrivals. To reflect the cash flows of the insurance company more realistically, Yang and Zhang [5] considered the renewal risk model with generalized Erlang (n) inter-claim times and random premium income based on Poisson process. They studied some ruin-related quantities through the well-known Gerber-Shiu function. Motivated by Yang and Zhang [5] and Zhang et al. [7], in this paper we will consider a risk model with two classes of claims and random income which will be given in the next section.

The rest of the paper is organized as follows. In section 2, we give the description of the risk model and derive the recursive formulae satisfied by the expected discounted penalty (Gerber-Shiu) functions. In section 3, the generalized Lundberg's equation is given by using the martingale argument and discussed the number of its roots. Probability generating functions of Gerber-Shiu functions and explicit expressions for the Gerber-Shiu functions when the initial surplus is zero are derived in section 4.

## II. THE RISK MODEL AND GERBER-SHIU FUNCTION

Consider the following surplus process with random premium income

$$U(t) = u + M(t) - S(t), \quad t \geq 0 \quad (1)$$

where  $u \in \mathbb{N}$  is a nonnegative integer representing the initial surplus, the numbering process  $\{M(t), t \geq 0\}$  is a Poisson process with intensity  $\alpha > 0$  reflecting the cash flows of the insurance company, and the aggregate-claim process  $\{S(t), t \geq 0\}$  is generated from two classes of insurance risks, i.e.,

$$S(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0 \quad (2)$$

where the  $X_i$ 's are i.i.d. positive integer-valued random variables with common probability function  $p_k = P(X = k)$ , for

$k \in \mathbb{N}^+ (= \mathbb{N} - 0)$ , mean  $\mu_X$  and p.g.f.  $\hat{p}(s) = \sum_{k=1}^{\infty} s^k p_k$ ,  $s \in \mathbb{C}$ , denoting first class claim sizes, and  $Y_i$ 's are also i.i.d. positive integer-valued random variables with common probability function  $q_k = P(Y = k)$ , for  $k = 1, 2, \dots$ , mean  $\mu_Y$  and p.g.f.  $\hat{q}(s) = \sum_{k=1}^{\infty} s^k q_k$ ,  $s \in \mathbb{C}$ , denoting second class claim sizes. The counting process  $\{N_1(t), t \geq 0\}$ , representing the number of claims from the first class up to time  $t$ , is a Poisson process with intensity  $\beta$ . While the counting process  $\{N_2(t), t \geq 0\}$ , representing the number of claims from the second class up to time  $t$ , is a generalized Erlang( $n$ ) process with i.i.d inter-claim times  $\{W_i, i \leq 1\}$ . More precisely, let  $W = W_1$ , then  $W = V_1 + V_2 + \dots + V_n$ , where  $\{V_i\}_{i=1}^n$  are  $n$  independent exponential distributed random variables with  $E[V_i] = 1/\lambda_i$ .

Further assume that  $\{M(t), t \geq 0\}$ ,  $\{N_1(t), t \geq 0\}$ ,  $\{N_2(t), t \geq 0\}$ ,  $\{X_i, i \geq 1\}$  and  $\{Y_i, i \geq 1\}$  are independent, and we assume the net profit condition

$$\alpha > \beta\mu_X + \frac{\mu_Y}{\sum_{i=1}^n \frac{1}{\lambda_i}}$$

holds in order to have a positive loading factor.

Note that if  $\beta = 0$ , (1) is reduced to

$$U(t) = u + M(t) - \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0,$$

the surplus process in the Andersen model with generalized Erlang( $n$ ) inter-claim times and random income, which was discussed by Yang and Zhang [5].

Now define random variable  $T = \min\{t \geq 0 : U(t) < 0\}$ ,  $\infty$  otherwise, be the time of ruin, and denote the corresponding ruin probability by  $\psi(u) = P(T < \infty | U(0) = u)$ .

Further define random variable  $J$  to be the cause-of-ruin:  $J = k$ , if the ruin is caused by a claim from class  $k$ ,  $k = 1, 2$ ,

$$\psi_k(u) = P(T < \infty, J = k | U(0) = u)$$

is the ruin probability due to a claim from class  $k$ .

Let  $w(x, y), x \in \mathbb{N}, y \in \mathbb{N}^+$ , be a non-negative penalty function. For  $\delta \geq 0$ , we define

$$\phi_k(u) = E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty, J = k) | U(0) = u]$$

to be the Gerber-Shiu function at ruin if the ruin is caused by a claim from class  $k$ , where  $U(T-)$  is the surplus immediately before ruin and  $|U(T)|$  is the deficit at ruin,  $I(\cdot)$  is an indicator function. And the p.g.f of  $\phi_k(u)$  is define as  $\hat{\phi}_k(s) = \sum_{u=0}^{\infty} s^u \phi_k(u)$ ,  $k = 1, 2$ .

