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On Frenet-Serret Invariants of Non-Null Curves in Lorentzian Space L^5

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Abstract**—**The aim of this paper is to determine Frenet-Serret invariants of non-null curves in Lorentzian 5-space. First, we define a vector product of four vectors, by this way, we present a method to calculate Frenet-Serret invariants of the non-null curves. Additionally, an algebraic example of presented method is illustrated.

Keywords**—**Lorentzian 5-space; Frenet-Serret Invariants; Non-null Curves.

I. INTRODUCTION

AT the beginning of the 20th century, A. Einstein's theory opened a door to new geometries such as Lorentzian Geometry, which is simultaneously the geometry of special relativity, was established. Thereafter, researchers discovered a bridge between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. They adapted the geometrical models to relativistic motion of charged particles. Consequently, the theory of the curves has been one of the most fascinating topic for such modeling process. As it stands, the Frenet-Serret formalism of a relativistic motion describes the dynamics of the charged particles.

In recent years, mentioned modelings have been extended to higher dimensional spaces. In the case of a differentiable curve, at each point a set of mutually orthogonal unit vectors was defined and constructed, and therefore, the rates of change of these vectors along the curve define its curvatures in the space. Since, a new area was given to geometers' hands. There exists a vast literature on this subject, for instance, one can see [1,2,3,5,6,8].

A curve of constant slope or general helix is defined by the property that the tangent lines make a constant angle with a fixed direction. A necessary and sufficient condition that a curve to be general helix is that ratio of curvature to torsion is constant. Indeed, a *helix* is a special case of the general helix. If both curvature and torsion are non-zero constants, it is called a helix or only a W-curve.

In this work, we develop a method to determine Frenet-Serret invariants (apparatus) of the non-null curves in five dimensional Lorentzian space in the spirit of the paper [7], which contains regular observations in five dimensional Euclidean space. We also present an algebraic example of the presented method.

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II. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbf{L}^5$ are briefly presented (A more complete treatment can be found in [5])

Lorentzian space $\mathbf{L}^5$ is a pseudo-Euclidean space $\mathbf{E}^5$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^3 + dx_3^2 + dx_4^2 + dx_5^2, \tag{1}$$

where $(x_1, x_2, x_3, x_4, x_5)$ is a rectangular coordinate system in $\mathbf{L}^5$. Since g is an indefinite metric, recall that a vector $\vec{v} \in \mathbf{L}^5$ can have one of the three causal characters; it can be space-like if $g(\vec{v},\vec{v})\rangle 0$ or $\vec{v}=0$, time-like if $g(\vec{v},\vec{v})\langle 0$ and null (light-like) if $g(\vec{v},\vec{v})=0$ and $\vec{v}\neq 0$. Similarly, an arbitrary curve $\vec{\gamma}=\vec{\gamma}(s)$ in $\mathbf{L}^5$ can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\vec{\gamma}'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{|g(\vec{v},\vec{v})|}$. Therefore, $\vec{v}$ is a unit vector if $g(\vec{v},\vec{v}) = \pm 1$. Next, vectors $\vec{v},\vec{w}$ in $\mathbf{L}^5$ are said to be orthogonal if $g(\vec{v},\vec{w})=0$. The velocity of the curve $\vec{\gamma}$ is given by $\|\vec{\gamma}'\|$. Thus, a space-like or a time-like curve $\vec{\gamma}$ is said to be parametrized by arclength function s, if $g(\vec{\gamma}',\vec{\gamma}')=\pm 1$. The Lorentzian hypersphere of center $\vec{m}=(m_1,m_2,m_3,m_4,m_5)$ and radius $r\in R^+$ in the space $\mathbf{L}^5$ defined by

$$S_1^3 = \left\{\vec{\alpha}=(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\in L^5 : g(\vec{\alpha}-\vec{m},\vec{\alpha}-\vec{m}) = r^2\right\}. \tag{2}$$