In what follows we need to introduce some auxiliary functions as in Zhang et al. [7]. Following the steps in Zhang et al. [7], we define the following modified claim number processes  $N_{2,j}(t)$  of  $N_2(t)$ . Let  $W_1^j = V_j + V_{j+1} + \dots + V_n$  be the time until the first claim occurs from the second class while the other inter-claim times are the same as that in  $N_2(t)$ . With all others being the same as in risk model (1), the only changes is to replace  $N_2(t)$  by  $N_{2,j}(t)$ . We define the modified risk process by  $U_j(t)$  with  $U_1(t) = U(t)$  and define the corresponding Gerber-Shiu function by  $\phi_{k,j}(u), k = 1, 2; j = 1, 2, \dots, n$ . Obviously, we have that  $\phi_{k,1}(u) = \phi_k(u), k = 1, 2$ .

Now we are ready to give the recursive equations for  $\phi_{k,j}(u), k = 1, 2$ .

For  $j = 1, 2, \dots, n-1$ , considering an infinitesimal time interval  $(0, dt)$  and using the total expectation formula, we have

$$\begin{aligned} e^{\delta dt} \phi_{1,j}(u) &= (1 - (\alpha + \beta + \lambda_j)dt) \phi_{1,j}(u) + \\ &\alpha dt (1 - \beta dt) (1 - \lambda_i dt) \phi_{1,j}(u+1) + \\ &\beta dt (1 - \alpha dt) (1 - \lambda_i dt) \times \\ &\left( \sum_{k=1}^u \phi_{1,j}(u-k) p_k + \sum_{k=u+1}^{\infty} w(u, k-u) p_k \right) \\ &+ \lambda_j dt (1 - \alpha dt) (1 - \beta dt) \phi_{1,j+1}(u) + o(dt). \end{aligned}$$

Substituting  $e^{\delta dt} = 1 + \delta dt + o(dt)$  into the above expression and then dividing both sides by  $o(dt)$  and letting  $o(dt) \rightarrow 0$ , one gets

$$\begin{aligned} e^{\delta dt} \phi_{1,j}(u) &= (1 - (\alpha + \beta + \lambda_j)dt) \phi_{1,j}(u) + \\ &\alpha dt (1 - \beta dt) (1 - \lambda_i dt) \phi_{1,j}(u+1) + \\ &\beta dt (1 - \alpha dt) (1 - \lambda_i dt) \times \\ &\left( \sum_{k=1}^u \phi_{1,j}(u-k) p_k + \sum_{k=u+1}^{\infty} w(u, k-u) p_k \right) \\ &+ \lambda_j dt (1 - \alpha dt) (1 - \beta dt) \phi_{1,j+1}(u) + o(dt). \end{aligned} \quad (3)$$

where  $A(u) = \sum_{k=u+1}^{\infty} w(u, k-u) p_k$ . But for  $\phi_{1,n}(u)$ , we have

$$\begin{aligned} e^{\delta dt} \phi_{1,n}(u) &= (1 - (\alpha + \beta + \lambda_n)dt) \phi_{1,n}(u) + \\ &\alpha dt (1 - \beta dt) (1 - \lambda_n dt) \phi_{1,n}(u+1) + \\ &\beta dt (1 - \alpha dt) (1 - \lambda_i dt) \times \\ &\left( \sum_{k=1}^u \phi_{1,j}(u-k) p_k + \sum_{k=u+1}^{\infty} w(u, k-u) p_k \right) + \\ &\lambda_n dt (1 - \alpha dt) (1 - \beta dt) \sum_{k=1}^u \phi_{1,1}(u-k) q_k + o(dt), \end{aligned}$$

which leads to

$$\begin{aligned} (\delta + \alpha + \beta + \lambda_n) \phi_{1,n}(u) &= \alpha \phi_{1,n}(u+1) + \beta \sum_{k=1}^u \phi_{1,n}(u-k) p_k \\ &+ \lambda_n \sum_{k=1}^u \phi_{1,1}(u-k) q_k + A(u). \end{aligned} \quad (4)$$

Similarly, we have

$$\begin{aligned} (\delta + \alpha + \beta + \lambda_j) \phi_{2,j}(u) &= \alpha \phi_{2,j}(u+1) + \beta \sum_{k=1}^u \phi_{2,j}(u-k) p_k \\ &+ \lambda_j \phi_{2,j+1}(u), \quad j = 1, 2, \dots, n-1, \end{aligned} \quad (5)$$

and

$$\begin{aligned} (\delta + \alpha + \beta + \lambda_n) \phi_{2,n}(u) &= \alpha \phi_{2,n}(u+1) + \beta \sum_{k=1}^u \phi_{2,n}(u-k) p_k \\ &+ \lambda_n \sum_{k=1}^u \phi_{2,1}(u-k) q_k + B(u), \end{aligned} \quad (6)$$

where  $B(u) = \sum_{k=u+1}^{\infty} w(u, k-u) q_k$ .

Now define  $\hat{\phi}_{k,j}(s) = \sum_{u=0}^{\infty} s^u \phi_{k,j}(u)$  to be the p.g.f of  $\phi_{k,j}(u), k = 1, 2; j = 1, 2, \dots, n$ , and  $\hat{A}(s) = \sum_{u=0}^{\infty} s^u A(u), \hat{B}(s) = \sum_{u=0}^{\infty} s^u B(u)$ . Then multiplying (3)-(4) by  $s^u$ , summing over  $u$  from 0 to  $\infty$  and applying some rearrangements, respectively, we obtain

$$\hat{\phi}_{1,j}(s) + \lambda_j s \hat{\phi}_{1,j+1}(s) = \alpha \hat{\phi}_{1,j}(0) - \beta \hat{A}(s), j = 1, 2, \dots, n-1,$$

and

$$\hat{\phi}_{1,n}(s) + \lambda_n s \hat{q}(s) \hat{\phi}_{1,1}(s) = \alpha \hat{\phi}_{1,n}(0) - \beta \hat{A}(s).$$

Put  $\hat{\Phi}_k(u) = (\phi_{k,1}(u), \phi_{k,2}(u), \dots, \phi_{k,n}(u))^T, \hat{\Phi}_k(s) = (\hat{\phi}_{k,1}(s), \hat{\phi}_{k,2}(s), \dots, \hat{\phi}_{k,n}(s))^T, k = 1, 2$ , and

$$\mathbf{B}(s) = \begin{bmatrix} \beta s \hat{p}(s) & \lambda_1 s & 0 & \cdots & 0 \\ 0 & \beta s \hat{p}(s) & \lambda_2 s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} s \\ \lambda_n s \hat{q}(s) & 0 & 0 & \cdots & \beta s \hat{p}(s) \end{bmatrix},$$

$\mathbf{A}_\delta(s) = \text{diag}(\alpha - s(\delta + \alpha + \beta + \lambda_1), \alpha - s(\delta + \alpha + \beta + \lambda_2), \dots, \alpha - s(\delta + \alpha + \beta + \lambda_n))$ . And set  $\mathbf{L}_\delta(s) = \mathbf{A}_\delta(s) + \mathbf{B}(s)$ . Then we can rewrite the equations about  $\hat{\phi}(s)$  in the following matrix form

$$\mathbf{L}_\delta(s) \hat{\Phi}_1(s) = \alpha \hat{\Phi}_1(0) - \beta s \hat{A}(s) \mathbf{e}, \quad (7)$$

where  $\mathbf{e}$  is a column vector of length  $n$  with all components being 1.

Similarly, from (5) and (6), we can obtain the following equation for  $\hat{\Phi}_2(s)$

$$\mathbf{L}_\delta(s) \hat{\Phi}_2(s) = \alpha \hat{\Phi}_2(0) - \lambda_n s \hat{q}(s) \mathbf{e}_n, \quad (8)$$

where  $\mathbf{e}_n$  is a column vector of length  $n$  with all the  $n$ th component being 1 and all others being 0.

Noting that we are only interested in  $\hat{\phi}_{k,1}(s) = \hat{\phi}_k(s), k = 1, 2$ . Eqs. (7) and (8) are two auxiliary equations that allow us to obtain the generating functions  $\phi_{k,1}(s), k = 1, 2$ , which heavily depend on the roots of a generalized Lundberg's equation to be discussed in the following section.

### III. A GENERALIZED LUNDBERG EQUATION

Let  $\tau_k = \sum_{i=1}^k W_i$  be the arrival time of the  $k$ th claim from the second class. Define  $U_0 = u$  and for  $k = 1, 2, \dots$ ,

$$U_k = U(\tau_k) = u + M(\tau_k) - \sum_{i=1}^k Y_i - \sum_{j=1}^{N_1(\tau_k)} X_j = u + \sum_{i=1}^k \left( M(W_i) - Y_i - \sum_{j=1}^{N_1(W_i)} X_j \right).$$

We seek a number  $s \in \mathbb{C}$  such that the process  $\{e^{-\delta \tau_k} s^{-U_k}; k \in \mathbb{N}\}$  will form a martingale. Here the martingale condition is

$$\begin{aligned} & E \left[ e^{-\delta W_1} s^{Y_1 + \sum_{j=1}^{N_1(W_1)} X_j - M(W_1)} \right] \\ &= E \left[ e^{-(\delta + \alpha + \beta(1 - \hat{p}(s)) - \frac{\alpha}{s}) W_1} \right] E [s^{Y_1}] \\ &= 1, \end{aligned}$$

which is equivalent to

$$s^n \hat{q}(s) \prod_{i=1}^n \lambda_i = \prod_{i=1}^n [s(\delta + \alpha + \beta + \lambda_i) - \alpha - \beta s \hat{p}(s)]. \quad (9)$$

Eq. (9) is a generalized version of Lundberg's equation.