Denote by $\left\{\vec{V}_1(s),\vec{V}_2(s),\vec{V}_3(s),\vec{V}_4(s),\vec{V}_5(s)\right\}$ the moving Frenet-Serret frame along the curve $\vec{\gamma}(s)$ in the space $\mathbf{L}^5$. Then, for a non-null unit speed curve of $\mathbf{L}^5$, the following Frenet-Serret equations are given in [2]:

$$\begin{bmatrix} \vec{V}_1' \\ \vec{V}_2' \\ \vec{V}_3' \\ \vec{V}_4' \\ \vec{V}_5' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & 0 \\ -\varepsilon_0\varepsilon_1\kappa_1 & 0 & \kappa_2 & 0 & 0 \\ 0 & -\varepsilon_1\varepsilon_2\kappa_2 & 0 & \kappa_3 & 0 \\ 0 & 0 & -\varepsilon_2\varepsilon_3\kappa_3 & 0 & \kappa_4 \\ 0 & 0 & 0 & -\varepsilon_3\varepsilon_4\kappa_4 & 0 \end{bmatrix} \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \\ \vec{V}_4 \\ \vec{V}_5 \end{bmatrix} \tag{3}$$

where $g(\vec{V}_j, \vec{V}_j) = \varepsilon_{j-1} = \mp 1$ for $1 \le j \le 5$, according to character of frame vector. Here, κ_i are the curvature functions, as well. And, we shall call the set whose elements are curvature functions and Frenet-Serret frame vector fields as *Frenet-Serret invariants* of the curves. Here, recall that, an arbitrary curve is called a *helix*, if it has constant Frenet-Serret curvatures.

III. A New Way to Determine Frenet-Serret Invariants of the Non-Null Curves in L^5

In this section, first, we define a vector product as follows:

Definition 1. Let $\vec{a}_1 = (a_{11}, a_{12}, a_{13}, a_{14}, a_{15})$, $\vec{a}_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25})$, $\vec{a}_3 = (a_{31}, a_{32}, a_{33}, a_{34}, a_{35})$ and $\vec{a}_4 = (a_{41}, a_{42}, a_{43}, a_{44}, a_{45})$ be vectors of $\mathbf{L}^5$. The vector product of $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{a}_4$ is defined with the determinant

$$\vec{a}_1 \wedge \vec{a}_2 \wedge \vec{a}_3 \wedge \vec{a}_4 = -\begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 & \vec{e}_5 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{vmatrix}, \tag{4}$$

where $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ and $\vec{e}_5$ are coordinate direction vectors of $\mathbf{L}^5$ which satisfy

$$\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 = -\vec{e}_5,\ \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 = -\vec{e}_1,$$
$$\vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_1 = -\vec{e}_2,$$
$$\vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_1 \wedge \vec{e}_2 = -\vec{e}_3,\ \vec{e}_5 \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 = -\vec{e}_4.$$

Remark 2. Let us denote above vector by $\vec{\Gamma} = \vec{a}_1 \wedge \vec{a}_2 \wedge \vec{a}_3 \wedge \vec{a}_4$. Suffice it to say that $g(\vec{a}_1, \vec{\Gamma}) = g(\vec{a}_2, \vec{\Gamma}) = g(\vec{a}_3, \vec{\Gamma}) = g(\vec{a}_4, \vec{\Gamma}) = 0.$

Let $\vec{\gamma} = \vec{\gamma}(s)$ be a non-null unit speed curve in $\mathbf{L}^5$. One can write the following differentiations with respect to s:

$$\begin{cases} \vec{\gamma}' = \vec{V}_1, \\ \vec{\gamma}'' = \kappa_1 \vec{V}_2, \\ \vec{\gamma}''' = -\varepsilon_0\varepsilon_1\kappa_1^2 \vec{V}_1 + \kappa_1' \vec{V}_2 + \kappa_1\kappa_2 \vec{V}_3, \\ \vec{\gamma}^{(IV)} = \begin{pmatrix} (-3\varepsilon_0\varepsilon_1\kappa_1\kappa_1')\vec{V}_1 + (-\varepsilon_0\varepsilon_1\kappa_1^3 - \varepsilon_1\varepsilon_2\kappa_1\kappa_2^2 + \kappa_1'')\vec{V}_2 \\ + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')\vec{V}_3 + (\kappa_1\kappa_2\kappa_3)\vec{V}_4 \end{pmatrix} \\ \vec{\gamma}^{(V)} = (\ldots)\vec{V}_1 + (\ldots)\vec{V}_2 + (\ldots)\vec{V}_3 + (\ldots)\vec{V}_4 + (\kappa_1\kappa_2\kappa_3\kappa_4)\vec{V}_5 \end{cases} \tag{5}$$