In particular, when  $\beta = 0$ , we can see that (9) is reduced to

$$s^n \hat{q}(s) \prod_{i=1}^n \lambda_i = \prod_{i=1}^n [s(\delta + \alpha + \lambda_i) - \alpha]. \quad (10)$$

corresponds to the Eq. (12) of Yang and Zhang [5], and Yang and Zhang show that Eq. (10) has exactly  $n$  roots within the unit circle. We will show in the following theorem 1 that Eq. (9) also has exactly  $n$  roots within the unit circle.

**Theorem 1** For  $\delta > 0$ , the generalized Lundberg equation (9) has exactly  $n$  roots, say,  $\rho_1(\delta), \rho_2(\delta), \dots, \rho_n(\delta)$  within the unit circle.

*Proof* Using the notations in Section II, let  $\mathbf{L}_\delta(s, u) = \mathbf{A}_\delta(s) + u \mathbf{B}_\delta(s)$ . Obviously,  $\mathbf{L}_\delta(s) = \mathbf{L}_\delta(s, 1)$ . Denote the determinant of a matrix  $\mathbf{D}$  by  $|\mathbf{D}|$ . It is easy to check that

$$|\mathbf{L}_\delta(s)| = |\mathbf{L}_\delta(s, 1)| = (-1)^{n+1} s^n \hat{q}(s) \prod_{i=1}^n \lambda_i + \prod_{i=1}^n [\alpha - s(\delta + \alpha + \beta \lambda_i) + \beta s \hat{p}(s)]$$

Thus the equation  $|\mathbf{L}_\delta(s)| = 0$  is equivalent to Eq. (9).

For  $\delta > 0$ , Let  $\mathcal{C}$  denote the circle with its center at  $(R, 0)$  and radius  $1 - R$ , where  $R = \min_{1 \leq i \leq n} \frac{\alpha}{\delta + \alpha + \beta + \lambda_i}$ , and  $\mathcal{C}^+$ , the interior of  $\mathcal{C}$ . Denote the area  $\{s : |s - R| \geq 1 - R, |s| \leq 1\}$  by  $\mathcal{C}_1$ . We first prove that for  $0 \leq u \leq 1$ ,

$$\det(\mathbf{L}_\delta(s, u)) \neq 0, s \in \mathcal{C}_1.$$

Thus we only prove that Matrix  $\mathbf{L}_\delta(s, u)$  is diagonally dominant for  $0 \leq u \leq 1$  and  $s \in \mathcal{C}_1$ . Since for  $i = 1, 2, \dots, n-1$ ,

$$\begin{aligned} & |\alpha - s(\delta + \alpha + \beta + \lambda_i) + u \beta s \hat{p}(s)| \\ & \geq (\delta + \alpha + \beta + \lambda_i) \left[ \left| s - \frac{\alpha}{\delta + \alpha + \beta + \lambda_i} \right| - \frac{\beta u}{\delta + \alpha + \beta + \lambda_i} \hat{p}(1) \right] \\ & \geq (\delta + \alpha + \beta + \lambda_i) \left[ |s - R| - \left| R - \frac{\alpha}{\delta + \alpha + \beta + \lambda_i} \right| - \frac{\beta u}{\delta + \alpha + \beta + \lambda_i} \right] \\ & \geq (\delta + \alpha + \beta + \lambda_i) \left[ 1 - R - \left( \frac{\alpha}{\delta + \alpha + \beta + \lambda_i} - R \right) - \frac{\beta u}{\delta + \alpha + \beta + \lambda_i} \right] \\ & = \delta + \beta + \lambda_i - \beta u > \lambda_i \geq |\lambda_i u s| \end{aligned} \quad (11)$$

Similarly,

$$|\alpha - s(\delta + \alpha + \beta + \lambda_n) + u \beta s \hat{p}(s)| > \lambda_n \geq |\lambda_n u s \hat{q}(s)|.$$

The diagonal dominance implies that  $\det(\mathbf{L}_\delta(s, u)) \neq 0$  for  $0 \leq u \leq 1, s \in \mathcal{C}_1$ . Now let  $f(u)$  denote the number of roots of equation  $|\mathbf{L}_\delta(s, u)| = 0$  in  $\mathcal{C}^+$ . Then

$$f(u) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\partial}{\partial s} \frac{\det(\mathbf{L}_\delta(s, u))}{\det(\mathbf{L}_\delta(s, u))} ds.$$

Hence  $f(u)$  is a continuous function on  $[0, 1]$ , integer valued, and therefore constant. We have  $f(0) = n$  since  $|\mathbf{A}_\delta(s)| = 0$  has  $n$  roots  $s_i = \frac{\alpha}{\delta + \alpha + \beta + \lambda_i}, i = 1, 2, \dots, n$ . Thus,  $f(1) = n$ . While from (11) with  $u = 1$ , we know that  $|\mathbf{L}_\delta(s)| \neq 0$  for  $s \in \mathcal{C}_1$ . This completes the proof.  $\square$

**Remark 1** We have proved that there exist  $n$  roots of Eq.(9) within the unit circle, but also they are located inside the

circle  $\mathcal{C}$  with its center at  $(\min_{1 \leq i \leq n} \frac{\alpha}{\delta + \alpha + \beta + \lambda_i}, 0)$  and radius  $1 - \min_{1 \leq i \leq n} \frac{\alpha}{\delta + \alpha + \beta + \lambda_i}$ .

**Remark 2** It is easy to check that  $0 < |\rho_i(\delta)| < 1$  for  $i = 1, 2, \dots, n$ . Denote the root with the largest module by  $\rho_n(\delta)$ . If  $\delta \rightarrow 0^+$ , then  $\rho_i(\delta) \rightarrow \rho_i(0)$ , for  $i = 1, 2, \dots, n$  with  $\rho_n(\delta) \rightarrow 1$ . In the rest of this paper, the  $n$  roots are assumed to be distinct and denoted by  $\rho_1, \rho_2, \dots, \rho_n$  for simplicity.

#### IV. GENERATING FUNCTIONS OF GERBER-SHIU FUNCTIONS

In this section, we will derive the analytic expressions of  $\hat{\phi}_1(s)$  and  $\hat{\phi}_2(s)$ . For convenience, we introduce the same operator  $T_r$  as in Li [2] on real-valued functions with domain being the set of positive integers. Define  $T_r$  to be an operator on any real-valued function  $f(x)$ ,  $x \in \mathbb{N}^+$ , by

$$T_r f(y) = \sum_{x=y}^{\infty} r^{x-y} f(x) = \sum_{x=0}^{\infty} r^x f(x+y), \quad r \in \mathbb{C}, y \in \mathbb{N}^+. \quad (12)$$

It is clear that the generating function of  $f$ ,  $\hat{f}(s) = sT_s f(1)$ , and for distinct  $r_1, r_2$ ,

$$T_{r_1} T_{r_2} f(y) = \frac{r_2 T_{r_2} f(y) - r_1 T_{r_1} f(y)}{r_2 - r_1}. \quad (13)$$

Other properties of this operator can be found in Li [2]. Using above notations, and solving Eqs. (7) and (8), we can get the explicit expressions for  $\phi_1(0)$  and  $\phi_2(0)$ , and hence the generating functions  $\hat{\phi}_1(s)$  and  $\hat{\phi}_2(s)$  can be obtained.

**Theorem 2** If the generalized Lundberg equation (9) has  $n$  distinct roots within unit circle, then the Gerber-Shiu functions with zero initial surplus,  $\phi_1(0)$  and  $\phi_2(0)$ , are given by

$$\phi_1(0) = \frac{\beta \prod_{k=1}^{n-1} \lambda_k \prod_{k=1}^n \rho_k}{\alpha} \sum_{m=1}^n \frac{\rho_m^{n-1} \hat{A}(\rho_m) d(\rho_m)}{\prod_{k=1, k \neq m}^n (\rho_k - \rho_m) [\beta \rho_k \rho_m T_{\rho_k} T_{\rho_m} p(1) - \alpha]}, \quad (14)$$

and

$$\phi_2(0) = \frac{\prod_{k=1}^n \lambda_k \rho_k}{\alpha} \sum_{m=1}^n \frac{\rho_m^{n-1} \hat{B}(\rho_m)}{\prod_{k=1, k \neq m}^n (\rho_k - \rho_m) [\beta \rho_k \rho_m T_{\rho_k} T_{\rho_m} p(1) - \alpha]}, \quad (15)$$

where  $d(s) = 1 + \sum_{k=2}^n \prod_{j=k}^n \frac{s(\delta + \alpha + \beta + \lambda_j) - \alpha - \beta s \hat{p}(s)}{\lambda_{j-1} s}$ .