From first equation of $(5)_1$, we know first vector field of Frenet-Serret frame. Thereafter, by means of $(5)_2$, we express

$$\|\vec{\gamma}''\| = \kappa_1 \tag{6}$$

and

$$\vec{V}_2 = \frac{\vec{\gamma}''}{\kappa_1}. \tag{7}$$

To determine the third vector field of the Frenet-Serret frame, we form

$$\|\vec{\gamma}''\|^2 (\varepsilon_1 \vec{\gamma}''' + \varepsilon_0 \|\vec{\gamma}''\|^2 \vec{\gamma}') - g(\vec{\gamma}'', \vec{\gamma}''')\vec{\gamma}'' = \varepsilon_1 \kappa_1^3 \kappa_2 \vec{V}_3. \tag{8}$$

Since, we immediately arrive at

$$\vec{V}_3 = \frac{\|\vec{\gamma}''\|^2 (\varepsilon_1 \vec{\gamma}''' + \varepsilon_0 \|\vec{\gamma}''\|^2 \vec{\gamma}') - g(\vec{\gamma}'', \vec{\gamma}''')\vec{\gamma}''}{\left\| \|\vec{\gamma}''\|^2 (\varepsilon_1 \vec{\gamma}''' + \varepsilon_0 \|\vec{\gamma}''\|^2 \vec{\gamma}') - g(\vec{\gamma}'', \vec{\gamma}''')\vec{\gamma}'' \right\|}. \tag{9}$$

Using (9), we write

$$\kappa_2 = \frac{\left\| \|\vec{\gamma}''\|^2 (\varepsilon_1 \vec{\gamma}''' + \varepsilon_0 \|\vec{\gamma}''\|^2 \vec{\gamma}') - g(\vec{\gamma}'', \vec{\gamma}''')\vec{\gamma}'' \right\|}{\|\vec{\gamma}''\|^3}. \tag{10}$$

Now, we shall investigate the fifth vector field with the following vector product with respect to Frenet-Serret frame. Thus, we express

$$\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} = -\kappa_1^2 \kappa_2^2 \kappa_3 \vec{V}_5. \tag{11}$$

Using obtained equations, we have

$$\vec{V}_5 = \mu \frac{\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)}}{\left\| \vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \right\|} \tag{12}$$

and

$$\kappa_3 = \frac{\left\| \vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \right\|}{\left[g(\vec{\gamma}''', \vec{V}_3) \right]^2}, \tag{13}$$

where μ is taken ∓ 1 to make $+1$ determinant of $[\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5]$ matrix. In terms of (11) and (13), we have the fourth curvature as follows:

$$\kappa_4 = \frac{\varepsilon_2 g(\vec{\gamma}''', \vec{V}_3) g(\vec{\gamma}^{(V)}, \vec{V}_5)}{\varepsilon_4 \left\| \vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} \right\|}. \tag{14}$$

Finally, considering above equations, one can calculate the fourth vector field by

$$\vec{V}_4 = \mu \vec{V}_2 \wedge \vec{V}_3 \wedge \vec{V}_1 \wedge \vec{V}_5. \tag{15}$$

IV. An Algebraic Example of the Presented Method

Let us consider the following curve

$$\vec{\delta} = \vec{\delta}(s) = (\sqrt{3}\sinh s, \sqrt{3}\cosh s, \sin s, s, \cos s). \tag{16}$$

Differentiating (16), we may write

$$\vec{\delta}'(s) = (\sqrt{3}\cosh s, \sqrt{3}\sinh s, \cos s, 1, -\sin s). \tag{17}$$