*Proof* From the proof of Theorem 1, we know that  $|\mathbf{L}_\delta(s)| = 0$  has  $n$  roots within unit circle, then we can determine nonzero

column vector  $\mathbf{d}_i$ ,  $i = 1, 2, \dots, n$  satisfying  $\mathbf{L}_\delta(\rho_i)^T \mathbf{d}_i = 0$ . Then from (7) and (8),

$$\mathbf{d}_i \Phi_1(0) = \frac{\beta}{\alpha} \rho_i \hat{A}(\rho_i) \mathbf{e}, \quad \mathbf{d}_i \Phi_2(0) = \frac{\lambda_n}{\alpha} \rho_i \hat{B}(\rho_i) \mathbf{e}_n. \quad (16)$$

It follows from the generalized Lundberg equation (9) that  $\mathbf{d}_i$  can be selected as follows

$$\mathbf{d}_i = \left( \prod_{k=2}^n \frac{\rho_i(\delta + \alpha + \beta + \lambda_k) - \alpha - \beta \rho_i \hat{p}(\rho_i)}{\lambda_{k-1} \rho_i}, \prod_{k=3}^n \frac{\rho_i(\delta + \alpha + \beta + \lambda_k) - \alpha - \beta \rho_i \hat{p}(\rho_i)}{\lambda_{k-1} \rho_i}, \dots, \frac{\rho_i(\delta + \alpha + \beta + \lambda_n) - \alpha - \beta \rho_i \hat{p}(\rho_i)}{\lambda_{n-1} \rho_i}, 1 \right)^T$$

Let  $\mathbf{D} := (d_{ij})_{n \times n} = (\mathbf{d}_1^T, \mathbf{d}_2^T, \dots, \mathbf{d}_n^T)$ , and let  $\mathbf{D}_{ij}$  denote the minor of  $\mathbf{D}$  with respect to row  $i$  and column  $j$ . Then (16) can be rewritten as

$$\mathbf{D} \Phi_1(0) = \frac{\beta}{\alpha} \begin{pmatrix} \rho_1 \hat{A}(\rho_1) d(\rho_1) \\ \rho_2 \hat{A}(\rho_2) d(\rho_2) \\ \vdots \\ \rho_n \hat{A}(\rho_n) d(\rho_n) \end{pmatrix}, \quad (17)$$

$$\mathbf{D} \Phi_2(0) = \frac{\lambda_n}{\alpha} \begin{pmatrix} \rho_1 \hat{B}(\rho_1) \\ \rho_2 \hat{B}(\rho_2) \\ \vdots \\ \rho_n \hat{B}(\rho_n) \end{pmatrix}. \quad (18)$$

Applying Cramér's rule to the above linear system gives

$$\phi_1(0) = \phi_{1,1}(0) = \frac{\beta}{\alpha} \frac{\begin{vmatrix} \rho_1 \hat{A}(\rho_1) d(\rho_1) & d_{12} & \cdots & d_{1n} \\ \rho_2 \hat{A}(\rho_2) d(\rho_2) & d_{2,2} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_n \hat{A}(\rho_n) d(\rho_n) & d_{n2} & \cdots & d_{nn} \end{vmatrix}}{|\mathbf{D}|}$$

Expanding the determinant in the numerator along the first column then yields

$$\phi_1(0) = \frac{\beta}{\alpha} \sum_{m=1}^n (-1)^{m+1} \frac{|\mathbf{D}_{m1}|}{|\mathbf{D}|} \rho_m \hat{A}(\rho_m) d(\rho_m), \quad (19)$$

After some careful calculations, we can get

$$\begin{aligned} & |\mathbf{D}| \\ &= \frac{1}{\lambda_1 \lambda_2^2 \cdots \lambda_{n-1}^{n-1}} \begin{vmatrix} 1 & \frac{\alpha}{\rho_1} + \beta \hat{p}(\rho_1) & \cdots & \left( \frac{\alpha}{\rho_1} + \beta \hat{p}(\rho_1) \right)^{n-1} \\ 1 & \frac{\alpha}{\rho_2} + \beta \hat{p}(\rho_2) & \cdots & \left( \frac{\alpha}{\rho_2} + \beta \hat{p}(\rho_2) \right)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\alpha}{\rho_n} + \beta \hat{p}(\rho_n) & \cdots & \left( \frac{\alpha}{\rho_n} + \beta \hat{p}(\rho_n) \right)^{n-1} \end{vmatrix} \\ &= \frac{1}{\lambda_1 \lambda_2^2 \cdots \lambda_{n-1}^{n-1}} \prod_{j,k=1, k>j}^n \left( \frac{\alpha}{\rho_k} + \beta \hat{p}(\rho_k) - \frac{\alpha}{\rho_j} - \beta \hat{p}(\rho_j) \right) \\ &= \frac{1}{\lambda_1 \lambda_2^2 \cdots \lambda_{n-1}^{n-1}} \prod_{j,k=1, k>j}^n \left( \frac{\rho_k - \rho_j}{\rho_k \rho_j} (\beta \rho_k \rho_j T_{\rho_k} T_{\rho_j} p(1) - \alpha) \right) \end{aligned} \quad (20)$$

and accordingly

$$|\mathbf{D}_{m1}| = \frac{1}{\lambda_2 \lambda_3^2 \cdots \lambda_{n-2}^{n-2}} \times \prod_{\substack{j, k=1, \\ j, k \neq m, k > j}}^n \left( \frac{\rho_k - \rho_j}{\rho_k \rho_j} (\beta \rho_k \rho_j T_{\rho_k} T_{\rho_j} p(1) - \alpha) \right). \quad (21)$$

Substituting the above two formulas into (19) yields (14).

Using similar arguments, we can obtain (15). This completes the proof.  $\square$

Next, we will derive the expressions of  $\hat{\phi}_k(s), k = 1, 2$ .

**Theorem 3** *If the generalized Lundberg equation (9) has  $n$  distinct roots within unit circle, then the generating functions of Gerber-Shiu functions,  $\hat{\phi}_1(s)$  and  $\hat{\phi}_2(s)$ , are given by*

$$\hat{\phi}_1(s) = \beta \prod_{k=1}^{n-1} \lambda_k \prod_{k=1}^n (s - \rho_k) \times \frac{\sum_{m=1}^n \prod_{k=1, k \neq m}^n \frac{\rho_m (\beta s \rho_k T_s T_{\rho_k} p(1) - \alpha)}{(\rho_m - \rho_k) (\beta \rho_m \rho_k T_{\rho_m} T_{\rho_k} p(1) - \alpha)} T_{\rho_m} T_s A(1) d(\rho_m)}{\prod_{i=1}^n (\alpha - s(\delta + \alpha + \beta + \lambda_i) + \beta s \hat{p}(s)) - s^n \hat{q}(s) \prod_{i=1}^n \lambda_i} \quad (22)$$

and

$$\hat{\phi}_2(s) = \prod_{k=1}^n (\lambda_k (s - \rho_k)) \times \frac{\sum_{m=1}^n \prod_{k=1, k \neq m}^n \frac{\rho_m (\beta s \rho_k T_s T_{\rho_k} p(1) - \alpha)}{(\rho_m - \rho_k) (\beta \rho_m \rho_k T_{\rho_m} T_{\rho_k} p(1) - \alpha)} T_{\rho_m} T_s B(1)}{\prod_{i=1}^n (\alpha - s(\delta + \alpha + \beta + \lambda_i) + \beta s \hat{p}(s)) - s^n \hat{q}(s) \prod_{i=1}^n \lambda_i} \quad (23)$$

*Proof* By (7), we have

$$\hat{\Phi}_1(s) = \frac{\mathbf{L}_{\delta}^*(s) (\alpha \Phi_1(0) - \beta s \hat{A}(s) \mathbf{e})}{|\mathbf{L}_{\delta}(s)|},$$

where  $\mathbf{L}_{\delta}^*(s)$  is the adjoint matrix of  $\mathbf{L}_{\delta}(s)$ . Then we have

$$\hat{\phi}_1(s) = \hat{\phi}_{1,1}(s) = \frac{\mathbf{L}_{\delta,1}^*(s) (\alpha \Phi_1(0) - \beta s \hat{A}(s) \mathbf{e})}{|\mathbf{L}_{\delta}(s)|}, \quad (24)$$

where

$$\begin{aligned} & \mathbf{L}_{\delta,1}^*(s) \\ &= \left( \prod_{k=2}^n (\alpha - s(\delta + \alpha + \beta + \lambda_k) + \beta s \hat{p}(s)), \right. \\ & \quad - \lambda_1 s \prod_{k=3}^n (\alpha - s(\delta + \alpha + \beta + \lambda_k) + \beta s \hat{p}(s)), \\ & \quad \lambda_1 \lambda_2 s^2 \prod_{k=4}^n (\alpha - s(\delta + \alpha + \beta + \lambda_k) + \beta s \hat{p}(s)), \\ & \quad \left. \dots, (-1)^{n+1} \prod_{k=1}^{n-1} \lambda_k s^{n-1} \right) \end{aligned} \quad (25)$$

is the first row of the adjoint matrix of  $\mathbf{L}_{\delta}^*(s)$ .