By the inner product, we may write $g(\vec{\delta}', \vec{\delta}') = -1$. One can easily see that $\vec{\delta} = \vec{\delta}(s)$ is an unit speed time-like curve. In this case, we may take $\varepsilon_0 = -1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$. Moreover, the equation (17) is congruent to $\vec{V}_1$. In order to determine Frenet-Serret invariants of this curve, we express the following differentiations:

$$\begin{cases} \vec{\delta}'' = (\sqrt{3}\sinh s, \sqrt{3}\cosh s, -\sin s, 0, -\cos s) \\ \vec{\delta}''' = (\sqrt{3}\cosh s, \sqrt{3}\sinh s, -\cos s, 0, \sin s) \\ \vec{\delta}^{(IV)} = (\sqrt{3}\sinh s, \sqrt{3}\cosh s, \sin s, 0, \cos s) \\ \vec{\delta}^{(V)} = (\sqrt{3}\cosh s, \sqrt{3}\sinh s, \cos s, 0, -\sin s) \end{cases}. \quad (18)$$

By the first equation of (18), we have the first curvature as

$$\left\|\vec{\delta}''\right\| = \kappa_1 = 2 = const. \quad (19)$$

Since, we get

$$\vec{V}_2 = \left(\frac{\sqrt{3}}{2}\sinh s, \frac{\sqrt{3}}{2}\cosh s, -\frac{1}{2}\sin s, 0, -\frac{1}{2}\cos s\right). \quad (20)$$

To determine the third vector field, we write

$$\begin{aligned} &\left\|\vec{\delta}''\right\|^2(\vec{\delta}''' - \left\|\vec{\delta}''\right\|^2\vec{\delta}') - g(\vec{\delta}'', \vec{\delta}''')\vec{\delta}'' = \\ &4(-3\sqrt{3}\cosh s, -3\sqrt{3}\sinh s, -5\cos s, -4{,}5\sin s). \end{aligned} \quad (21)$$

Thus, we have, respectively,

$$\vec{V}_3 = \frac{1}{\sqrt{14}}(-3\sqrt{3}\cosh s, -3\sqrt{3}\sinh s, -5\cos s, -4{,}5\sin s) \quad (22)$$

and

$$\kappa_2 = \sqrt{14} = const. \quad (23)$$

The vector product $\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)}$ gives us

$$\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)} = (-\sqrt{3}\cosh s, -\sqrt{3}\sinh s, 3\cos s, -6, -3\sin s). \quad (24)$$

Furthermore, the norm of (24) can be calculated as $\left\|\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{\gamma}''' \wedge \vec{\gamma}^{(IV)}\right\| = \sqrt{42}$. Thereby, $\vec{V}_5$ may be formed

$$\vec{V}_5 = \mu\left(-\frac{1}{\sqrt{14}}\cosh s, -\frac{1}{\sqrt{14}}\sinh s, \sqrt{\frac{3}{14}}\cos s, -\sqrt{\frac{2}{7}}, -\sqrt{\frac{3}{14}}\sin s\right). \quad (25)$$

In terms of above equations, we can write third and fourth curvatures of the curve, respectively,

$$\kappa_3 = \sqrt{\frac{3}{14}} = const. \quad (26)$$

and

$$\kappa_4 = \sqrt{\frac{2}{7}} = const. \quad (27)$$

Finally, we have

$$\vec{V}_4 = \mu\left(-\frac{1}{2}\sinh s, -\frac{1}{2}\cosh s, -\frac{3}{2}\sin s, 0, -\frac{3}{2}\cos s\right). \quad (28)$$

Corollary 3.

i) $\left\{\vec{V}_1(s), \vec{V}_2(s), \vec{V}_3(s), \vec{V}_4(s), \vec{V}_5(s)\right\}$ is an orthonormal frame of $\mathbf{L}^5$.

ii) $\vec{\delta} = \vec{\delta}(s)$ is a time-like helix.

V. Conclusion and Further Remarks

Throughout the presented paper, one of the recent topic in the theory of the curves in Lorentzian space was treated. In the recent papers, although Frenet-Serret frame vectors and curvatures are defined, there was not an explicit method with a vector product to calculate Frenet-Serret invariants of an unit speed non-null curve which lies fully in $\mathbf{L}^5$. We have determined in an explicit way. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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