From (17), one gets

$$\Phi_1(0) = \frac{\beta}{\alpha |\mathbf{D}|} \begin{pmatrix} \sum_{m=1}^n (-1)^{m+1} \rho_m \hat{A}(\rho_m) d(\rho_m) |\mathbf{D}_{m1}| \\ \sum_{m=1}^n (-1)^{m+2} \rho_m \hat{A}(\rho_m) d(\rho_m) |\mathbf{D}_{m2}| \\ \vdots \\ \sum_{m=1}^n (-1)^{m+n} \rho_m \hat{A}(\rho_m) d(\rho_m) |\mathbf{D}_{mn}| \end{pmatrix}. \quad (26)$$

By similar matrix treatment as in Zhang et al. [7], by virtue of (20), (21), (25) and (26) we can verify that

$$\begin{aligned} & \alpha \mathbf{L}_{\delta,1}^*(s) \Phi_1(0) \\ &= (-1)^{n+1} \beta \prod_{i=1}^{n-1} \lambda_i \sum_{m=1}^n \rho_m \hat{A}(\rho_m) d(\rho_m) \times \\ & \quad \prod_{k=1, k \neq m}^n \frac{(s - \rho_k) (\beta s \rho_k T_s T_{\rho_k} p(1) - \alpha)}{(\rho_m - \rho_k) (\beta \rho_m \rho_k T_{\rho_m} T_{\rho_k} p(1) - \alpha)}, \end{aligned}$$

and

$$\beta s \hat{A}(s) \mathbf{L}_{\delta,1}^*(s) \mathbf{e} = (-1)^{n+1} \beta s^n \hat{A}(s) d(s) \prod_{i=1}^{n-1} \lambda_i.$$

Substituting the above two expressions into (24) and using the expression of  $|\mathbf{L}_{\delta}(s)|$ , we can get

$$\begin{aligned} \hat{\phi}_1(s) &= \beta \prod_{i=1}^{n-1} \lambda_i \times \\ & \frac{\sum_{m=1}^n \rho_m \hat{A}(\rho_m) d(\rho_m) \prod_{k=1, k \neq m}^n \frac{(s - \rho_k) (\beta s \rho_k T_s T_{\rho_k} p(1) - \alpha)}{(\rho_m - \rho_k) (\beta \rho_m \rho_k T_{\rho_m} T_{\rho_k} p(1) - \alpha)} - s^n \hat{A}(s) d(s)}{s^n \hat{q}(s) \prod_{i=1}^n \lambda_i - \prod_{i=1}^n (\alpha - s(\delta + \alpha + \beta + \lambda_i) + \beta s \hat{p}(s))} \end{aligned} \quad (27)$$

Since  $d(s) = 1 + \sum_{k=2}^n \prod_{j=k}^n \frac{\delta + \alpha + \lambda_j + \beta(1 - \hat{p}(s)) - \alpha/s}{\lambda_{j-1}}$ , by a change of variables  $y = \beta \hat{p}(s) + \alpha/s, y_k = \beta \hat{p}(\rho_k) + \alpha/\rho_k, k = 1, 2, \dots, n$ , the function  $d(s)$  is reduced to

$$l(y) = d(s) = 1 + \sum_{k=2}^n \prod_{j=k}^n \frac{\delta + \alpha + \lambda_j + \beta - y}{\lambda_{j-1}},$$

which is a polynomial function of  $y$  with degree  $n - 1$ . Using Lagrange interpolation formula yields

$$l(y) = \sum_{m=1}^n \prod_{k=1, k \neq m}^n \frac{y - y_k}{y_m - y_k} l(y_k),$$

which leads to

$$\begin{aligned} d(s) &= \sum_{m=1}^n d(\rho_m) \times \\ & \quad \prod_{k=1, k \neq m}^n \frac{\beta \hat{p}(s) + \alpha/s - \beta \hat{p}(\rho_k) - \alpha/\rho_k}{\beta \hat{p}(\rho_m) + \alpha/\rho_m - \beta \hat{p}(\rho_k) - \alpha/\rho_k} \\ &= \frac{1}{s^{n-1}} \sum_{m=1}^n \rho_m^{n-1} \times \\ & \quad \prod_{k=1, k \neq m}^n \frac{(s - \rho_k) (\beta s \rho_k T_s T_{\rho_k} p(1) - \alpha)}{(\rho_m - \rho_k) (\beta \rho_m \rho_k T_{\rho_m} T_{\rho_k} p(1) - \alpha)}, \end{aligned}$$

Substituting the above equation into (27) and after some calculations, we can get (22). Similar to the proof of  $\hat{\phi}_1(s)$ ,

by (8), (18), (20) and (21) we can obtain

$$\hat{\phi}_2(s) = \prod_{i=1}^n \lambda_i \frac{\sum_{m=1}^n \rho_m^n \hat{B}(\rho_m) \prod_{k=1, k \neq m}^n \frac{(s-\rho_k)(\beta s \rho_k T_s T_{\rho_k} p(1)-\alpha)}{(\rho_m-\rho_k)(\beta \rho_m \rho_k T_{\rho_m} T_{\rho_k} p(1)-\alpha)} - s^n \hat{B}(s)}{s^n \hat{q}(s) \prod_{i=1}^n \lambda_i - \prod_{i=1}^n (\alpha - s(\delta + \alpha + \beta + \lambda_i) + \beta s \hat{p}(s))},$$

which can be rewritten as (23). This ends the proof.

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